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Classification of sequentially weakly continuous JB^*-triples

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Abstract. Let D be the open unit ball of a JB^*-triple A and let $Aut(D)$ be the group of all biholomorphic automorphisms of D. It is shown that every element of $Aut(D)$ is sequentially weakly continuous if and only if every primitive ideal of A is a maximal closed ideal and A^{**} is a type I JBW^*-triple without infinite-spin part. Implications for general structure theory are explored. In particular, it is deduced that every JB^*-triple A contains a smallest ideal J for which the sequentially weakly continuous biholomorphic automorphisms of the open unit ball of A/J are all linear.

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1 Introduction

Kaup and Upmeier [31] (see also [39]) analysed complete holomorphic vector fields on the open unit ball D of a complex Banach space A to uncover a closed subspace V of A and partial Jordan triple product $\{\ \}: A \times V \times A \to A$ which, via the group $Aut(D)$ of all biholomorphic automorphisms of D, they use to show that A is completely determined as a Banach space by the holomorphic structure of D. When $A = V$, A is said to be a JB^*-triple and, by a deep result of Kaup [27] is characterised by a certain normed ternary Jordan algebraic structure (see Sect. 2). A Gelfand-Naimark type theorem due to Friedman and Russo [15] proves that most JB^*-triples are the J^*-algebras (hereafter referred to as JC^*-triples) of Harris [20] which are, for arbitrary Hilbert spaces H and K, the norm closed subspaces of $B(H,K)$ algebraically closed under the triple product

$$\{a\ b\ c\} = \frac{1}{2}(ab^*c + cb^*a)\ .$$

The class of JC^*-triples is stable under the action of contractive projections [14] (as is the category of all JB^*-triples by a result of Kaup [28] and Stachó [36]) and contains all Hilbert spaces, spin factors, C^*-algebras and most Jordan C^*-algebras.

Weak continuity and sequential weak continuity of elements in $Aut(D)$ and of certain natural maps on A, where A is a JB^*-triple, have been considered in a number of recent papers [37, 24, 30, 25, 10] variously to investigate weakly continuous 1-parameter subgroups of $Aut(D)$ and to explore structure in A. The JB^*-triple A is said to be *weakly continuous* if all elements in $Aut(D)$ are weakly continuous and is said to be *sequentially weakly continuous* accordingly.

Kaup and Stachó [30], in equivalent terms, prove that A is weakly continuous if and only if all primitive ideals of A are maximal closed ideals and A^{**} is an ℓ^∞-sum of Cartan factors none of which are infinite dimensional spin factors. In particular, for a locally compact Hausdorff space X, $C_0(X)$ is weakly continuous precisely when X is scattered.

On the other hand results of Isidro and Kaup [24] show that every abelian JB^*-triple is sequentially weakly continuous.

Our purpose in this paper is to classify sequentially weakly continuous JB^*-triples and to consider implications for general structure theory. We show that the sequentially weakly continuous JB^*-triples A are precisely those for which primitive ideals are maximal closed ideals and A^{**} is a type I JBW^*-triple without infinite spin part. We further show that every JB^*-triple A contains a smallest closed ideal J such that the sequentially weakly continuous biholomorphic automorphisms of the open unit ball of A/J are all linear. It follows that the sequentially weakly continuous C^*-algebras are precisely the liminal C^*-algebras.

We make use of recent results in representation theory [9] and we introduce and exploit as a useful device the class of JB^*-triples whose second dual is a type I JBW^*-triple.

2 Notations and preliminaries

A JB^*-triple is a complex Banach space with a continuous ternary product $(a,b,c) \mapsto \{a\,b\,c\}$ symmetric and bilinear in a and c and conjugate linear in b, for which $\|\{a\,a\,a\}\| = \|a\|^3$ and $x \mapsto \{a\,a\,x\}$ is positive hermitian operator on A, satisfying

$$\{a\,b\{x\,y\,z\}\} = \{\{a\,b\,x\}y\,z\} + \{x\,y\{a\,b\,z\}\} - \{x\{b\,a\,y\}z\}\ .$$

A subspace I of A is said to be an ideal of A if $\{A\,I\,A\} + \{A\,A\,I\} \subset I$ and to be an *inner ideal* of A if $\{I\,A\,I\} \subset I$. The norm closed ideals of A are its M-ideals [4]. A JBW^*-triple is a JB^*-triple with a (unique) predual in case of which the triple product is separately weak * continuous [4], [11], [21].

Recall [40] that JB^*-algebras are the complexifications of JB-algebras. We use [19] as our standard reference for JB-algebras and JB^*-algebras.

A tripotent e of A is an element satisfying $e = \{e\,e\,e\}$ the inner ideal of A generated by which, $A(e) = \{eA\,e\}\ \ (= \{e\{eA\,e\}e\})$, is a JB^*-algebra with product $a \circ b = \{a\,e\,b\}$ and involution $a^{\#} = \{e\,a\,e\}$; it is a JBW^*-algebra if A is a JBW^*-triple. A tripotent e of A is said to be *complete* if $\{e\,e\,x\} = 0$ implies $x = 0$, to be unitary if $A(e) = A$ and to be *minimal* if non-zero and $A(e) = \mathbb{C}e$. Given $\rho \in \partial_e(A_1^*)$ (the extreme boundary of A_1^*) there is a unique minimal tripotent e of A^{**} with $\rho(e) = 1$, called the support $s(\rho)$ of ρ [13].

The JBW^*-triples of premier importance and which are fundamental to representation theory ([9]) are the Cartan factors. Let H and K be Hilbert spaces, let $j\ :\ H \to H$ be a conjugation and let $\mathbb{O}$ denote the complex octonians. The six kinds of Cartan factors are described as follows.

(1) Rectangular: $B(H, K)$
(2) Hermitian: $\{x \in B(H) \mid x = j\,x^* j\}$
(3) Symplectic: $\{x \in B(H) \mid x = -j\,x^* j\}$
(4) Spin factor: H with $\dim(H) \geq 3$ with product $\{x\,y\,z\} = \frac{1}{2}[\langle x, y\rangle z + \langle z, y\rangle x - \langle x, j\,z\rangle j\,y]$ and norm given by $\|x\|^2 = \langle x, x\rangle + (\langle x, x\rangle^2 - |\langle x, j\,x\rangle|^2)^{1/2}$
(5) $B_{1,2}$: The 1×2 matrices over $\mathbb{O}$
(6) M_3^8: The hermitian 3×3 matrices over $\mathbb{O}$.

A JB^*-triple is said to be *elementary* if it is isometric (hence isomorphic [27]) to the norm closed ideal, $K(M)$, generated by the minimal tripotents in a Cartan factor M. We have $K(M)^{**} = M$ and that A is elementary if and only if A^{**} is a Cartan factor [7]. By a Cartan factor representation, $\pi\ :\ A \to M$, we mean a (triple) homomorphism from a JB^*-triple A into a Cartan factor M such that $\overline{\pi A} = M$, where the bar denotes weak * closure. The weak * closed ideal A_ρ^{**} of A^{**} generated by $s(\rho)$ when $\rho \in\ \partial\, e(A_1^*)$ is a Cartan factor and the restriction to A of the natural projection (cf [21]) $A^{**} \to A_\rho^{**}$ is a Cartan factor representation, $\pi_\rho : A \to A_\rho^{**}$. The primitive ideals of A (i.e. primitive M-ideals) of A are the kernels of the Cartan factor representations of A. The set of all primitive ideals of A, $\mathrm{Prim}(A)$, is regarded as a topological space in the usual way via the hull-kernel topology. See [9] for further details. $\mathrm{Max}(A)$ denotes the set of all maximal M-ideals of A.

As indicated above we habitually regard a JB^*-triple A as being contained in A^{**} and we identify the weak * closure, in A^{**}, of a JB^*-subtriple B of A with B^{**}. In this way $B = A \cap B^{**}$ by the Hahn Banach Theorem.

(2.1) Lemma. *Let A be a weak * dense JB^*-triple in a Cartan factor M such that $A \cap K(M) \neq \{0\}$ and let I be a norm closed inner ideal of A. Then $K(M) \subset A$ and $\bar{I}$ is a Cartan factor with $K(\bar{I}) = K(M) \cap I$.*

Proof. We have $K(M)^{**} = M$. So, with $J = K(M) \cap A$, we have $\bar{J} = J^{**}$ which, being a non-zero weak * closed ideal of $\bar{A} = M$, equals $K(M)^{**}$. Hence, $J = K(M)$.

As $\bar{I}$ is a weak * closed inner ideal of M, it is a Cartan factor. Further, $E = K(M) \cap I$ is an inner ideal of $K(M)$ and hence of M [9, 2.3] and therefore is an inner ideal of $\bar{I}$. Moreover, $E = \{I\ K(M) I\}$, so that $E^{**} = \bar{E} = \bar{I}$. Thus, E is an inner ideal of E^{**}. Hence, $E = K(E^{**})$ [8, 3.4] as required. □

(2.2) Lemma [9, 3.2, 3.3]. *Let A be a JB^*-triple with a norm closed inner ideal I.*

(a) For each Cartan factor representation $\pi : A \to M$ there exists $\rho \in \partial e(A_1^*)$ and a surjective isometry $\varphi : M \xrightarrow{\sim} A_\rho^{**}$ with $\pi_\rho = \varphi\pi$.
(b) For each $\rho \in \partial e(I_1^*)$, with extension $\bar{\rho} \in \partial e(A_1^*)$, I_ρ^{**} is a weak * closed inner ideal of $A_{\bar{\rho}}^{**}$ and $\pi_\rho = \pi_{\bar{\rho}}|I$. □

If X is a compact Hausdorff space and D is a finite dimensional Cartan factor, the JB^*-triple of all continuous functions from X to D, $A = C(X, D) = C(X) \otimes D$, has only Cartan factor representations onto D as is easily seen, and all Cartan factor representations of each JB^*-subtriple of A are onto Cartan subfactors of D.

(2.3) Lemma. *Let A be a JB^*-triple and let D be a finite dimensional Cartan factor. All Cartan factor representations of A are onto D if and only if $A^{**} = C(X) \otimes D$ for some compact hyperstonean space X.*

Proof. Let all Cartan factor representations of A be onto D. Then A is isometric to a subtriple of $\ell^\infty(I) \otimes D$ for some set I, by [15, Proposition 1], so that as D is finite dimensional A^{**} is realised as a JBW^*-subtriple of $(\ell^\infty(I) \otimes D)^{**} = \ell^\infty(I)^{**} \otimes D$. Therefore, by [22, (1.7)] together with the above remarks, if A^{**} is not of the form stated it contains a weak * closed ideal $J = C(Z) \otimes E$ where Z is compact hyperstonean and E is a proper subfactor of D. But then, letting $P : A^{**} \to J$ be the natural projection, the non-zero quotient $P(A)$ of A has no Cartan factor representations onto D, a contradiction.

Conversely, if A^{**} is of the form stated, the atomic part of A^{**} is of the form $\ell^\infty(I) \otimes D$ and the result follows. □

For a locally compact Hausdorff space X with positive Radon measure μ and a JBW^*-triple M we shall denote by $L^\infty(X, \mu, M)$ the JB^*-triple of all essentially bounded weak * measurable (with respect to M_*) M-valued functions on X.

3 JB^*-triples with type I second dual

A JBW^*-triple M is defined to be type I if $M(e)$ is a type I JBW^*-algebra for a complete tripotent e of M. The type I JBW^*-triples are characterised (cf. [22]) as the ℓ^∞-sums of the form $\sum^{\oplus} L^\infty(X_\alpha,\ \mu_\alpha,\ C_\alpha)$ where each C_α is a Cartan factor. A general classification of JBW^*-triples is given in [23]. We are interested in JB^*-triples A for which A^{**} is a type I JBW^*-triple and, as a helpful medium for subsequent investigation of sequential weak continuity we are led to introduce the analogues of postliminal C^*- and JB^*-algebras ([12, Sect. 4], [30, Sect. 6], [5,6]).

Let A be a JB^*-triple. We define A to be

(a) *liminal* if $\pi(A) = K(M)$, for each Cartan factor representation $\pi : A \to M$;
(b) *postliminal* if $K(M) \subset \pi(A)$, for each Cartan factor representation $\pi : A \to M$.

By (2.1) the condition $K(M) \subset \pi(A)$ in (b) is the same as $\pi(A) \cap K(M) \neq \{0\}$. Equivalent formulations of (a) and (b) are that A is liminal or postliminal, respectively, if A/P is elementary or contains a non-zero elementary ideal (for $A \neq \{0\}$) for each $P \in \mathrm{Prim}(A)$. We note that conditions (a) and (b) are inherited by JB^*-triple quotients and that a JB^*-algebra is liminal or postliminal, respectively, if and only if it is liminal or postliminal as a JB^*-triple.

Given an element x in a JB^*-triple A we let $A(x)$ denote the norm closed inner ideal of A generated by x. If A is a JB^*-subtriple of JBW^*-triple M, the weak * closure $\overline{A(x)} = \bar{A}(e) \subset M(e)$ for some tripotent e of $\bar{A}$ such that $A(x)$ is a JB^*-subalgebra of the JBW^*-algebra $M(e)$ with $x \in A(x)_+$ (details of this and the following statement can be found in [9]). Moreover, the JB^*-algebra structure of $A(x)$ is independent of M in the following sense. Let B be a JB^*-subtriple of a JBW^*-triple N and let $\pi : A \to B$ be a triple homomorphism with $y = \pi(x)$. Then $\pi : A(x) \to B(y)$ is a * homomorphism of JB^*-algebras and is a surjective * isomorphism if $\pi : A \to B$ is a surjective isometry (the respective weak * closures $\overline{A(x)}$, $\overline{B(y)}$ need not be isomorphic).

Recall that A is *abelian* if and only if A satisfies $\{xy\{abc\}\} = \{\{xya\}bc\}$ and that this is the same as A being isometric to a subtriple of an abelian C^*-algebra [29, (6.2)].

An element x in a JB^*-triple A is defined to be *abelian* if $A(x)$ is abelian. When realised as a JB^*-algebra this is equivalent to $A(x)$ being an abelian C^*-algebra. A JB^*-triple A is said to be *antiliminal* if it contains no non-zero abelian elements.

(3.1) Lemma. *Let A be a JB^*-triple. Then*

(a) A is postliminal if and only if $A(x)$ is postliminal for each x in A.
(b) A is liminal if and only if $A(x)$ is liminal for each x in A.

Proof. (a) Let A be postliminal, $x \in A$ and let $\rho \in \partial e(I_1^*)$ where $I = A(x)$, and let $\tau \in \partial e(A_1^*)$ extend ρ. By (2.2), the Cartan factor representation $\pi_\tau : A \to A_\tau^{**} = M$ extends the Cartan factor representation $\pi_\rho : I \to I_\rho^{**} = N$ and N is a weak * closed inner ideal of M. By (2.1) we have $\pi_\rho(I) \cap K(M) = K(N)$, whence I is postliminal.

Conversely, let $\pi : A \to M$ be a Cartan factor representation and let $x \in A$ with $\pi(x) \neq 0$. Then $\pi : A(x) \to \overline{M(\pi(x))} = N$ is a Cartan factor representation. So if $A(x)$ is postliminal, $K(N) \subset \pi(A(x))$ so that $\pi(A) \cap K(M) \neq \{0\}$.
(b) This is similar. □

(3.2) Proposition. *Let A be a JB^*-subtriple of a JB^*-triple B.*
(a) *If B is postliminal then A is postliminal.*
(b) *If B is liminal then A is liminal.*

Proof. (a) Given x in A there is a tripotent e of $A^{**} \subset B^{**}$ such that $A(x)^{**} = A^{**}(e)$ is a JBW^*-subalgebra of the JBW^*-algebra $B(x)^{**} = B^{**}(e)$. In particular, $A(x)$ is a JB^*-subalgebra of $B(x)$. Hence, as (a) is true for JB^*-algebras [5], it follows from (3.1) that it holds for JB^*-triples too.
(b) This is similar. □

(3.3) Theorem. *The following are equivalent for a JB^*-triple A.*

(a) A is postliminal
(b) Each non-zero quotient of A contains a non-zero abelian element.
(c) A^{**} is a type I JBW^*-triple.

Proof.
(a) $\Rightarrow$ (b): Let A be postliminal with a non-zero element x. By (3.1) $A(x)$ is a postliminal JB^*-algebra and therefore by [5] contains a non-zero abelian element $y \in A(x)$. The norm closed inner ideal of $A(x)$ generated by y is $A(y)$ and so y is an abelian element of A. As condition (a) passes to quotients (b) follows.
(b) $\Rightarrow$ (c): Assume (b) and let J be a non-zero weak * closed ideal of A^{**}. The restriction, $\varphi : A \to J$ of the natural projection $P : A^{**} \to J$ is a triple homomorphism with $\overline{\varphi(A)} = J$. By assumption $\varphi(A)$ contains

a non-zero abelian element x. Therefore, $\overline{\varphi(A)(x)} = J(e)$ where e is an abelian tripotent of J. Hence, A^{**} is a type I JBW^*-triple by [21, (4.13)].
(c) ⇒ (a): Let A^{**} be a type I JBW^*-triple and let x be in A. The weak * closed inner ideal $A(x)^{**}$ of A^{**} is type I as a JBW^*-triple and hence as a JBW^*-algebra. Therefore, by [5, Theorem 5.6], $A(x)$ is a postliminal JB^*- algebra and we have that A is postliminal by (3.1). □

(3.4) Lemma. *Let x be an abelian element in a JB^*-triple A. The norm closed ideal J of A generated by x is liminal.*

Proof. We may suppose that $x \neq 0$. Let $\pi : J \to M$ be a Cartan factor representation. Let e be the tripotent of M with $\overline{M(\pi(x))} = M(e)$. Then $M(e)$ is an abelian JBW^*-algebra factor and so $M(e) = \mathbb{C}e \subset K(M)$. As $K(M)$ is the norm closed ideal of M generated by e, $\pi(J)$ is contained in $K(M)$ and so must be equal to it by (2.1). □

It is easy to see that the largest liminal ideal of a JB^*-triple A is the set of all elements x in A for which $\pi(x) \in K(M)$ for every Cartan factor representation $\pi : A \to M$. By (3.4), the largest liminal ideal is zero if and only if A is antiliminal.

A composition series in a JB^*-triple A is an increasing family $\{I_\lambda | \, 0 \leq \lambda \leq \alpha\}$ of norm closed ideals indexed by a segment $[0, \alpha]$ of the ordinals such that $I_0 = \{0\}$, $I_\alpha = A$ and for each limit ordinal λ I_λ is the norm closure of $\bigcup\{J_\mu \,|\, \mu < \lambda\}$. Using the above a standard argument (cf. [12, 4.3.3–4.3.6]) gives the following.

(3.5) Proposition. *Let A be a JB^*-triple. Then A*
(a) *is postliminal if and only if A has a composition series $\{I_\lambda \,|\, 0 \leq \lambda \leq \alpha\}$ such that $I_{\lambda+1}/I_\lambda$ is liminal for each $\lambda < \alpha$;*
(b) *has a largest postliminal ideal J, and J is the smallest norm closed ideal I of A for which A/I is antiliminal.* □

4 Spin structure

By [24, (3.8)] infinite dimensional spin factors form an obstruction to the sequential weak continuity of biholomorphic automorphisms on the open unit ball. Knowledge of spin structure in JB^*-triples is therefore desirable. Below in (4.4) we show that spin factors intrude into antiliminal JB^*-triples, an observation that follows from a real version, (4.3), of [33, (6.7.4)]. We remark that if the latter is a little more than is strictly needed, the exposition benefits from transparency of transfer from the complex realm (as elucidated in [33, Sect. 6.6, Sect. 6.7]) to the real. By a real C^*- algebra we understand a norm closed real * subalgebra of a (complex) C^*-algebra.

Given a real C^*-algebra direct limit, G, of a unital system of * homomorphisms $\pi_n : R_n \to R_{n+1}$, where each R_n is isomorphic to $M_{k_n}(\mathbb{R})$ where

$k_n \geq 2$ there is, by [16, Proposition 17.2] a sequence $(m(n))$ in $\mathbb{N}\setminus\{1\}$ such that, with $m(n)! = m(1)\ldots m(n)$, G is isomorphic (as a real C^*-algebra) to the direct limit of the unital system $\varphi_n : M_{m(n)!}(\mathbb{R}) \to M_{m(n+1)!}(\mathbb{R})$ of standard maps. By analogy with the complex case let the latter be called a *real Glimm algebra* of rank $(m(n))$. The notion of a *quasi-matrix system* of rank $(m(n))$ is defined in [33, p.215].

(4.1) Proposition. *If R is a real C^*-algebra containing a quasi-matrix system of rank $(m(n))$, R contains a real C^*-subalgebra with a quotient isomorphic to a real Glimm algebra of rank $(m(n))$.*

Proof. Using the fact that R_{sa} is a JC-algebra, this is obtained as in [33, 6.6.5]. □

(4.2) Proposition. *Let R be a real C^*-subalgebra of $B(H)$, let e be a finite dimensional projection in $\bar{R}$ (weak * closure) and let x be in $e\,\bar{R}\,e$. Then there exists y in R with $\|y\| = \|x\|$ and ye = x. If x is self-adjoint or positive y can be chosen self-adjoint or positive accordingly.*

Proof. The complexification $\bar{R} \oplus i\bar{R} = W$ is a von Neumann algebra and $R \oplus iR = A$ is weak * dense in W. Hence, by [33, 2.7.5], $\|a\| = \|x\|$ with $ae = x$ for some a in A. We have $a = y + iz$ where $y,\ z \in R$, giving $x = ae = ye + ize$ so that $x = ye$ and $\|\,x\,\| \ \leq\ \|\,y\,\| \leq \|\,a\,\| = \|\,x\,\|$. The final part of the statement also follows from [33, 2.7.5] because $y = y^*$ if $a = a^*$ and $y \geq 0$ if $a \geq 0$. □

Let R be a real C^*-algebra, $A = R \oplus iR$ and let $\pi : A \to B(H)$ be an irreducible * representation. By the proof of [1, Theorem 3.1] $\overline{\pi R}$ is realised as $B(H_0)$ where H_0 is a real, complex or quaternionic Hilbert space derived from H. For x in R, $\pi(x)$ is a compact operator on H_0 in this realisation if and only if $\pi(x)$ is compact on H.

Suppose that $x \in R_+$ with $\|x\| = 1$ such that $\pi(x)$ is not compact and that $y \in R_+$ with $\|y\| = 1$ and $xy = x$. For each $m \in \mathbb{N}$ the eigen 1-space of $\pi(x)$ in H_0 contains an orthonormal sequence $h_1,\ \ldots,\ h_m$. Therefore, by (4.2), there exists a in R_+ and $u_1,\ \ldots,\ u_m$ of norm 1 in R such that

$$\pi(a)h_n = nh_n\ , \quad n = 1, \ldots, m\ ; \quad \pi(u_n)h_1 = h_n\ , \quad n = 2, \ldots, m\ .$$

By these remarks together with (4.1), just as in (6.7.1) and the opening seven lines of (6.7.2) of [33], we have the following real analogue of [33, (6.7.4)].

(4.3) Proposition. *If G is a real Glimm algebra (of any prescribed rank) and R is a real C^*-algebra such that $R \oplus iR$ is an antiliminal C^*-algebra then R contains a real C^*-subalgebra with a quotient isomorphic to G.* □

Identify a JC-algebra A with its image in it universal enveloping C^*-algebra $C^*(A)$ and let φ be the canonical involutory * antiautomorphism of

$C^*(A)$ (pointwise fixing A) [19, Sect. 7]. Then $R^*(A) = \{a \in C^*(A) \mid \varphi(a) = a*\}$ is the universal enveloping real C^*-algebra of A. We have $R^*(A) \cap iR^*(A) = \{0\}$, $R^*(A) \oplus iR^*(A) = C^*(A)$ and each self adjoint Jordan homomorphism of A into a real C^*-algebra extends to a real * homomorphism on $R^*(A)$. Further, just as $C^*(\cdot)$ is, $R^*(\cdot)$ is a functor preserving direct limits.

For $2 \leq n < \infty$, let $U_n = \mathbb{R}1 \oplus H_n$ be the real spin factor where H_n is the real Hilbert space of dimension n and let Cliff(H_n) be the real Clifford algebra of H_n, with respect to the Hilbert form on H_n, considered as a real * algebra with respect to its main involution (cf. [26, p. 75]). By the universal property of the (self-adjoint) Clifford representation $H_n \hookrightarrow$ Cliff (H_n) we obtain that $R^*(U_n)$ is isomorphic to Cliff(H_n). In particular, by the middle column of the table in [2, p. 11], we see that $R^*(U_{2+8n}) \simeq M_{2^{4n+1}}(\mathbb{R})$ for all $n \geq 0$. As the infinite dimensional separable real spin factor U_∞ is the norm closure of unital inclusions $U_n \hookrightarrow U_{n+1}$, $R^*(U_\infty)$ is the real C^*-algebra direct limit of the induced unital system $\pi_n : R^*(U_n) \to R^*(U_{n+1})$. Telescoping modulo eight we see that $R^*(U_\infty)$ is a real Glimm algebra. We have the following consequence.

(4.4) Theorem. *Let A be an antiliminal JB^*-triple. Then A contains a JB^*-subtriple with a JB^*-triple quotient containing an infinite dimensional spin factor as a JB^*-subtriple.*

Proof. Let x be a non-zero element in A. Then $A(x)$ is an antiliminal JC^*-algebra and so contains a non-zero norm closed ideal J with no spin factor representations so that with $B = J_{sa}$ we have $B = \{b \in C^*(B)_{sa} \mid \varphi(b) = b\}$ where φ is the canonical * antiautomorphism of $C^*(B)$. (cf [5, Lemma 4.3], [18, Theorem 2.2, Lemma 4.2]). If I is the largest liminal ideal of $C^*(B)$ we have $B \cap I = \{0\}$ [6] so that $I = \{0\}$ by [18, Lemma 4.3] because we must have $\varphi(I) = I$. Therefore, $C^*(B) = R^*(B) \oplus iR^*(B)$ is antiliminal and by (4.3) together with above remarks there is a real C^*-algebra $R \subset R^*(B)$ with a quotient isomorphic to $R^*(U_\infty)$. Now $R_{sa} \oplus iR_{sa}$ is a JC^*- subalgebra of $A(x)$ with a quotient containing the complex spin factor $U_\infty \oplus i\, U_\infty$ as a JC^*-subalgebra. □

We next investigate spin structure in the second dual. Let V_α, where $\alpha \geq 2$, denote the complex spin factor of dimension $\alpha + 1$ if α is finite and of dimension α if α is an infinite cardinal. By [21, 22], a JBW^*-triple $M = M_{sp} \oplus N$ where M_{sp} is an ℓ^∞-sum $\sum^\oplus M_\alpha$ where $M_\alpha = L^\infty(X_\alpha,\ \mu_\alpha,\ V_\alpha)$ and where N has no weak * closed ideals of this form. We refer to M_{sp} as the *spin part* of M. By the *infinite spin part* of M we understand the ℓ^∞-sum of those M_α's where α is infinite. We note that M_{sp} is a JW^*-algebra and that all of its Cartan factor representations are onto spin factors. It is easy to see that if a JB^*-triple A has an infinite dimensional spin factor representation then A^{**} has non-zero infinite spin part. The converse seems to require delicate arguments. (It is not immediately clear that A^{**} has *weak * continuous* homomorphisms onto an infinite dimensional spin factor when A^{**} has non-zero infinite spin part). We shall need:

(4.5) Proposition. *Let A be a JB^*-triple. Then* $\mathrm{Prim}(A)$ *is a Baire space.*

Proof. In order to obtain a contradiction assume that $\mathrm{Prim}(A)$ contains a non-empty meagre open subset U. Choose

$$P \in U \quad \text{and} \quad x \in A \setminus P \ .$$

By [9, Proposition 3.3]

$$V = \{P \in \mathrm{Prim}\,(A) \mid A(x) \not\subset P\}$$

is an open neighbourhood of P and is homeomorphic to $\mathrm{Prim}\,(A(x))$. As $A(x)$ is a JB^*-algebra Prim$(A(x))$ and hence, V, is a Baire space by [17, Corollary 4.2]. Therefore, being meagre and open in V, $U \cap V$ must be empty. But P lies in $U \cap V$ and we have the required contradiction. □

(4.6) Lemma. *Let A be a JB^*-triple. Then A has a spin factor representation if and only if A^{**} has non-zero spin part.*

Proof. A spin factor representation $\pi : A \to V$ extends to a weak * continuous homomorphism from A^{**} onto V so that V is isomorphic to a weak * closed ideal of A^{**}.

Conversely, let $P : A^{**} \to M$ be the projection onto the non-zero spin part M of A^{**} and consider the weak * dense JB^*-subtriple of M, $B = P(A)$. Suppose that B has no spin factor representations and let $x \in B$. Then for each spin factor representation $\pi : M \to V$ we have $\pi(B) = \mathbb{C} \oplus \mathbb{C}$ or a Hilbert space so that $\pi\,(B(x))$ is abelian. As M has a faithful family of spin factor representations we have that $B(x)$ is abelian, as therefore is $M(x)$. Choose, as we may, a unitary tripotent e of M. Now choose a net (x_λ) in the unit ball of B such $x_\lambda \to e$ in the strong * topology on M (see [3, Definition 3.1, Corollary 3.3]). Given $y = u$, v, a, b or c in M put $y_\lambda = \{x_\lambda\, y\, x_\lambda\}$ and $y_1 = \{e\, y\, e\}$. Since each x_λ is abelian in M and the triple product is jointly strong * continuous on bounded nets [35, Theorem] upon taking limits we see that

$$\{u_1\, v_1 \{a_1\, b_1\, c_1\}\} = \{\{u_1\, v_1\, a_1\} b_1\, c_1\} \ .$$

It follows that $M(e)$ is abelian, a contradiction. So A has a spin factor representation because B does. □

(4.7) Theorem. *Let A be a JB*-triple. Then A has an infinite dimensional spin factor representation if and only if A^{**} has non-zero infinite spin part.*

Proof. Necessity being clear we prove sufficiency. Assume that all spin factor representations of A are finite dimensional. It follows from [9, Theorem 5.2] that there are norm closed ideals $I \subset J$ of A such that I and A/J have no spin factor representations and all Cartan factor representations of J/I (if any) are of rank 2. As $A^{**} = I^{**} \oplus (J/I)^{**} \oplus (A/J)^{**}$, via (4.6) we may suppose that $A = J/I$. In which case, by [9, Theorem 5.9], we have norm closed ideals $J_1 \subset J_2 \subset J_3 \subset J_4 \subset J_5$ in A such that all Cartan factor representations of $J_1, J_3/J_2, J_4/J_3$ and A/J_5 are, respectively, onto V_α's where $\alpha > 5$, $V_5,\ V_4$ and V_2; J_2/J_1 has no spin factor representations and, using [9, Lemma 5.6], $B = J_5/J_4$ contains a norm closed ideal J with no spin factor representations such that all Cartan factor representations of B/J are onto V_3. By (2.3) together with (4.6), it follows that the infinite spin part of A^{**} resides in J_1^{**} so that we may assume $A = J_1$. But then [9, Theorem 5.9] gives that

$$F\ :\ \mathrm{Prim}(A) \to \mathbb{N}\ , \quad \text{where} \quad f(P) = n \quad \text{if} \quad A/P = V_n\ ,$$

is lower semicontinuous. As $\mathrm{Prim}(A)$ is a Baire space, by (4.5), f is continuous at some point $Q \in \mathrm{Prim}(A)$. Hence, with $m = f(Q)$, $f^{-1}(\{m\})$ contains an open neighbourhood of Q, which gives a norm closed ideal J of A for which all Cartan factor representations are onto V_m. Passing to A/J and proceeding, by transfinite induction we obtain a composition series $\{J_\lambda |\ 0 \le \lambda \le \alpha\}$ of A such that for each $\lambda < \alpha\,,\ \ J_{\lambda+1}/J_\lambda$ has only Cartan factor representations onto a fixed spin factor V_{n_λ}, where $n_\lambda < \infty$. Now,

$$A^{**} = \sum_{\lambda<\alpha}^{\oplus} (J_{\lambda+1}/J_\lambda)^{**} \quad (\ell^\infty - \mathrm{sum})\ ,$$

which by (2.3) implies that A^{**} has zero infinite spin part. □

We conclude this section with remarks on JBW^*-triples. A spin system $(s_i)_{i\in I}$ of order α in a JC^*-algebra is a family of anticommuting symmetries (that is $s_i^2 = 1$ for all i and $s_i s_j + s_j s_i = 0$ wherever $i \neq j$) with $\mathrm{card}\,(I) = \alpha$. The Banach space generated by such and 1 is V_α.

(4.8) Remarks

(a) If $A = \sum\limits_{\alpha\in S}^{\oplus} A_\alpha$, where $A_\alpha = L^\infty(X_\alpha,\ \mu_\alpha,\ V_\alpha) \neq \{0\}$ and S is a set of distinct cardinals with least member α_0 each A_α contains a spin system of order α_0 and hence, summing over S, so does A. Therefore, for any spin factor representation $\pi\ :\ A \to V_\beta$ we have $\beta \ge \alpha_0$. It follows that if S is infinite and $\pi(A_\alpha) = \{0\}$ for all $\alpha \in S$, then β is infinite and hence that, with $J = (\Sigma\, A_\alpha)_0\ \ (c_0 -\ \mathrm{sum})$, all Cartan factor representations of A/J are onto infinite dimensional spin factors.

(b) Let A be a JBW^*-triple without spin part and let e be a complete tripotent of A. The JBW^*-algebra $A(e)$ has no spin part by the results of

[22] and therefore has no spin factor representation by the structure theory in [19, Sect. 5.3], for example. By (4.9) (for which we shall have further use) below it follows that A has no spin factor representations.

(4.9) Lemma. *Let $\pi : A \to B$ be a surjective triple homomorphism where A is a JBW^*-triple and B is a JB^*-triple containing a tripotent f. Then $\pi(e) = f$ for some tripotent e in A. If f is complete then e can be chosen to be complete.*

Proof. Choose x in A with $\pi(x) = f$, let p be the tripotent of A with $\overline{A(x)} = A(p)$ and consider the * Jordan homomorphism $\pi : A(p) \to B(q)$ where $q = \pi(p)$. We have that $x \in A(p)_+$, f is a projection in $B(q)$ and that if W is the abelian von Neumann subalgebra of $A(p)$ generated by x we have a * homomorphism $\pi : W \to C$ onto an abelian C^*-subalgebra of $B(q)$. Now the usual Borel functional calculus gives a projection e of W with $\pi(e) = f$.

If f is complete, choose a complete tripotent e_1 of A such that e is a projection in the JBW^*-algebra $A(e_1)$ [21, (3.12)]. We have $\{e\, e(e_1 - e)\} = 0$ so that $\{f\, f\ (\pi(e_1) - f)\} = 0$ and hence $\pi(e_1) = f$. □

5 Sequential weak continuity

Given a subset S of a JB*-triple A a function $f : S \to A$ is said to be *sequentially weakly continuous* if whenever a sequence $x_n \to x$ weakly in S we have $f(x_n) \to f(x)$ weakly.

Let D denote the open unit ball of A. The class of bijections $f : D \to D$ for which f and f^{-1} are Frechet differentiable is the real Banach Lie group [39], $Aut(D)$, of biholomorphic automorphisms of D. As shown in [37] (see also [10]) given $a \in A$, the one-parameter subgroup of $Aut(D)$, $\exp t\, X_a$, where the vector field $X_a \equiv (a - \{xax\})\, \frac{\partial}{\partial x}$, consists of weakly (sequentially weakly) continuous automorphisms if and only if the quadratic map on A, $x \mapsto \{xax\}$, is weakly (sequentially weakly) continuous. The structure of A when every $g \in Aut(D)$ is weakly continuous is completely solved in [30].

Let $Aut_\sigma(D)$ denote the sequentially weakly continuous members of the group $Aut(D)$ of biholomorphic automorphisms of the open unit unit ball D of A. Denote by $\sigma(A)$ the set of elements a of A for which the quadratic map $x \mapsto \{x\, a\, x\}$ is sequentially weakly continuous. By [24, (2.6)] $\sigma(A)$ is a norm closed ideal of A and by [10, page 517] $Aut_\sigma(D)$ is a subgroup of $Aut(D)$ with $Aut_\sigma(D) = Aut(D)$ if and only if $\sigma(A) = A$ and, moreover, $Aut_\sigma(D)$ is the group of restrictions to D of the linear isometries of A if and only if $\sigma(A) = \{0\}$.

A is defined to be *sequentially weakly continuous* if $Aut_\sigma(D) = Aut(D)$ or, equivalently, if $\sigma(A) = A$.

(5.1) Lemma (Isidro-Kaup [24]). *Let M be a Cartan factor.*
(a) $\sigma(M) = K(M)$ *if M is not an infinite dimensional spin factor*
(b) $\sigma(M) = \{0\}$ *if M is an infinite dimensional spin factor.* □

(5.2) Lemma. *Let A be a JB^*-algebra, J a norm closed ideal of A and let (a_n) be a sequence in A such that $a_n + J \to 0$ weakly (in A/J). There is a sequence (b_n) in A such that $b_n \to 0$ weakly and $b_n - a_n \in J$ for all n.*

Proof. Passing to the JB^*-subalgebra of A generated by the a_n we may suppose that A is separable in which case J has a sequential increasing approximate identity so that $x_n \to e$ strongly where e is the central projection in A^{**} with $J^{**} = A^{**} \circ e$ [19, 4.4.15]. Put $b_n = a_n \circ (1 - x_n)$ for each n. Then $b_n - a_n \in J$ for each n and for any positive linear functional ρ of J we have, via the Cauchy-Schwarz inequality, $\rho(b_n) = \rho\left(a_n \circ ((1 - x_n) \circ e)\right) \to 0$. Hence, $b_n \to 0$ weakly. □

(5.3) Lemma. *Let A be a JB^*-algebra such that $x_n^2 \to 0$ weakly whenever (x_n) is a sequence in A such that $x_n \to 0$ weakly. Then all Cartan factor representations of A are finite dimensional.*

Proof. The condition is inherited by JB^*-subalgebras and, by (5.2), by all quotients too. Hence, if A has an antiliminal quotient the condition is satisfied by the infinite separable spin factor V_∞ by (4.4) implying that $\sigma(V_\infty) = V_\infty$ in contradiction to (5.1). Therefore, A is postliminal. Via (5.2), passing to a primitive quotient of A we may suppose that

$$K(M) \subset A \subset M \ ,$$

where M is a type I JBW^*-algebra factor, but not an infinite dimensional spin factor. If M is infinite dimensional there is an infinite dimensional real Hilbert space H_0 such that $K(H_0)_{sa}$ embeds in $K(M)_{sa}$ as a JC-subalgebra so that if (h_n) is an infinite orthonormal sequence in H_0, $x_n = h_1 \otimes h_n + h_n \otimes h_1 \to 0$ weakly but, for $n \geq 2$, $x_n^2 = h_1 \otimes h_1 + h_n \otimes h_n$ does not converge weakly to zero. Therefore M is finite dimensional whence the result. □

For a JB^*-triple A let $J_\sigma(A)$ denote the norm closed ideal of all elements x of A for which $\pi(x) \in \sigma(M)$ for all Cartan factor representations $\pi : A \to M$. Let $\mathrm{Max}(A)$ denote the set of all maximal norm closed ideals of A.

(5.4) Lemma. *Let A be a JB^*-triple. Then $J_\sigma(A) \subset \sigma(A)$.*

Proof. Let $a \in J_\sigma(A)$ and let (x_n) be a sequence in A such that $x_n \to 0$ weakly. Given $\rho \in \partial_e(A_1^*)$ we have $\pi_\rho\left(\{x_n\, a_n\, x_n\}\right) = \{\pi_\rho(x_n)\, \pi_\rho(a)\, \pi_\rho(x_n)\} \to 0$ weakly. But $\rho = \rho\, \pi_\rho$ and so $\pi\left(\{x_n\, a\, x_n\}\right) \to 0$ weakly. As $(\{x_n\, a\, x_n\})$ is bounded, it follows that $\{x_n\, a\, x_n\} \to 0$ weakly by Rainwater's Theorem [34]. Hence, a lies in $\sigma(A)$. □

We can now give our classification of sequentially weakly continuous JB^*-triples.

(5.5) Theorem. *The following are equivalent for a JB^*-triple A.*
(a) *A is sequentially weakly continuous.*
(b) $\mathrm{Prim}(A) = \mathrm{Max}(A)$ *and A^{**} is type I with no infinite spin part.*
(c) *A is liminal with no infinite dimensional spin factor representations.*

Proof. The equivalence of (b) and (c) follows from (3.3) and (4.7) together with the fact that elementary JB^*-triples have no proper norm closed ideals.

(a) $\Rightarrow$ (c): Assume (a). A spin factor representation $\pi : A \to V$ induces a spin factor representation of JB^*-algebras $\pi : A(x) \to V(e) = V$, where e is a unitary tripotent in V and x is in A with $\pi(x) = e$. By assumption and (5.2) we must have $\sigma(V) = V$ so that V is finite dimensional by (5.1).

Via (3.1) (b), in order to show that A is liminal we may suppose that it is a JC^*-algebra and, by a further invocation of (5.2), that A is weak * dense in a type I JW^*-factor $M \subset B(H)$, for some complex Hilbert space H. We must show that $A = K(M)$.

Suppose that there exists $a \geq 0$ in A lying outside $K(M)$. The JB^*-subalgebra of A generated by a is a commutative C^*-algebra isomorphic to $C_0\,(sp(a) \setminus \{0\})$. By functional calculus this contains a sequential increasing approximate identity (a_n) such that $a_n\, a_{n+1} = a_n$ for all n. As $\|\, a - a\, a_n\| \to 0$, for some n a_n and a_{n+1} are not in $K(M)$. In particular, there exist $x \geq 0$ and $y \geq 0$ in A lying outside $K(M)$ such that in the operator product on $B(H)$ we have $xy = yx$ and $xy = y$. It follows that $xz = z$ for all $z \in A(y)$. Let e be the projection in M for which the weak * closure $\overline{A(y)} = M(e)$.

Let (z_n) be a sequence in $A(y)$ such that $z_n \to 0$ weakly. As A and hence, $A(y)$, is sequentially weakly continuous the above implies that

$$z_n^2 = (z_n\, x) z_n = \{z_n\, x\, z_n\} \to 0 \text{ weakly.}$$

So $A(y)$ satisfies the condition of (5.3). This implies that $M(e)$ $(= eMe)$ is finite dimensional and so must be contained in $K(M)$. Hence, $x \in K(M)$ and we have arrived at a contradiction, proving (c).

(c) $\Rightarrow$ (a): By (5.1) the condition (c) implies that $J_\sigma(A) = A$ so that $\sigma(A) = A$, by (5.4). $\square$

(5.6) Corollary. *Let A be a JB^*-triple.*
(a) If A is sequentially weakly continuous then so is every JB^*-triple quotient of A.
(b) A is sequentially weakly continuous if and only if $A(x)$ is sequentially weakly continuous for all x in A. $\square$

(5.7) Theorem. *Let A be a JB^*-triple, let I and J be, respectively, the largest liminal and largest postliminal ideal of A and let*

$$V(A) = \bigcap \{P \in \mathrm{Prim}(A) \mid \ A/P \textit{ is an infinite dimensional spin factor}\}\,.$$

Then $\sigma(A) = I \cap V(A) = J_\sigma(A)$.

Further $J \cap V(A)$ is the smallest norm closed ideal L of A for which $\sigma(A/L) = \{0\}$.

Proof. By construction $I \cap V(A)$ is the largest liminal ideal of A without infinite dimensional spin factor representations and equals $J_\sigma(A)$ by (5.1). Therefore, as $\sigma(A)$ is sequentially weakly continuous, (5.5) (a) $\Leftrightarrow$ (c) implies that $\sigma(A) \subset J_\sigma(A)$. Hence, $\sigma(A) = J_\sigma(A)$ by (5.4).

We note that $J/(J \cap V(A))$ is the largest postliminal ideal of $A/(J \cap V(A))$ and that $V(A/J \cap V(A)) = V(A)/(J \cap V(A))$. Hence, by the first part, we have that $\sigma(A/(J \cap V(A))) = \{0\}$. Now let L be a norm closed ideal of A with $\sigma(A/L) = \{0\}$. If $J \cap V(A)$ is not contained in L there is a norm closed ideal M of A with $J \cap V(A) \cap L \subset M \subset J \cap V(A)$ such that $M/(J \cap V(A) \cap L)$ is a non-zero liminal ideal without infinite dimensional spin factor representations, in $(J \cap V(A))/(J \cap V(A) \cap L)$, giving rise to a non-zero ideal of A/L with the same properties. In which case the first part gives $\sigma(A/L) \neq \{0\}$ and so the required contradiction. □

(5.8) Corollary. *A C^*-algebra A is sequentially weakly continuous if and only if it is liminal. Further, $\sigma(A)$ is the largest liminal ideal of A, and $\sigma(A) = \{0\}$ if and only if A is antiliminal.* □

For $n \in \mathbb{N}$ we say that a JBW^*-triple A is of *rank type* n if A is an ℓ^∞-sum of $L^\infty(X_\alpha, \mu_\alpha, C_\alpha)$'s where each C_α is a Cartan factor of rank n. Given a complete tripotent e in a rank type n JBW-triple, A, $A(e)$ is a type I_n JBW^*-algebra: in the case when $n = 2$ but A has no spin part, $A(e) \simeq L^\infty(X_a, \mu_a, V_3) \oplus L^\infty(X_b, \mu_b, V_5)$. We note that a rank type n JBW^*-triple is generated as a Banach space by its abelian tripotents and if it is without spin part it is sequentially weakly continuous (by (4.8) and (5.6)). We say that M is *homogeneous of finite rank type* if it is of rank type n for some $n \in \mathbb{N}$.

A JBW^*-triple is said to be *continuous* if it has no type I part. By (5.6) or (5.7), $\sigma(A) = \{0\}$ for continuous JBW^*-triples A. Therefore, questions of sequential weak continuity for JBW^*-triples devolve to the type I case.

Let A be a type I JBW^*-triple. We have $A = A_{sp} \oplus B = S_f \oplus S_\infty \oplus B$, where B is type I and without spin part, S_∞ is the infinite spin part of A and $S_f = \Sigma^\oplus A_{n_i}$ where each $A_{n_i} = L^\infty(X_{n_i},\ \mu_{n_i},\ V_{n_i})$ and (n_i) is a strictly increasing sequence (possibly finite) in $\mathbb{N}$.

This notation is retained for the next two results.

(5.9) Theorem. *For the type I JBW^*-triple $A = A_{sp} \oplus B$ we have*
(a) $\sigma(A) = (\Sigma^{\oplus} A_{n_i})_0 \oplus B_{ab}$, where $(\ \ )_0$ denotes c_0-sum and where B_{ab} is the norm closed ideal of B generated by its abelian tripotents;
(b) $\sigma(A/\sigma(A)) = \{0\}$.

Proof. The ideal J on the right hand side of the equation in (a) has no infinite dimensional spin factor representations and is clearly liminal so that $J \subset \sigma(A)$ by (5.7). Also by (5.7) we have $\sigma(A)/J \subset \sigma(A/J)$. Therefore parts (a) and (b) will follow if $\sigma(A/J) = \{0\}$. If $A = A_{sp}$ this is immediate from (4.8) (a) and (5.7). Suppose that A = B and that $\pi\ :\ A \to M$ is a Cartan factor representation with $\pi(J) = \{0\}$. If $K(M) \subset \pi(A)$ (4.9) implies that there is a tripotent e of A such that $\pi(e) = f \neq 0$ is a minimal tripotent of M. This gives a Jordan * homomorphism $\pi\ :\ A(e) \to \mathbb{C}f$. Thus, if $A(e) = I \oplus K$ where I is the type I_1 part of $A(e)$ [19, (5.3.5)], we have $\pi(K) = \{0\}$. But $I \subset J$ by construction so that $\pi(I) = \{0\}$ also. This is a contradiction and it proves that A/J is antiliminal in this case so that $\sigma(A/J) = \{0\}$ as before. The result follows. □

(5.10) Corollary. *The type I JBW^*-triple $A = A_{sp} \oplus B$ is sequentially weakly continuous if and only if A_{sp} is a direct sum of finitely many $L^\infty(X_\alpha, \mu_\alpha, V_\alpha)'s$ where $\alpha < \infty$ and B is a direct sum of finitely many JBW-triples that are homogeneous of finite rank type.* □

We close with an example of a JB^*-triple A with only linear sequentially weakly continuous biholomorphic automorphisms on its open unit ball but which has a faithful family of finite dimensional spin factor representations.

(5.11) Example. Consider the finite dimensional spin factors V_n, $2 \leq n < \infty$, realised as JC^*-algebras with a common identity so that $V_n \subset V_{n+1}$ and V_∞ is the norm closure of $\bigcup_1^\infty V_n$. Let $B = C\left([0,\ 1],\ V_\infty\right)$ be the JC^*-algebra, with supremum norm, of all continuous functions $f\ :\ [0,\ 1] \to V_\infty$. Now let (r_n) be an enumeration of the rationals in [0,1] and define

$$A = \{f \in B \,|\, f(r_n) \in V_{n+1}, \quad \text{for all} \quad n\}\ .$$

Then A is a JC^*-subalgebra of B and the evaluations, $f \mapsto f(t)$, $t \in [0,1]$, are * Jordan homomorphisms $\pi_t\ :\ A \to V_\infty$, with $\pi_{r_n}(A) \subset V_{n+1}$ for each n. Given $n \in \mathbb{N}$, choose $g \in C[0,1]$ with $g(r_1) = \ \ldots\ = g(r_n) = 0\,,\ \ g(r_{n+1}) = 1$. For each x in V_{n+2} define G_x in B by $G_x(t) = f(t)x$. Then $G_x \in A$ and $\pi_{r_{n+1}}(G_x) = x$. Hence, $\pi_{r_{n+1}}(A) = V_{n+2}$. Since for f in A, $f(r_n) = 0$ for all $n \geq 2$ implies $f = 0$, $\{\pi_{r_n} \,|\, n \geq 2\}$ is a faithful family of finite dimensional spin factor representations of A. On the other hand for irrational s in [0,1] we have, for $n \in \mathbb{N}$, $h \in C\,[0,1]$ with $h(r_1) = \ \ldots\ = h(r_n) = 0\,,\ \ h(s) = 1$. As before, for each x in V_{n+2}, $H_x \in A$ where $H_x(t) = f(t)x$ for all t, and $\pi_s(H_x) = x$. Therefore,

$V_{n+2} \subset \pi_s(A)$, for all n. Hence, $V_\infty = \pi_s(A)$ for all irrational s in [0,1], giving a faithful family of representations of A onto V_∞. By (5.7) we have $\sigma(A) = \{0\}$, as required.

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