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Generalized bi-circular projections on spaces of analytic functions

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Abstract. We characterize the generalized bi-circular projections on various Banach spaces of both scalar and vector valued analytic functions, including the Bergman, Bloch, and Hardy spaces. We also establish that the only projections in the convex hull of two isometries on a Hardy space are generalized bi-circular projection.

1. Introduction

Stacho and Zalar introduced the notion of a bi-circular projection in a Banach space, see [31] and [32]. A projection P on a Banach space X is said to be a *bi-circular projection* if $e^{ia}P+e^{ib}(I-P)$ is an isometry for all choices of real numbers a and b. These projections are norm hermitian in a Banach space setting, see [12]. In a Hilbert space the bi-circular projections are precisely the orthogonal projections. A generalization of this concept was introduced by Fosner, Illisevic, and Li in [14]. They investigated projections P with the property that $P+\lambda(I-P)$ is an isometry of X for some $\lambda \in \mathbb{T} \setminus \{1\}$, where $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$. In addition, any projection on a Banach space which can be written as the average of the identity operator and an isometric reflection is also a generalized bi-circular projection. We have characterized generalized bi-circular projections in various settings, cf. [4], [5], and [6].

In this paper, we give a characterization of these more general projections on various Banach spaces of analytic functions. We consider both scalar and vector

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valued functions. We designate these projections introduced by Fosner, Illisevic, and Li generalized bi-circular projections.

2. Generalized bi-circular projections on some functional Banach spaces

The following lemma from [14] will be used several times throughout the paper. It gives a simple characterization of when a projection is a generalized bi-circular projection in terms of an equation that the associated isometry must satisfy.

Lemma 2.1. *Let $\lambda \in \mathbb{T}$ with $\lambda \neq 1$. Suppose that $P_\lambda \in B(X)$ is a projection different from* 0 *or I. Let $\mathcal{I}(X)$ denote the group of surjective isometries of X. Then the following two statements are equivalent*:
(i) $P_\lambda + \lambda(I - P_\lambda) = T_\lambda \in \mathcal{I}(X)$.
(ii) $P_\lambda = \frac{(T_\lambda - \lambda I)}{1-\lambda}$ *and* $(T_\lambda - \lambda I)(T_\lambda - I) = 0$.

We observe that T_λ is a surjective isometry since

$$[P_\lambda + \lambda(I - P_\lambda)][P_\lambda + \lambda^{-1}(I - P_\lambda)] = I.$$

We start by considering a class of functional Banach spaces on the disk in which all the surjective isometries are of some predesignated form. We give a complete classification of the generalized bi-circular projections in this general setting.

Definition 2.2. (cf. [9]) A Banach space of complex valued functions on a set X is called a *functional Banach space* on X if the vector operations are pointwise operations, $f(x) = g(x)$ for each $x \in X$ implies $f = g$, $f(x) = f(y)$ for every f in the space implies $x = y$, and for each x in X, the linear functional $f \to f(x)$ is continuous.

Examples of functional Banach spaces are the Bergman spaces, the Bloch spaces, and the Hardy spaces. We denote by $\mathcal{F}(\Delta)$ a functional Banach space on the unit open disk $\Delta = \{z \in C \,|\, |z| < 1\}$ with the property that it contains sufficiently many polynomial functions to interpolate four points. This is certainly the case for the Hardy, Bergman and Bloch spaces. In addition, we also assume that the surjective isometries on $\mathcal{F}(\Delta)$ are of the form

$$\mathcal{J}(f)(z) = \beta[\varphi'(z)]^{\frac{r}{p}} f(\varphi(z)) - \beta\alpha f(\varphi(0)),$$

where φ is a disk automorphism (i.e. $\varphi(z) = \mu\frac{a-z}{1-\overline{a}z}$, with $|a| < 1$ and $|\mu| = 1$), $|\beta| = 1$, $0 \leq |\alpha| \leq 1$, p a positive integer, and r a nonnegative real number.

We observe that the surjective isometries on each of the spaces listed above are of this form for convenient choices of the parameters and disk automorphisms, cf. [8], [13] and [18].

A generalized bi-circular projection is given by $P(f)(z) = \frac{\mathcal{J}(f(z))-\lambda f(z)}{1-\lambda}$, for some $\lambda \neq 1$ and of modulus 1. Lemma 2.1 implies that $\mathcal{J}^2(f(z))-(\lambda+1)\mathcal{J}(f(z))+\lambda f(z) = 0$, for every $f \in \mathcal{F}(\Delta)$ and $z \in \Delta$. This equation is equivalent to

(2.1)
$$\beta^2[(\varphi^2)'(z)]]^{\frac{r}{p}}f(\varphi^2(z)) - \beta^2\alpha[\varphi'(z)]^{\frac{r}{p}}f(\varphi(0)) - \beta^2\alpha[\varphi'(\varphi(0))]^{\frac{r}{p}}f(\varphi^2(0))+ \\ +(\beta\alpha)^2f(\varphi(0)) - (\lambda+1)[\beta[\varphi'(z)]^{\frac{r}{p}}f(\varphi(z)) - \beta\alpha f(\varphi(0))] + \lambda f(z) = 0.$$

We assume first that $\varphi(0) = 0$. This implies that $\varphi(z) = -\mu z$. If, in addition, $\varphi(z) \neq z$ (or $\mu \neq -1$), then there exists a point $z_0 \in \Delta$ so that $\varphi(z_0) \neq z_0$. We choose a polynomial f such that $f(\varphi(z_0)) = 1$ and $f(0) = f(z_0) = f(\varphi^2(z_0)) = 0$. The equation (2.1), evaluated at f and $z = z_0$, yields $(\lambda + 1)\beta[-\mu]^{\frac{r}{p}} = 0$. Hence $\lambda = -1$ and $\mathcal{J}^2 = I$. Under these assumptions, (2.1) reduces to

$$\beta^2[\mu^2]^{\frac{r}{p}}f(\mu^2 z) - 2\beta^2\alpha[-\mu]^{\frac{r}{p}}f(0) + (\beta\alpha)^2f(0) = f(z).$$

This last equation implies that $\beta^2 = 1$ and $\mu = 1$. Moreover, we also conclude that $\alpha = 0$ or $f(0) = 0$. In both cases, the isometries are of the form $\mathcal{J}(f)(z) = \pm[-1]^{\frac{r}{p}}f(-z)$ and thus the corresponding projections are given by

$$P(f)(z) = \frac{f(z) \pm [-1]^{\frac{r}{p}}f(-z)}{2}.$$

We assume that $\varphi(z) = z$, hence (2.1) reduces to

$$\beta^2 f(z) - 2\beta^2\alpha f(0) + (\beta\alpha)^2f(0) - (\lambda+1)[\beta f(z) - \beta\alpha f(0)] + \lambda f(z) = 0.$$

This last equation yields the following six cases:
(1) $\beta = 1$ and $f(0) = 0$ or $\alpha = 0$;
(2) $\beta = 1$ and $\lambda = 1 - \alpha$;
(3) $\beta = \lambda$ and $f(0) = 0$ or $\alpha = 0$;
(4) $\beta = \lambda$ and $\alpha = 1 - \overline{\lambda}$

with corresponding projections:
(1) $P(f)(z) = f(z)$;
(2) $P(f)(z) = f(z) - f(0)$ and $|1 - \alpha| = 1$
(3) $P(f)(z) = 0$;
(4) $P(f)(z) = f(0)$, $0 < |1 - \overline{\lambda}| \leq 1$.

This concludes the analysis for disk automorphisms with $\varphi(0)=0$. Now, we assume that $\varphi(0)\neq 0$ but $\varphi^2(0)=0$. This implies that $\varphi(z)=\frac{a-z}{1-\bar{a}z}$ and $\varphi^2(z)=z$. Consequently (2.1) becomes

$$\begin{aligned}\beta^2\{f(z)-\alpha[\varphi'(z)]^{\frac{r}{p}}f(a)\}-\beta^2\alpha\{[\varphi'(a)]^{\frac{r}{p}}f(0)-\alpha f(a)\}-\\ -(\lambda+1)\beta\{[\varphi'(z)]^{\frac{r}{p}}f(\varphi(z))-\alpha f(a)\}+\lambda f(z)=0.\end{aligned} \tag{2.2}$$

As described before, a convenient choice of a polynomial function implies that $\lambda=-1$ and $\mathcal{J}^2=I$. Thus, we write (2.2) as

$$\beta^2\{f(z)-\alpha[\varphi'(z)]^{\frac{r}{p}}f(a)\}-\beta^2\alpha\{[\varphi'(a)]^{\frac{r}{p}}f(0)-\alpha f(a)\}=f(z).$$

In particular, for $z=a$, we have

$$\beta^2\{f(a)-\alpha[\varphi'(a)]^{\frac{r}{p}}f(a)\}-\beta^2\alpha\{[\varphi'(a)]^{\frac{r}{p}}f(0)-\alpha f(a)\}=f(a).$$

This implies that $f(0)=0$, for every $f\in\mathcal{F}(\Delta)$, or $\alpha=0$. If for every $f\in\mathcal{F}(\Delta)$ we have $f(0)=0$, then (2.2) also implies that $\beta^2=1$ and $[\varphi'(z)]^{\frac{r}{p}}\alpha-\alpha^2=0$. Therefore $\alpha=0$ or $\alpha=1$ whenever $r=0$. These considerations imply that $\mathcal{J}(f)(z)=\pm[\varphi'(z)]^{\frac{r}{p}}f(\varphi(z))$ or $\mathcal{J}(f)(z)=\pm\left(f(\varphi(z))-f(\varphi(0))\right)$, with φ an automorphism of the disk of the form $\varphi(z)=\frac{a-z}{1-\bar{a}z}$. The associated projections are given by

$$P(f)(z)=\frac{f(z)\pm[\varphi'(z)]^{\frac{r}{p}}f(\varphi(z))}{2}\quad\text{or}\quad P(f)(z)=\frac{f(z)\pm[f(\varphi(z))-f(\varphi(0))]}{2},$$

with $\varphi(z)=\frac{a-z}{1-\bar{a}z}$. Using the techniques applied before we conclude that whenever $\varphi(0)\neq 0$ and $\varphi^2\neq id$, equation (2.1) has no solutions. In fact, it is enough to select $z_0\in\Delta$ so that $z_0\notin\{0,\varphi(0),\varphi^2(0),\varphi(z_0),\varphi^2(z_0)\}$, then choose $f\in\mathcal{F}(\Delta)$ such that $f(z_0)=1$ and $f(\varphi(z_0))=f(\varphi^2(z_0))=f(0)=f(\varphi(0))=f(\varphi^2(0))=0$. Therefore (2.1) implies $\lambda=0$. This contradicts that λ has modulus 1. We now summarize the conclusion of this analysis in the following theorem.

Theorem 2.3. *Let $\mathcal{F}(\Delta)$ be a functional Banach space on the unit disk containing all the polynomial functions, where all surjective isometries are of the form*

$$\mathcal{J}(f)(z)=\beta[\varphi'(z)]^{\frac{r}{p}}f(\varphi(z))-\beta\alpha f(\varphi(0)),$$

with $\varphi(z)=\mu\frac{a-z}{1-\bar{a}z}$, $|a|<1$, $|\mu|=|\beta|=1$, $0\leq|\alpha|\leq 1$, p a positive integer, and r a nonnegative integer. Then generalized bi-circular projections on $\mathcal{F}(\Delta)$ are of the following forms:

$$P(f)=0,\ P(f)=f(0),\ P(f)=f,\ P(f)=f-f(0),\ \textit{or}\ P(f)=\frac{f\pm\mathcal{J}(f)}{2},$$

with $\mathcal{J}(f)(z)=[\varphi'(z)]^{\frac{r}{p}}f(\varphi(z))$ or $\mathcal{J}(f)(z)=f(\varphi(z))-f(\varphi(0))$ and φ is a disk automorphism such that $\varphi^2(z)=z$.

Remark 2.4. The projections listed above are in fact generalized bi-circular projections. Consider $\varphi = id$, $\alpha = 0$, and $\beta = 1$ (or λ). Then $\mathcal{J}(f) = \beta f$ and $P(f) = f$ (or 0, respectively). If $\alpha = 1 - \lambda$ (or $\alpha = 1 - \overline{\lambda}$) then $\mathcal{J}(f) = f(z) - \alpha f(0)$ and $P(f) = f - f(0)$ (or $\mathcal{J}(f) = \lambda f - (\lambda - 1)f(0)$ and $P(f) = f(0)$, respectively). If $\varphi(z) = \frac{a-z}{1-\overline{a}z}$, then $\alpha = 0$ and $P(f) = \frac{f \pm \mathcal{J}(f)}{2}$, we have $\mathcal{J}(f)(z) = \beta[\varphi'(z)]^{\frac{r}{p}} f(\varphi(z))$ whenever $\alpha = 0$ or $\mathcal{J}(f)(z) = [\varphi'(z)]^{\frac{r}{p}} f(\varphi(z)) - f(\varphi(0))$, whenever $f(0) = 0$ for every f in the functional Banach space under consideration.

The Bloch space is a special case of the previous theorem (cf. [8]):

$$\mathcal{B} = \{f : f \text{ is holomorphic on } \Delta,\ f(0) = 0 \text{ and } \sup_{|z|<1} |f'(z)|(1-|z|^2) \equiv M(f) < \infty\}.$$

This space equipped with the pointwise operations and the norm $\|f\| = M(f)$ becomes a nonseparable Banach space. The space $\mathcal{B}_0$ denotes the closed subspace of $\mathcal{B}$ spanned by the polynomials. In [8], it was shown that every isometry S of $\mathcal{B}_0$ is surjective. Furthermore, there exist λ of modulus 1 and φ an automorphism of the disk, Δ, such that

$$S(f) = \mu(f \circ \varphi - f(\varphi(0))),$$

for all $f \in \mathcal{B}_0$. Hence $\mathcal{B}_0$ satisfies the hypothesis of Theorem 2.3.

Corollary 2.5. *P is a generalized bi-circular projection on $\mathcal{B}_0$ if and only if*

$$Pf(z) = \frac{f(z) \pm [f(\varphi(z)) - f(\varphi(0))]}{2},$$

where φ is a disk automorphism such that $\varphi^2 = id$.

A consequence of the proof of the theorem above and Corollary 3 in [8] is the following statement.

Corollary 2.6. *P is a generalized bi-circular projection on $\mathcal{B}$ if and only if*

$$Pf(z) = \frac{f(z) \pm [f(\varphi(z)) - f(\varphi(0))]}{2},$$

where φ is a disk automorphism such that $\varphi^2 = id$.

3. Generalized bi-circular projections and integral operators

We consider a class of Banach spaces where the surjective isometries are integral operators rather than the more standard weighted composition operators. We show that generalized bi-circular projections on such spaces are also given by integral operators.

We consider $\mathcal{K}$ a separable complex Hilbert space and the spaces S^p and $S^p_{\mathcal{K}}$ of analytic functions f defined on Δ such that f' is in H^p and in $H^p(\mathcal{K})$, respectively. These spaces are equipped with the norm (see [26] and [17])

$$\|f\| = |f(0)| + \|f'\|_p.$$

Surjective linear isometries on these spaces have an integral representation as described in the next theorem.

Theorem 3.1.

(1) (*Cf.* [26]) *Let T be a linear isometry of S^p onto S^p, $p \neq 2$. Then there exist unimodular complex numbers α and β and a disk automorphism φ such that*

$$Tf(z) = \alpha[f(0) + \beta \int_0^z [\varphi'(\xi)]^{1/p} f'(\varphi(\xi))]d\xi, \text{ for all } f \in S^p \text{ and } z \in \Delta.$$

(2) (*Cf.* [17]) *Let T be a linear isometry of $S^p_{\mathcal{K}}$ onto $S^p_{\mathcal{K}}$, $p \neq 2$. Then there exist unitary operators on $\mathcal{K}$, U and V, and a disk automorphism φ such that*

$$Tf(z) = Vf(0) + U \int_0^z [\varphi'(\xi)]^{1/p} f'(\varphi(\xi))d\xi, \text{ for all } f \in S^p_{\mathcal{K}} \text{ and } z \in \Delta.$$

We characterize the generalized bi-circular projections on these spaces by using the form of the isometries stated in the previous theorem. The next theorem gives a characterization of the generalized bi-circular projection on $S^p_{\mathcal{K}}$. We only prove the theorem for the more general case of vector-valued functions.

Theorem 3.2. *P is a generalized bi-circular projection on $S^p_{\mathcal{K}}$ ($p \neq 2$), with $\mathcal{K}$ a separable complex Hilbert space, if and only if P is represented in one of the following ways:*

(1) $P(f)(z) = \dfrac{Vf(0) + U\int_0^z [\varphi'(\xi)]^{1/p} f'(\varphi(\xi))d\xi \, + f(z)}{2}$, *for some disk automorphism φ so that $\varphi^2 = id$ and unitary operators on $\mathcal{K}$, U and V so that $U^2 = V^2 = I$.*

(2) $P(f)(z) = \dfrac{(U - \lambda I)(f(z) - f(0)) + (V - \lambda I)f(0)}{1 - \lambda}$, *for some unimodular complex number λ and unitary operators on $\mathcal{K}$, U and V both satisfying the operator equation $X^2 - (\lambda + 1)X + \lambda I = 0$.*

Proof. Given a generalized bi-circular projection P we denote by T its associated isometry given by

$$Tf(z) = Vf(0) + U\int_0^z [\varphi'(\xi)]^{1/p} f'(\varphi(\xi))d\xi, \text{ for all } f \in S^p_{\mathcal{K}}, \quad \text{and} \quad z \in \Delta,$$

where U and V are unitary operators and φ is a disk automorphism. We also have that

$$T^2(f) - (\lambda+1)Tf + \lambda f = 0,$$

for every $f \in S^p_{\mathcal{K}}$. This equation becomes

$$\begin{aligned} V^2 f(0) + U^2 \int_0^z [(\varphi^2)'(\xi)]^{\frac{1}{p}} f'(\varphi^2(\xi))\, d\xi - (\lambda+1)Vf(0) - \\ -(\lambda+1)U\int_0^z [\varphi'(\xi)]^{\frac{1}{p}} f'(\varphi(\xi))\, d\xi + \lambda f(z) = 0. \end{aligned} \tag{3.1}$$

Differentiating (3.1) we get the following:

$$U^2[(\varphi^2)'(z)]^{\frac{1}{p}} g((\varphi^2)(z)) - (\lambda+1)U[\varphi'(z)]^{\frac{1}{p}} g(\varphi(z)) + \lambda g(z) = 0, \tag{3.2}$$

where $g(z) = f'(z)$. We set $g(z) = z\cdot\nu$ and $g(z) = z^2\cdot\nu$ for a given vector $\nu \in \mathcal{K}$. These choices determine the equations

$$[(\varphi^2)'(z)]^{\frac{1}{p}}((\varphi^2)(z))U^2\nu - (\lambda+1)[\varphi'(z)]^{\frac{1}{p}}(\varphi(z))U\nu + \lambda z\cdot\nu = 0,$$

and

$$[(\varphi^2)'(z)]^{\frac{1}{p}}[(\varphi^2)(z)]^2 U^2\nu - (\lambda+1)[\varphi'(z)]^{\frac{1}{p}}[\varphi(z)]^2\, U\nu + \lambda z^2\cdot\nu = 0.$$

Subtracting the second equation from the first one multiplied by z we have

$$[(\varphi^2)'(z)]^{\frac{1}{p}}[\varphi^2(z)][z-\varphi^2(z)]U^2\nu - (\lambda+1)[\varphi'(z)]^{\frac{1}{p}}\varphi(z)[z-(\varphi(z))]U\nu = 0. \tag{3.3}$$

If φ is not the identity but $\varphi^2 = id$ we have $\lambda = -1$. Hence $T^2 f = f$ and (3.1) reduces to

$$V^2 f(0) + U^2\int_0^z f'(\xi)\, d\xi = f(\xi).$$

This last equation implies that $U^2 = V^2 = I$. Therefore

$$P(f)(z) = \frac{Vf(0) + U\int_0^z [\varphi'(\xi)]^{1/p} f'(\varphi(\xi))d\xi\, + f(z)}{2},$$

as stated in the theorem. If $\varphi = id$, then (3.1) becomes

$$V^2 f(0) + U^2[f(z) - f(0)] - (\lambda + 1)V f(0) - (\lambda + 1)U[f(z) - f(0)] + \lambda f(z) = 0.$$

Therefore we have that $U^2 - (\lambda + 1)U + \lambda I = 0$ and $V^2 - (\lambda + 1)V + \lambda I = 0$. In this case, the projection P has the form

$$P(f)(z) = \frac{(U - \lambda I)(f(z) - f(0)) + (V - \lambda I)f(0)}{1 - \lambda}.$$

It is left to consider when φ and φ^2 are both different from id. We show that these conditions lead to $\lambda = 0$ and hence no solutions exist. In fact, since there exists $z_0 \in \Delta$ such that $\varphi(z_0) \neq z_0 \neq \varphi^2(z_0)$, there exists a polynomial p that assigns the value zero at $\varphi(z_0)$ and $\varphi^2(z_0)$ and the value 1 at z_0. We set $g(z) = p(z) \cdot \nu$, then (3.2) evaluated at $g(z_0)$ yields $\lambda = 0$.

Conversely, it is easy to check that projections of the form described in the theorem are in fact generalized bi-circular projections.

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Remark 3.3. The unitary operator in the scalar case (i.e. $\dim \mathcal{K} = 1$) is just a multiplication by a modulus 1 complex number.

We generalize the previous class of spaces but restrict our attention to just scalar valued functions, since descriptions of surjective isometries in the vector-valued case are not known.

As introduced in [18], let N represent a norm on $H(\Delta)$, the space of analytic and complex valued functions with domain Δ. Let H_N and S_N be the space of functions in $H(\Delta)$ such that $N(f) < \infty$ and $N(f') < \infty$, respectively. Moreover, we also denote by H_N the space H_N when equipped with the norm N and we denote by S_N the space S_N with the norm $\|f\| = |f(0)| + N(f')$. For $1 \leq p < \infty$ and $x > -1$, we consider the norm on $H(\Delta)$

$$N(f) = \left(\frac{1}{\pi} \int_0^{2\pi} \int_0^1 |f(re^{i\theta})|^p (1 - r^2)^x r \, dr \, d\theta \right)^{1/p}. \tag{3.4}$$

The space H_N is a weighted Bergman space (cf. [18]). We cite the following Theorem 3 from [18] concerning the space S_N with norm (3.4).

Theorem 3.4. *Let $x > -1$ and $1 \leq p < \infty$, $p \neq 2$. If T is a surjective isometry of S_N, then there is β and α of modulus 1, an automorphism φ of Δ, and $g = \alpha(\varphi')^{\frac{2+x}{p}}$ so that*

$$T(f)(z) = \beta\left[f(0) + \int_0^z g(\xi)(f' \circ \varphi)(\xi)\, d\xi\right], \text{ for all } f \in S_N,\ z \in \Delta.$$

The following corollary characterizes the generalized bi-circular projection on S_N, and it is a consequence of Theorem 3.4 and the proof of Theorem 3.2.

Corollary 3.5. *Let $x > -1$ and $1 \leq p < \infty$, $p \neq 2$. P is a generalized bi-circular projection on S_N if and only if P is represented in one of the following ways:*

(1) $P(f)(z) = \dfrac{\beta\left[f(0) + \alpha\int_0^z [\varphi'(\xi)]^{(2+x)/p} f'(\varphi(\xi))d\xi\right] + f(z)}{2}$, *for some disk automorphism φ so that $\varphi^2 = id$, α and β are unimodular.*

(2) $P(f)(z) = \dfrac{(\alpha - \lambda)I\, f(z) + (\beta - \alpha)f(0)}{1 - \lambda}$, *for some unimodular complex number λ, β and α.*

Remark 3.6. Corollary 3.5 can be generalized as follows. The Bergman space can be replaced by $\mathcal{F}(\Delta)$, a smooth functional Banach space on the disk which contains the polynomials and in which every surjective isometry is of the form $\mathcal{J}f(z) = \beta[\varphi(z)]^{r/p}f(\varphi(z))$, where φ is a disk automorphism. Theorem 1 in Kolaski's paper [18] applies and the proof of the generalized corollary follows as in Theorem 3.2

4. Generalized bi-circular projections on vector-valued Hardy spaces

In this section we characterize the generalized bi-circular projections on some Hardy spaces of vector valued functions. For $p = \infty$, we consider functions with values in a Banach space satisfying a multiplier condition. For $1 \leq p < \infty$ we restrict to functions that take their values in a separable Hilbert space.

The following theorem is due to Cambern, Jarosz, and Lin [7], [24] and provides a description of surjective isometries necessary for our study of generalized bi-circular projections on $H^\infty(E)$. This theorem is crucial to derive our results. The case when E is the scalar field can be found in Hoffman's book, [16].

Theorem 4.1. *Suppose that E is a Banach space with trivial multiplier algebra. Let T be a surjective linear isometry of $H^\infty(E)$. Then there is a disk automorphism φ and a surjective isometry $\mathcal{J}$ of E such that*

$$(TF)(z) = \mathcal{J}(F(\varphi(z))).$$

The multiplier algebra of the Banach space E will play no significant role in our proofs so we refer the reader to Behrends' book, [1], for the definition and basic results. Among the Banach spaces with trivial multiplier algebras are Hilbert spaces, uniformly convex and uniformly smooth spaces.

Proposition 4.2. *P is a generalized bi-circular projection on $H^\infty(E)$ if and only if there exist a modulus 1 complex number λ $(\neq 1)$ and a surjective isometry $\mathcal{J}$ of E such that $\mathcal{J}^2 - (\lambda+1)\mathcal{J} + \lambda I = 0$, and either*

(1) $PF(z) = \dfrac{\mathcal{J} - \lambda I}{1-\lambda} F((z))$ *or*

(2) $PF(z) = \dfrac{F(z) + \mathcal{J}F((\varphi(z)))}{2}$, *where $\mathcal{J}^2 = I$ and φ is a disk automorphism of the form $\frac{z-a}{\bar{a}z-1}$ $(|a|<1)$.*

Proof. Suppose that P is a generalized bi-circular projection on $H^\infty(E)$. Lemma 2.1 asserts the existence of a surjective isometry T such that

$$P = \frac{T-\lambda I}{1-\lambda},$$

with $|\lambda| = 1$ and $\lambda \neq 1$. The isometry T is represented by

$$(TF)(z) = \mathcal{J}(F(\varphi(z))),$$

where $\mathcal{J}$ is a surjective isometry on E and $\varphi(z) = \mu\frac{a-z}{1-\bar{a}z}$ $(|\mu| = 1$ and $|a|<1)$. Furthermore,

$$T^2F - (\lambda+1)TF + \lambda F = 0,$$

for every $F \in H^\infty(E)$. Hence,

$$\mathcal{J}^2F(\varphi(\varphi(z))) - (\lambda+1)\mathcal{J}F(\varphi(z)) + \lambda F(z) = 0 \tag{4.1}$$

for every $F \in H^\infty(E)$. Let $\nu \in E$ and set $F(z) = 1\cdot\nu$ for $|z|<1$. Then $\mathcal{J}^2\nu$ $-(\lambda+1)\mathcal{J} + \lambda\nu = 0$, as stated in the proposition. We write equation (4.1) as follows:

$$[(\lambda+1)\mathcal{J} - \lambda I]\, F(\varphi(\varphi(z))) - (\lambda+1)\mathcal{J}F(\varphi(z)) + \lambda F(z) = 0$$

or equivalently

$$(\lambda+1)\mathcal{J}\left[F(\varphi(\varphi(z))) - F(\varphi(z))\right] = \lambda\left[F(\varphi(z)) - F(z)\right]. \tag{4.2}$$

Equation (4.2) evaluated at the functions $F(z) = z\cdot\nu$ and $F(z) = z^2\cdot\nu$ ($\nu \in E$) yields the following:

$$(\lambda+1)(\varphi(\varphi(z)) - \varphi(z))\mathcal{J}\nu = \lambda(\varphi(\varphi(z)) - z)\nu$$

and

$$(\lambda+1)([\varphi(\varphi(z))]^2 - [\varphi(z)]^2)\mathcal{J}\nu = \lambda([\varphi(\varphi(z))]^2 - z^2)\nu,$$

respectively. Combining these last two equations we have

$$[\varphi(\varphi(z)) - z]\,[\varphi(z) - z]\,\nu = 0.$$

This implies that $\varphi^2(z) = z$ (for every z). Equation (4.2) reduces to

$$[(\lambda+1)\mathcal{J} - \lambda I]\,F(z) - (\lambda+1)\mathcal{J}F(\varphi(z)) + \lambda F(z) = 0, \tag{4.3}$$

and therefore

$$(\lambda+1)\mathcal{J}F(z) = (\lambda+1)\mathcal{J}F(\varphi(z)).$$

If $\lambda = -1$, then $\mathcal{J}^2 = I$ and it follows that

$$PF(z) = \frac{F(z) + \mathcal{J}F(\varphi(z))}{2}$$

with $\varphi(\varphi(z)) = z$, as stated in the item 2. If $\lambda \neq -1$, then equation (4.3) evaluated at $F(z) = z\cdot\nu$, with $z = 0$, yields

$$-(\lambda+1)\mu a\mathcal{J}\nu = 0. \tag{4.4}$$

This implies $a = 0$, therefore $\varphi(z) = -\mu z$. Since $\varphi(\varphi(z)) = z$, we have $\mu^2 = 1$. It also follows that $F(z) = F(-\mu z)$, for every $F \in H^\infty(E)$. Hence $\mu = -1$ and

$$PF(z) = \frac{\mathcal{J} - \lambda I}{1-\lambda}F(z) = Q_\lambda F(z), \ \ (\lambda \neq -1).$$

The condition $\mathcal{J}^2 - (\lambda+1)\mathcal{J} + \lambda I = 0$ implies that Q_λ is a generalized bi-circular projection on E. Conversely, we suppose that $\mathcal{J}^2 - (\lambda+1)\mathcal{J} + \lambda I = 0$ on E. If $PF(z) = \frac{\mathcal{J}-\lambda I}{1-\lambda}F((z))$ then clearly $P^2 = P$ and $P + \lambda(I-P) = \mathcal{J}$. If $PF(z) = \frac{\mathcal{J}F(\varphi(z))+F(z)}{2}$ then $P(F)(z) - (I-P)(F)(z) = \mathcal{J}F(\varphi(z))$. Since $\mathcal{J}^2 = I$ and $\varphi^2 = id$, we have that $2P - I$ is an isometric reflection. This concludes the proof of the proposition.

■

Remark 4.3. This theorem is also true in the scalar case, i.e. E is the set of complex numbers. The isometry characterization for the scalar case is due to Forelli, cf. [13].

We note that generalized bi-circular projections on $H^\infty(E)$ are bi-contractive, i.e. $\|P\| = \|I - P\| = 1$. To our knowledge the general form of the bi-contractive projections is not known for many spaces of analytic functions. Since the convex combination of two isometries has norm less than or equal to one, it is natural to ask when such a combination is a projection. If it is, must it be a generalized bi-circular projection? If not, must it be bi-contractive? We begin this analysis by establishing when the convex combination can be a projection.

We consider the operators that are a convex combination of two isometries, i.e. $T_\alpha(F)(z) = [\alpha I_1 + (1-\alpha)I_2](F)(z)$, where I_i represents an isometry on $H^\infty(E)$ and α a real number in the open interval (0, 1). The isometries I_i act on a function $f \in H^\infty(E)$ as follows:

$$\mathcal{J}_i f((\varphi_i(z))),$$

where φ_i represents a disk automorphism.

Theorem 4.4. *Given two distinct surjective isometries I_1 and I_2 on $H^\infty(E)$ and E is a Banach space for which the identity is an extreme point of the operator unit ball. If an operator T_α in the convex combination of I_1 and I_2 is a projection, then $\alpha = 1/2$.*

Proof. The operator T_α is a projection if and only if

$$\begin{aligned}\alpha^2\mathcal{J}_1^2F(\varphi_1^2(z)) + \alpha(1-\alpha)\mathcal{J}_2\mathcal{J}_1F(\varphi_1\varphi_2(z)) + \alpha(1-\alpha)\mathcal{J}_1\mathcal{J}_2F(\varphi_2\varphi_1(z)) +\\ +(1-\alpha)^2\mathcal{J}_2^2F(\varphi_2^2(z)) = \alpha\mathcal{J}_1F(\varphi_1(z)) + \alpha\mathcal{J}_2F(\varphi_2(z)).\end{aligned} \tag{4.5}$$

Without loss of generality we assume that $\alpha \geq 1/2$. In addition, we first assume that there exists a point $z_0 \in \Delta$ such that $z_0 \neq \varphi_1(z_0) \neq \varphi_2(z_0) \neq z_0$. We consider a Lagrange polynomial p on Δ such that $p(\varphi_1(z_0)) = 1$ and $p(\varphi_2(z_0)) = p(\varphi_1^2(z_0)) = p(\varphi_1\varphi_2(z_0)) = 0$. We set $F(z) = p(z)\nu$, for some arbitrary $\nu \in E$. The equation (4.5) at $F(z_0)$ is

$$\alpha(1-\alpha)p(\varphi_2\varphi_1(z_0))\mathcal{J}_1\mathcal{J}_2\nu + (1-\alpha)^2p(\varphi_2^2(z_0))\mathcal{J}_2^2\nu = \alpha\mathcal{J}_1\nu,$$

$(\varphi_2\varphi_1(z_0) \neq \varphi_2^2(z_0))$. If both $\varphi_2^2(z_0)$ and $\varphi_2\varphi_1(z_0)$ were not equal to $\varphi_1(z_0)$ then the Langrange polynomial p could be chosen so that $p(\varphi_2^2(z_0)) = p(\varphi_2\varphi_1(z_0)) = 0$ and hence $\mathcal{J}_1$ would be the zero operator. If $\varphi_1(z_0) = \varphi_2^2(z_0) \neq \varphi_2\varphi_1(z_0)$, then p can be chosen so that $p(\varphi_2\varphi_1(z_0)) = 0$ and equation (4.5) reduces to $(1-\alpha)^2\mathcal{J}_2^2\nu = \alpha\mathcal{J}_1\nu$.

Hence $(1-\alpha)^2 = \alpha$, contradicting our initial assumption on α. Now, $\varphi_1(z_0) = \varphi_2\varphi_1(z_0) \neq \varphi_2^2(z_0)$, then p can be chosen so that $p(\varphi_2^2(z_0)) = 0$ and equation (4.5) reduces to $\alpha(1-\alpha)\mathcal{J}_1\mathcal{J}_2\nu = \alpha\mathcal{J}_1\nu$ and $1-\alpha = 1$. This contradiction implies that there exists no such z_0.

We assume that there exists $z_0 \in \Delta$ so that $z_0 \neq \varphi_1(z_0) = \varphi_2(z_0)$. We choose p so that $p(\varphi_1(z_0)) = p(\varphi_2(z_0)) = 1$ and $p(\varphi_1^2(z_0)) = p(\varphi_2^2(z_0)) = p(\varphi_1\varphi_2(z_0)) = p(\varphi_2\varphi_1(z_0)) = 0$. Equation (4.5) reduces to

$$\alpha\mathcal{J}_1\nu + (1-\alpha)\mathcal{J}_2\nu = 0.$$

This implies that $\alpha = 1/2$.

We assume that there exists $z_0 \in \Delta$ so that $z_0 = \varphi_1(z_0) \neq \varphi_2(z_0)$. Since $\varphi_1\varphi_2(z_0) \neq z_0$, we choose a polynomial so that $p(z_0) = 1$ and $p(\varphi_2(z_0)) = p(\varphi_1\varphi_2(z_0)) = 0$. We set $F(z) = p(z)\nu$. Equation (4.5) at $F(z_0)$ reduces to

$$\alpha^2\mathcal{J}_1^2\nu + (1-\alpha)^2 p(\varphi_2^2(z_0))\mathcal{J}_2^2\nu = \alpha\mathcal{J}_1\nu.$$

This equation implies that $\varphi_2^2(z_0) = z_0$. Therefore

$$\alpha^2\mathcal{J}_1^2\nu + (1-\alpha)^2\mathcal{J}_2^2\nu = \alpha\mathcal{J}_1\nu,$$

for all $\nu \in E$. Consequently we have that $\alpha^2 + (1-\alpha)^2 \geq \alpha$ and $\alpha = 1/2$.

Similarly if there exists $z_0 \in \Delta$ so that $z_0 = \varphi_2(z_0) \neq \varphi_1(z_0)$, we must have that $\alpha = 1/2$. In fact, let p be such that $p(\varphi_2(z_0)) = 1$ and $p(\varphi_1(z_0)) = p(\varphi_1^2(z_0)) = 0$, then given $F(z) = p(z)\nu$ we have that $\varphi_1^2(z_0) = z_0$ and $\alpha^2 + (1-\alpha)^2 \geq 1-\alpha$. This implies that $\alpha = 1/2$.

Now we assume that there exists z_0 so that $z_0 = \varphi_1(z_0) = \varphi_2(z_0)$. Therefore for every $\nu \in E$ we have

$$\alpha^2\mathcal{J}_1^2\nu + \alpha(1-\alpha)\mathcal{J}_1\mathcal{J}_2\nu + \alpha(1-\alpha)\mathcal{J}_2\mathcal{J}_1\nu + (1-\alpha)^2\mathcal{J}_2^2\nu = \alpha\mathcal{J}_1\nu + (1-\alpha)\mathcal{J}_2\nu,$$

or equivalently

$$[\alpha\mathcal{J}_1 + (1-\alpha)\mathcal{J}_2]^2\nu = [\alpha\mathcal{J}_1 + (1-\alpha)\mathcal{J}_2]\nu.$$

If $\mathcal{J}_1 = \mathcal{J}_2$ then $\mathcal{J}_1^2 = \mathcal{J}_1$. Hence for every $\nu \in E$ $\mathcal{J}_1\nu - \nu \in \mathrm{Ker}(\mathcal{J}_1)$ and $\mathcal{J}_1 = I$. If $\mathcal{J}_1 \neq \mathcal{J}_2$, let $Q_\alpha = \alpha\mathcal{J}_1 + (1-\alpha)\mathcal{J}_2$. The space E is decomposed into the direct sum $E = Q_\alpha(E) \oplus \mathrm{Ker}(Q_\alpha)$. If there exists a nontrivial vector in $\mathrm{Ker}(Q_\alpha)$, say v, then we have that $\alpha\mathcal{J}_1 v = -(1-\alpha)\mathcal{J}_2 v$ and $\alpha = 1/2$. If Q_α is one-to-one then Q_α is onto and since $Q_\alpha^2 = Q_\alpha$ then $Q_\alpha = I$. Therefore $\mathcal{J}_1 = \mathcal{J}_2 = I$, since I is an extreme point of the operator ball.

■

Corollary 4.5. *A projection given by the average of two distinct isometries I_1 and I_2 on $H^\infty(E)$ is a generalized bi-circular projection.*

Proof. The proof of the theorem defines the following partition of Δ:

$$\Delta_0 = \{z : \varphi_1(z) = z = \varphi_2(z)\}, \qquad \Delta_1 = \{z : \varphi_1(z) = z \neq \varphi_2(z)\},$$

and

$$\Delta_2 = \{z : \varphi_2(z) = z \neq \varphi_1(z)\}.$$

Moreover, given $z \in \Delta_1$ ($z \in \Delta_2$) we must have that $\varphi_2^2(z) = z$ ($\varphi_1^2(z) = z$, respectively). We define the disk automorphism $\varphi(z) = \varphi_1(z)$, if $z \in \Delta_2$ and $\varphi(z) = \varphi_2(z)$, if $z \in \Delta_1$. We observe that $\varphi^2 = id$. If $\Delta_0 = \Delta$, then $\mathcal{J}_1 = \pm\mathcal{J}_2 = I$, defining the identity and the zero projections. Otherwise $\mathcal{J}_1 = \mathcal{J}_2 = I$ and the projection can be written as the average of I with the reflection $R(f) = f(\varphi)$. ∎

We now investigate the form of the generalized bi-circular projections on vector valued Hardy spaces ($1 \leq p < \infty$) $H^p(K)$. The surjective isometries of $H^p(K)$ with K a separable Hilbert space are given in the following theorem due to Pei-Kee Lin (cf. [23]).

Theorem 4.6. *Let $1 \leq p < \infty$, $p \neq 2$, and let K be any complex Hilbert space. If $T\colon H^p(K) \to H^p(K)$ is a surjective isometry, then there exist a unitary operator $\mathcal{U}$ on K, and a disk automorphism φ such that*

$$(TF)(z) = (\varphi'(z))^{1/p}\mathcal{U}(F \circ \varphi(z))$$

for $F \in H^p(K)$ and $|z| < 1$.

We now suppose that T is a surjective isometry associated with a generalized bi-circular projection P on $H^p(K)$. From Lemma 2.1 we know that

$$T^2F - (\lambda + 1)TF + \lambda F = 0 \tag{4.6}$$

for every $F \in H^p(K)$. Theorem 4.6 implies

$$[\varphi'(\varphi(z))\varphi'(z)]^{1/p}\mathcal{U}^2F(\varphi(\varphi(z))) - (\lambda + 1)[\varphi'(z)]^{1/p}\mathcal{U}F(\varphi(z)) + \lambda F(z) = 0. \tag{4.7}$$

Proposition 4.7. *Let $1 \leq p < \infty$, $p \neq 2$, and let K be any complex Hilbert space. Then P is a generalized bi-circular projection on $H^p(K)$ if and only if there exist a modulus 1 complex number $\lambda \neq 1$ and a surjective isometry $\mathcal{U}$ of K such that $\mathcal{U}^2 - (\lambda+1)\mathcal{U} + \lambda I = 0$, and*

$$P_\lambda F(z) = \frac{\mathcal{U} - \lambda I}{1-\lambda} F((z)) \quad or \quad P_\lambda F(z) = \frac{F(z) + \mathcal{U}F((\varphi(z)))}{2},$$

where φ is a disk automorphism of the form $\frac{z-a}{\bar{a}z-1}$ ($|a| < 1$).

Proof. If $\varphi(z_0) \neq z_0$, then we choose a polynomial p so that $p(z_0) = p(\varphi^2(z_0)) = 0$ and $p(\varphi(z_0)) = 1$. Therefore equation (4.7) reduces to the following:

$$(\lambda+1)\varphi'(z_0)^{\frac{1}{p}}\mathcal{U}\nu = 0.$$

This implies that $\lambda = -1$ and equation 4.7 is now written as follows:

$$[\varphi'(\varphi(z_0))\varphi'(z_0)]^{1/p}\mathcal{U}^2\, F(\varphi(\varphi(z_0))) = \lambda F(z_0). \tag{4.8}$$

This implies that $\varphi(\varphi(z_0)) = z_0$ and $|\varphi'(\varphi(z_0))\varphi'(z_0)]^{1/p}| = 1$. Therefore $\varphi^2 = id$ and $\mathcal{U}^2 = I$. If $\varphi = id$, then equation 4.7 reduces to $\mathcal{U}^2 - (\lambda+1)\mathcal{U} + \lambda I = 0$. This last equation is equivalent to saying that $\frac{\mathcal{U}-\lambda I}{1-\lambda}$ is a generalized bi-circular projection on K.

■

Remark 4.8. Let $P = \frac{\mathcal{U}-\lambda I}{1-\lambda}$, defined on the Hilbert space K, be such that $\mathcal{U}^2 - (\lambda+1)\mathcal{U} + \lambda I = 0$. It is easy to check that P is Hermitian, since $\lambda I - U = \lambda U^* - I$.

5. Spaces of analytic functions of several variables

In this section we consider three classical types of spaces of holomorphic functions on the unit ball B of $\mathbb{C}^n$, i.e. the set of all $z \in \mathbb{C}^n$ so that $|z| = \langle z, z\rangle^{\frac{1}{2}} < 1$, we represent by $\mathbb{S}$ the boundary of B:

(1) The weighted Bergman spaces, $A^{p,r}(B)$, of holomorphic functions on B, for which

$$\|f\|_{p,r}^p = \int_B |f(z)|^p [1 - \langle z, z\rangle]^{-r}\, dm(z) < \infty$$

where $dm(z)$ represents the Lebesgue measure in B.

(2) The Banach space of E-valued bounded holomorphic functions on the open disc B of $\mathbb{C}^n$, $H_E^\infty(B)$, and

(3) The Hardy spaces $H^p(B)$ with $0 < p < \infty$, of all holomorphic functions on B such that

$$\sup_{0<r<1} \int_{\mathbb{S}} |f_r|^p d\varphi < \infty,$$

where $d\varphi$ represents the volume measure on $\mathbb{S}$.

Surjective isometries on these spaces have a representation that allows us to characterize the generalized bi-circular projections. A surjective isometry T on a weighted Bergman space (or on $H^p(B)$) is given by (cf. [29])

$$T(f)(z) = \theta\Big[\frac{1-|a|^2}{(1-\langle z,a\rangle)^2}\Big]^{\frac{n+1+x}{p}} f(\phi(z)) \ \Big(\text{or } T(f)(z) = \theta\Big[\frac{1-|a|^2}{(1-\langle z,a\rangle)^2}\Big]^{\frac{n}{p}} f(\phi(z))\Big),$$

where θ is a complex number of modulus 1, $\phi \in \mathrm{Aut}(B)$ such that $\phi(a) = 0$. If E has trivial multiplier, a surjective isometry on $H^\infty_E(B)$ is given by (cf.[25])

$$T(F)(z) = \mathcal{J}F(\phi(z)),$$

where $\mathcal{J}$ is a surjective isometry of E and $\phi \in \mathrm{Aut}(B)$.

Disk automorphisms are described in [29], they are obtained from the unitary operators and from a class of basic automorphisms, defined as follows

$$\varphi_a(z) = \frac{a - \frac{\langle z,a\rangle}{\langle a,a\rangle}a - \sqrt{1-|a|^2}\left(z - \frac{\langle z,a\rangle}{\langle a,a\rangle}a\right)}{1-\langle z,\, a\rangle},$$

for some $a \in B$. Given $\psi \in \mathrm{Aut}(B)$ with $\psi(a) = 0$, there exists a unique unitary operator of B and φ_a such that $\psi = U\varphi_a$.

The next lemma will be used in the proof of the forthcoming theorem.

Lemma 5.1. *Let a, b and c be three points in B so that $a \neq b \neq c$. Then there exists a polynomial p such that $p(a) = p(c) = 1$ and $p(b) = 0$.*

Proof. Without loss of generality we assume that $a_1 \neq b_1$, then let

$$f_a(z) = \frac{z_1 - b_1}{a_1 - b_1} + \prod_{i=2}^{n}(z_i - b_i)(z_i - a_i)$$

be such that $f_a(a) = 1$ and $f_a(b) = 0$. The statement is proven if $a = c$. Hence we assume that $a \neq c$. Similarly we define f_c so that $f_c(c) = 1$ and $f_c(b) = 0$. If $f_a(c) \cdot f_c(a) \neq 1$, then the polynomial function

$$p(z) = \frac{1 - f_c(a)}{1 - f_a(c)f_c(a)} f_a(z) + \frac{1 - f_a(c)}{1 - f_a(c)f_c(a)} f_c(z)$$

satisfies the statement in the lemma. If $f_a(c)\cdot f_c(a)=1$, then we set

$$p(z)=\left\{\frac{-1+\frac{1}{f_a(c)}}{\sum_{i=1}^{n}|a_i-c_i|}\sum_{i=1}^{n} sign(c_i-a_i)(z_i-a_i)+1\right\}\frac{f_c(z)}{f_c(a)},$$

so that $p(b)=0$ and $p(a)=p(c)=1$.

∎

Theorem 5.2. *P is a generalized bi-circular projection on $A^{p,r}(B)$ (or $H^p(B)$) if and only if P is the identity, the zero projection or given by*

$$P(f)(z)=\frac{1}{2}\Big\{\pm\Big[\frac{1-|a|^2}{(1-\langle z,a\rangle)^2}\Big]^{y/p}f(\phi(z))+f(z)\Big\},$$

where $a\in B$, $\phi\in \mathrm{Aut}(B)$ so that $\phi(a)=0$, $\phi^2=id$, and $y=n+1+x$ (or $y=n$, respectively).

Proof. We consider a generalized bi-circular projection P on $A^{p,r}(B)$. We observe that for a generalized bi-circular projection on $H^p(B)$ we follow a similar reasoning and take $x=-1$. The projection P is given by $P(f)(z)=\frac{1}{1-\lambda}[T(f)(z)-\lambda f(z)]$ $(\lambda\neq 1)$ and

$$T(f)(z)=\theta\Big[\frac{1-|a|^2}{1-\langle z,a\rangle)^2}\Big]^{\frac{n+1+x}{p}}f(\phi(z)).$$

Therefore we have

$$T^2f(z)-(\lambda+1)T(f)(z)+\lambda f(z)=0,$$

or equivalently

$$\begin{aligned}\theta^2[1-|a|^2]^{2\frac{n+1+x}{p}}f(\phi^2(z))-\theta(\lambda+1)[1-|a|^2)^{\frac{n+x+1}{p}}f(\phi(z))+\\ +\lambda(1-\langle z,a\rangle)^{2\frac{n+1+x}{p}}(1-\langle\phi(z),a\rangle)^{2\frac{n+1+x}{p}}f(z)=0.\end{aligned}\tag{5.9}$$

In particular, for $z=a$ we have

$$\theta^2[1-|a|^2]^{2\frac{n+1+x}{p}}f(\phi(0))-\theta[1-|a|^2]^{\frac{n+1+x}{p}}f(0)+\lambda[1-|a|^2]^{2\frac{n+1+x}{p}}f(a)=0\tag{5.10}$$

for all $f\in A^{p,r}(B)$. Suppose $a\neq 0$, then $0=\phi(a)\neq\phi(0)$. Lemma 5.1 implies the existence of a polynomial function p such that $p(\phi(0))=p(a)=0$ and $p(0)=1$. We evaluate equation (5.10) for the polynomial function $1-p$ which implies that $\lambda=-1$. Hence $T^2=I$ and $\theta^2=1$. Equation (5.9) reduces to the following:

$$[1-|a|^2]^{2\frac{n+1+x}{p}}f(\phi^2(z))-(1-\langle z,a\rangle)^{2\frac{n+1+x}{p}}(1-\langle\phi(z),a\rangle)^{2\frac{n+1+x}{p}}f(z)=0.$$

If there exists z_0 such that $\phi^2(z_0) \neq z_0$, then Lemma 5.1 implies the existence of a polynomial function p such that $p(\phi^2(z_0)) = 1$ and $f(z_0) = 1$. This leads to a contradiction and thus $\phi^2 = id$. Therefore we have that $T(f)(z) = \pm[\frac{1-|a|^2}{(1-\langle z,a\rangle)^2}]^{\frac{n+1+x}{p}} f(\phi(z))$ and

$$P(f)(z) = \frac{1}{2}\Big\{ \pm \Big[\frac{1-|a|^2}{(1-\langle z,\, a\rangle)^2}\Big]^{y/p} f(\phi(z)) + f(z)\Big\},$$

as stated in the theorem. Now suppose $a = 0$, then $\phi(z) = -\mathcal{U}(z)$, with $\mathcal{U}$ a unitary operator on B. Equation (5.9) reduces to the following:

$$\theta^2 f(\mathcal{U}^2 z) - \theta(\lambda+1) f(-\mathcal{U}z) + \lambda f(z) = 0,$$

for all f. If there exists $z_0 \in B$ so that $\mathcal{U}z_0 \neq -z_0$ then Lemma 5.1 assures the existence of a polynomial function that implies $\lambda = -1$, $\theta^2 = 1$, and $\mathcal{U}^2 = I$. Therefore $P(f)(z) = \frac{1}{2}[\pm f(-\mathcal{U}z) + f(z)]$. If $\mathcal{U} = -I$ then $\theta^2 f(z) - \theta(\lambda+1)f(z) + \lambda f(z) = 0$, and then $\theta = \lambda$ or $\theta = 1$. These imply that $P(f)(z) = 0$, for all $f \in$ or $P = I$, respectively. Conversely, it is easy to check that the projections given are in fact generalized bi-circular projections. This concludes the proof of the theorem. ∎

Similar techniques to those considered in this paper will provide a characterization of generalized bi-circular projections on the vector valued Hardy space $H^\infty_E(B)$, with E a Banach space with trivial multipliers. It is a known result that surjective isometries on this space are of the form $TF(z) = \mathcal{J}F(\phi(z))$ for some disk automorphism ϕ and surjective isometry $\mathcal{J}$ on E, see [25].

Theorem 5.3. *Let E be a complex Banach space with trivial multipliers. Then P is a generalized bi-circular projection on $H^\infty_E(B)$ if and only if one of the following holds.*

(1) *There exists a surjective isometry of the valued space, $\mathcal{J}$, and an automorphism ϕ so that $\mathcal{J}^2 = I$ and $\phi^2 = id$ and*

$$PF(z) = \frac{1}{2}[\mathcal{J}F(\phi(z)) + F(z)],$$

(2) *There exists a surjective isometry of the valued space, $\mathcal{J}$, so that*

$$PF(z) = \frac{\mathcal{J} - \lambda I}{1-\lambda} F(z),$$

where $\frac{\mathcal{J}-\lambda I}{1-\lambda}$ is a generalized bi-circular projection on E.

We conjecture that a more general result stated for a functional Banach space on the unit ball also holds, provided the form of surjective isometries is as described in our previous Theorem 2.3. Such a setting will include the space $\mathcal{B}_0(B)$ considered by Krantz and Ma in [20] where surjective isometries of $\mathcal{B}_0(B)$ will fit as a particular case of the specified form. It is not clear whether Krantz and Ma's isometry characterization extends to the Bloch space as in Corollary 3 in Cima and Wogen's paper, see [8, p. 316].

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