Surjective isometries on spaces of differentiable vector-valued functions

by

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Abstract. This paper gives a characterization of surjective isometries on spaces of continuously differentiable functions with values in a finite-dimensional real Hilbert space.

1. Introduction. We consider the space of continuously differentiable functions on the interval \([0, 1]\) with values in a Banach space \(E\). This function space, equipped with the norm \(\|f\|_1 = \max_{x \in [0, 1]} (\|f(x)\|_E + \|f'(x)\|_E)\), is a Banach space, denoted by \(C^{(1)}([0, 1], E)\).

Banach and Stone obtained the first characterization of the isometries between spaces of scalar-valued continuous functions (see [2, 15]). Several researchers derived extensions of the Banach–Stone theorem to a variety of different settings. For a survey of this topic we refer the reader to [7]. Cambern and Pathak [4, 5] considered isometries on spaces of scalar-valued differentiable functions and gave a representation for the surjective isometries of such spaces. In this paper, we extend their result to the vector-valued function space \(C^{(1)}([0, 1], E)\), for \(E\) a finite-dimensional Hilbert space. We also characterize the generalized bi-circular projections on \(C^{(1)}([0, 1], E)\).

The characterization of the extreme points of the dual unit ball of a closed subspace of the continuous functions a compact Hausdorff space due to Arens and Kelley [6, p. 441] plays a crucial role in our proofs. In addition, the following result by de Leeuw which gives a converse of the Arens–Kelley theorem, for a closed subspace \(\mathcal{X}\) of \(C(\Omega)\) (cf. [11]), is also essential to our methods. To state de Leeuw’s result we need the following definition.

Definition 1.1. The point \(\omega \in \Omega\) is said to be a peak point for \(h \in \mathcal{X}\) if \(h(\omega) = 1\), \(|h(\omega_1)| \leq 1\) for every \(\omega_1 \in \Omega\), and \(|h(\omega_1)| = 1\) at some \(\omega_1 \neq \omega\) if and only if \(|g(\omega_1)| = |g(\omega)|\) for all \(g \in \mathcal{X}\).

2000 Mathematics Subject Classification: 46B04, 46E15.
Key words and phrases: surjective isometries, generalized bi-circular projections.
DOI: 10.4064/sm192-1-4
Theorem 1.2 (cf. [11, p. 61]). If \( \omega \in \Omega \) is a peak point for some \( h \in X \), then the functional \( \Phi \in X^* \) defined by \( \Phi(g) = g(\omega) \) is an extreme point of the unit ball in \( X^* \).

We construct an isometric embedding of \( C^1([0,1], E) \) onto a closed subspace of the space of scalar-valued continuous functions on a compact set. This allows us to describe the form of the extreme points of \( C^1([0,1], E)^*_1 \). We denote by \( B_1 \) the unit ball in a Banach space \( B \). We consider the isometry \( F \) from \( C^1([0,1], E) \) onto a subspace \( \mathcal{M} \) of the scalar-valued continuous functions on \( \Omega = [0,1] \times E_1^* \), with \( E^* \) equipped with the weak* topology, \( F(f) = F_f(x, \varphi, \psi) = \varphi(f(x)) + \psi(f'(x)) \).

The surjective isometry on the dual spaces \( F^*(F_f^*)^*(g) = F_f^*(F_f^*) ) \) maps the extreme points of \( \mathcal{M}^*_1 \) onto the extreme points of \( C^1([0,1], E)^*_1 \). It follows from the Arens--Kelley lemma [6, p. 441] that

\[
\text{ext}(\mathcal{M}^*_1) \subseteq \{ \Phi_{\varphi, \psi} : \Phi_{\varphi, \psi}(F_f^*) = \varphi(f(x)) + \psi(f'(x)), \forall f \in C^1([0,1], E) \}.
\]

Proposition 1.3. If \( E \) is a smooth, separable and reflexive Banach space, over the reals or complex numbers. Then \( \Phi \) is an extreme point of \( \mathcal{M}^*_1 \) if and only if there exists \( (x, \varphi, \psi) \in \Omega \), with \( \varphi \) and \( \psi \) extreme points of \( E_1^* \), such that

\[
\Phi(f) = \varphi(f(x)) + \psi(f'(x)).
\]

Proof: If \( \Phi \) is an extreme point of \( \mathcal{M}^*_1 \), then \( \Phi = \Phi_{\varphi, \psi} \) for some \( \omega = (x, \varphi, \psi) \in \Omega \). If \( \varphi \) (or \( \psi \)) is not an extreme point of \( E_1^* \), then there must exist distinct functionals \( \varphi_1 \) and \( \varphi_2 \) in \( E_1^* \) such that \( \varphi = (\varphi_1 + \varphi_2)/2 \). For \( i = 1, 2 \), we set \( \omega_i = (x, \varphi_i, \psi) \) and

\[
\Phi_{\varphi_i}(F_f^*) = \varphi_i(f(x)) + \psi(f'(x)).
\]

We have \( \Phi = (\Phi_{\varphi_1} + \Phi_{\varphi_2})/2 \) and

\[
|\Phi_{\varphi_i}(F_f^*)| \leq |\varphi_i(f(x))| + |\psi(f'(x))| \leq \|f(x)\|_E + \|f'(x)\|_E \leq \|f\|_1 = \|F_f^*\|_\infty.
\]

On the other hand, there exist \( a_1 \in E_1 \) (where \( i = 1, 2 \)) such that \( \varphi(a_i) = 1 \). Thus, \( f \) is the constant function equal to \( a_i \), then \( \Phi_{\varphi_i}(F_f^*) = 1 \) and \( \Phi_{\varphi_i} \in \mathcal{M}^*_1 \). Thus \( \Phi \) is not an extreme point of \( \mathcal{M}^*_1 \), contradicting our initial assumption. Similar reasoning applies if \( \Phi \notin \text{ext}(\mathcal{E}_1) \).

Now we show that \( \Phi \) given by

\[
\Phi(f) = \varphi(f(x)) + \psi(f'(x)),
\]

with \( \omega = (x, \varphi, \psi) \in \Omega \) and \( \varphi, \psi \in \text{ext}(E_1^*) \), is an extreme point of \( \mathcal{M}^*_1 \).

There exist \( a_1 \) and \( a_2 \) in \( E_1 \) such that \( \varphi(a_1) = e^{i\alpha_1} \) and \( \psi(a_2) = e^{i\alpha_2} \). We define \( f \in C^1([0,1], E) \) by

\[
f(t) = \frac{e^{-i\alpha_1}a_1 + \lambda(t)e^{-i\alpha_2}a_2}{2}
\]

with

\[
\lambda(t) = \begin{cases} \frac{-\lambda (x^2 - t^2) + (x - 1)(x - t)}{2} & \text{for } 0 \leq t \leq x, \\ \frac{-\lambda (t^2 - x^2) + (x + 1)(t - x)}{2} & \text{for } x \leq t \leq 1. \end{cases}
\]

We observe that \( \lambda(x) = 0, \lambda'(x) = 1, \) and \( |\lambda(t)| + |\lambda'(t)| = 1 - \frac{1}{2}(x-t)^2 < 1 \) for all \( t \neq x \). Therefore

\[
|F_f(\omega)| = |\varphi(f(x)) + \psi(f'(x))| = 1.
\]

If \( \omega_1 \neq \omega \) with \( \omega_1 = (x_1, \varphi_1, \psi_1) \) and \( x_1 \neq x \), we have

\[
|F_f(\omega_1)| = |\varphi_1(f(x_1)) + \psi_1(f'(x_1))| = \left| \varphi_1 \left( \frac{e^{-i\alpha_1}a_1 + \lambda(x_1)e^{-i\alpha_2}a_2}{2} \right) \right| + \psi_1 \left( \frac{\lambda'(x)(x)_{2}}{2} \right) 
\leq \frac{1}{2} + \frac{|\lambda(x_1)| + |\lambda'(x_1)|}{2} < 1.
\]

If \( x_1 = x \), and \( \varphi_1 \neq \varphi \) or \( \psi_1 \neq \psi \), then

\[
|F_f(\omega_1)| = |\varphi_1(f(x_1)) + \psi_1(f'(x_1))| = \left| \varphi_1 \left( \frac{e^{-i\alpha_1}a_1}{2} \right) + \psi_1 \left( \frac{e^{-i\alpha_2}a_2}{2} \right) \right| < 1,
\]

unless \( |\varphi_1(e^{-i\alpha_1}a_1)| = 1 \) and \( |\psi_1(e^{-i\alpha_2}a_2)| = 1 \). The conclusion now follows from Theorem 1.2. 

An extreme point of \( \mathcal{M}^*_1 \) is therefore represented by a triplet \( (x, \varphi, \psi) \in \Omega, \) with \( x \in [0,1] \) and \( \varphi, \psi \) extreme points of \( E_1 \). Given the hypothesis on \( E \) we know that \( \text{ext}(E_1^*) = E_1^* \). If \( T \) is a surjective isometry of \( C^1([0,1], E) \), then \( T^* \) maps extreme points to extreme points. Hence Proposition 1.3 asserts that given \( \omega = (x, \varphi, \psi) \) there exists \( \omega_1 = (x_1, \varphi_1, \psi_1) \) such that

\[
\varphi(T_f)(x) + \psi(T_f)(x) = \varphi_1(f(x_1)) + \psi_1(f'(x_1)) \quad \text{for every } f \in C^1([0,1], E).
\]

This determines a transformation \( \tau \), on \( \Omega = [0,1] \times E_1^* \times E_1^* \), associated with the isometry \( T \) and given by

\[
\tau(x, \varphi, \psi) = (x_1, \varphi_1, \psi_1).
\]

Lemma 1.4. \( \tau \) is a homeomorphism.

Proof: We first observe that \( \tau \) is well defined. Suppose there exist two triplets \( \omega_1 = (x_1, \varphi_1, \psi_1) \) and \( \omega_2 = (x_2, \varphi_2, \psi_2) \), both corresponding to \( \omega = (x, \varphi, \psi) \). Then

\[
\varphi_1(f(x_1)) + \psi_1(f'(x_1)) = \varphi_2(f(x_2)) + \psi_2(f'(x_2)).
\]

If \( x_1 \neq x_2 \), we select a function \( f \in C^1([0,1], E) \) constant equal to \( a \), an arbitrary vector in \( E_1 \), on a neighborhood of \( x_1 \), say \( \mathcal{O}_{a_1} \), and equal to
zero on a neighborhood of \( x_2 \), say \( C_{x_2} \), with \( C_{x_2} \cap \Omega_{x_2} = \emptyset \). Equation (1.3) implies that \( \varphi_1(a) = 0 \), so \( \varphi = 0 \). This contradicts \( \varphi \in E_1 \) and shows that \( x_1 = x_2 \). If \( f \) is now chosen to be constant equal to \( a \), an arbitrary vector in \( E_1 \), then (1.3) reduces to \( \varphi_1(a) = \varphi_2(a) \), thus \( \varphi_1 = \varphi_2 \). If \( f \) is given by \( f(x) = (x-x_2)a \) then (1.3) implies that \( \psi_1 = \psi_2 \). Therefore \( \tau \) is well defined. Similar arguments and the invertibility of \( T \) imply that \( \tau \) is a bijection. The continuity of \( \tau \) follows from the weak* continuity of \( T^* \).

2. Properties of the homeomorphism \( \tau \). In this section we explore further properties of the homeomorphism \( \tau \) to be used in our characterization of surjective isometries on \( C^1([0,1],E) \), with \( E \) a real and finite-dimensional Hilbert space.

For a fixed \( x \in [0,1] \) we define the map \( \tau_x : E_1^* \times E_1^* \to [0,1] \) by \( \tau_x(\varphi, \psi) = \pi_1 \tau(x, \varphi, \psi) \), with \( \pi_1 \) representing the projection on the first component.

The next lemma holds for a finite-dimensional Banach range space, the proof does not require an inner product structure.

**Lemma 2.1.** If \( E \) is a finite-dimensional Banach space, then \( \tau_x \) is a constant function.

**Proof.** If \( \tau_x \) is not constant, then its image is a non-degenerate subinterval of \([0,1] \). We select a basis for \( E^* \), say \( \{ \varphi_1, \ldots, \varphi_k \} \), consisting of functionals of norm 1. We select an element \( y \in \tau_x(E_1^* \times E_1^*) \subseteq \tau_x(\varphi_i, \varphi_j), \tau_x(\varphi_i, -\varphi_j) \}_{i,j=1,\ldots,k} \).

Then we set \( \tau(x, \varphi, \varphi) = (x_1, \xi_1, \xi_2), \tau(x, \varphi, -\varphi) = (y_1, \alpha_1, \beta_1), \) and \( \tau(x, \varphi_0, \psi_0) = (y, \eta, \xi) \). We select \( g \in C^1([0,1],E) \) such that, for all \( i = 1, \ldots, k \), \( g(x_i) = g(y) = g'(x_i) = g'(y) = 0 \). Then \( g = u + v \), where \( u \) and \( v \) are such that \( \eta(u) = 1 \) and \( \xi(v) = 1 \). Therefore we have

\[
\varphi_i((Tg)(x))+ \varphi_i((Tg)(x)) = \eta_i(g(x_i)) + \xi_i(g'(x_i)) = 0
\]

and

\[
\varphi_i((Tg)(x)) - \varphi_i((Tg)(x)) = \alpha_i(g(y_i)) + \beta_i(g'(y_i)) = 0.
\]

These equations imply that \( \varphi_i(Tg)(x) = 0 \) and \( \varphi_i(Tg)(x) = 0 \) for all \( i \). Hence \( Tg(x) = Tg(x) = 0 \), implying that \( 2 = \eta(g(y)) + \xi(g'(y)) = 0 \). This contradiction establishes the claim.

For fixed \( x \in [0,1] \) and \( \varphi \in E_1^* \), we define the map \( \tau_x(\varphi) : E_1^* \to E_1^* \) by \( \tau_x(\varphi)(\psi) = \varphi \). Provided that \( \tau(x, \varphi, \psi) = (x_1, \varphi, \psi) \).

**Lemma 2.2.** If \( E \) is a finite-dimensional real Hilbert space then, for any fixed \( x \in [0,1] \) and \( \varphi \in E_1^* \), \( \tau_x(\varphi) \) is constant.

**Proof.** The Riesz Representation Theorem allows us to set notation as follows: \( \varphi, \psi \in E_1^* \) are completely determined by the inner product with a single vector \( u, v \) respectively. Hence we define \( \tau : [0,1] \times E_1 \times E_1 \to [0,1] \times E_1 \times E_1 \) by \( \tau(x,u,v) = (x_1, u_1, v_1) \), and for every \( f \in C^1([0,1],E) \),

\[
(\tau f)(x), u + (\tau f)(x), v = (f(x_1), u_1) + (f(x_1), v_1).
\]

For fixed \( x \) and \( u \), let \( F_{(x,u)} : E_1 \to E_1 \) be given by \( F_{(x,u)}(v) = \pi_0(\tau(x,u,v)) \), where \( \pi_0 \) is the projection on the second component. We prove the lemma by showing that \( F_{(x,u)} \) is constant. For simplicity we denote \( F_{(x,u)} \) by \( F \), unless the dependence on \( x, u \) has to be emphasized.

We choose \( f(t) = a \), a unit vector. Then

\[
(\tau f)(x), u + (\tau f)(x), v = (a, F(\pm v)).
\]

This implies that

\[
(\tau f)(x), u = \left( a, \frac{F(v) + F(-v)}{2} \right)
\]

for every \( v \in E_1 \). The function \( G : E_1 \to E_1 \) defined by \( G(v) = F(v) + F(-v) \) is therefore constant, denoted by \( w \). As a consequence, for every \( v_0 \) and \( v_1 \) in \( E_1 \), we have

\[
(F(v_0), F(-v_0)) = (F(v_1), F(-v_1)), \quad (F(v_0), F(v_1)) = (F(-v_0), F(-v_1)).
\]

Therefore

\[
\| F(v_0) - F(-v_0) \|^2 = F(v_0) - F(-v_0) = 2 - 2(F(v_0), F(-v_0)) = 2 - 2(F(v_1), F(-v_1))
\]

and

\[
\| F(v_0) - F(-v_0) \|^2 = F(v_1) - F(-v_1) \|^2.
\]

Moreover, the function \( H : E_1 \to \mathbb{R} \) given by \( H(v) = \| F(v) - F(-v) \| \) is also constant. This implies that

\[
(F(v_0) - F(-v_0), F(v_0) + F(-v_0)) = 0.
\]

If \( v \in E_1 \) is such that \( (F(v), F(-v)) \) is linearly independent, we set \( \Pi_v \) to be the two-dimensional space spanned by \( F(v) \) and \( F(-v) \). Clearly \( w \in \Pi_v \). In the plane \( \Pi_v \), we represent \( F(v) \) by \( |w| \|w\| e^{i\alpha} \) and \( F(-v) \) by \( |w| \|w\| e^{-i\alpha} \).

This is the polar representation of \( F(v) \) and \( F(-v) \) in \( \Pi_v \), with \( w \) identified with the positive direction of the z-axis. Without loss of generality, we choose \( \alpha = (0, \pi) \). This, in particular, implies that \( F(v) = F(v) = (2 \cos(\alpha) \|w\|) e^{i\alpha} \) and \( 2 \cos(\alpha) = \|w\| \). The value of \( \alpha \) is then uniquely determined, so \( \{ F(v), F(-v) \} \) are the only two values in the range of \( F \) belonging to the plane \( \Pi_v \). The line \( \partial w \) divides the line segment \( F(v)F(-v) \) into two equal parts. Since \( G \) is a constant function we have

\[
F(E_1) \subseteq \left( \frac{w}{2} + \{w\} \right) \cap S \left( \frac{w}{2}, \frac{\| F(v) - F(-v) \|}{2} \right)
\]

with \( S(x, \delta) \) representing the set of points in \( E \) at distance \( \delta \) from \( x \in E \).
and \( \{w\} \) representing the space orthogonal to the span of \( w \). We also notice that \( F(v) \neq \pm F(-v) \) for every \( v \).

These considerations imply that \( F \) maps the \( n - 1 \)-sphere \( \text{ext}(E_1) \) to a set homeomorphic to a subset of an \( n - 2 \)-sphere, and \( F \) sends antipodal points to antipodal points. We now show that such a map cannot exist.

First, for \( n = 2 \) this would mean that \( F \) would map \( S^1 \) onto two points, which is impossible since \( S^1 \) is connected and \( F \) is continuous. The general case is a consequence of the Borsuk–Ulam Theorem (see [13, p. 266]).

Therefore \( \{F(v), F(-v)\} \) is linearly dependent, and as a consequence, we consider the following two possibilities:

(i) \( F(v) = F(-v) \) for every \( v \),
(ii) \( F(-v) = -F(v) \) for every \( v \).

In case (i), we have \( F(v) = w/2 \) for every \( v \), so \( F \) is constant.

In case (ii), given two different vectors \( v_0 \) and \( v_1 \) in \( E_1 \) we have

\[
\langle (TF)(x), u \rangle + \langle (TF)(x), \frac{v_0 + v_1}{\|v_0 + v_1\|} \rangle = \langle a, F\left( \frac{v_0 + v_1}{\|v_0 + v_1\|} \right) \rangle,
\]

\[
\langle (TF)(x), u \rangle - \langle (TF)(x), \frac{v_0 + v_1}{\|v_0 + v_1\|} \rangle = \langle a, F\left( -\frac{v_0 + v_1}{\|v_0 + v_1\|} \right) \rangle.
\]

Hence \( \langle (TF)(x), u \rangle = 0 \) and

\[
\langle (TF)(x), \frac{v_0 + v_1}{\|v_0 + v_1\|} \rangle = \langle a, F\left( \frac{v_0 + v_1}{\|v_0 + v_1\|} \right) \rangle.
\]

This implies that

\[
(F(v_0) + F(v_1)) \bigg/ \|v_0 + v_1\| \bigg/ \|v_0 + v_1\| \bigg/ \|v_0 + v_1\| = F\left( \frac{v_0 + v_1}{\|v_0 + v_1\|} \right),
\]

\[
\|F(v_0) + F(v_1)\| = \|v_0 + v_1\|.
\]

Equation (2.2) implies that

\[
(F(v_0), F(v_1)) = (v_0, v_1),
\]

or \( F \) is norm preserving. We define a map \( \Theta : [0, 1] \times E_1 \to C(E_1, E_1) \) by \( \Theta(x, u)(v) = F_{x,u}(v) \). It follows from Lemma 1.4 that \( \Theta \) is continuous. Furthermore, we have shown that, for each \( (x, u) \in [0, 1] \times E_1 \), \( \Theta(x, u) \) is either constant or an isometry in \( E_1 \).

The continuity of \( \Theta \) and the connectedness of the domain \([0, 1] \times E_1 \) implies that the range of \( \Theta \) consists only of constant functions on \( C(E_1, E_1) \) or only of norm preserving functions on \( E_1 \) that map antipodal points onto antipodal points. This last assertion follows from the fact that the distance between one such norm preserving map on \( E_1 \) and a constant function is greater than or equal to \( \sqrt{2} \). In fact, let \( F_{x_0, u_0} = \Theta(x_0, u_0) \) be a constant function, everywhere equal to \( a \), and \( F_{x_1, u_1} = \Theta(x_1, u_1) \) be norm preserving on \( E_1 \) with \( F_{x_1, u_1}(-v) = -F_{x_1, u_1}(v) \) for all \( v \in E_1 \). Then we have

\[
\|F_{x_0, u_0} - F_{x_1, u_1}\|_\infty = \max_{v \in E_1}\{|F_{x_0, u_0}(v) - F_{x_1, u_1}(v)|\}.
\]

Furthermore,

\[
\|F_{x_0, u_0}(v) - F_{x_1, u_1}(v)|_E = \|a - F_{x_1, u_1}(v)|_E,
\]

\[
\|F_{x_0, u_0}(-v) - F_{x_1, u_1}(-v)|_E = \|a + F_{x_1, u_1}(v)|_E,
\]

implying that

\[
d = \|a - F_{x_1, u_1}(v)|_E^2 + \|a + F_{x_1, u_1}(v)|_E^2 \leq 2 \max\{\|a \pm F_{x_1, u_1}(v)|_E^2\}.
\]

Consequently,

\[
\|F_{x_0, u_0} - F_{x_1, u_1}\|_\infty \geq \sqrt{2}.
\]

As mentioned before, this implies that the range of \( \Theta \) contains only constant functions or only norm preserving maps. Now we show that the assumption that the range of \( \Theta \) contains only norm preserving maps that send antipodal points to antipodal points leads to a contradiction.

In fact, if the range of \( \Theta \) contains only such maps, then for a fixed constant function \( f \) on \([0, 1] \) equal to \( a \in E_1 \), we have

\[
\langle (TF)(x), u \rangle + \langle (TF)(x), v \rangle = \langle a, F_{x_0}(u) \rangle
\]

and

\[
\langle (TF)(x), u \rangle - \langle (TF)(x), v \rangle = \langle a, F_{x_0}(v) \rangle.
\]

Therefore \( \langle (TF)(x), u \rangle = 0 \) for all \( u \) and \( x \), and so \( TF \) is zero. This completes the proof.

REMARK 2.3. We mention that we can also prove, following a similar strategy, that for a fixed \( x \in [0, 1] \) and \( \psi \in E_1 \), the map \( \tau_{x, \psi} : E_1^* \to E_1^* \) such that

\[
\tau_{x, \psi}(\varphi) = \psi_1
\]

is constant. This result is stated in Lemma 3.2 of the next section.

3. Surjective isometries of \( C^1([0, 1], E) \). In this section we establish that surjective isometries on \( C^1([0, 1], E) \) are composition operators. First, we prove preliminary results about surjective isometries on these spaces.

The space \( E \) is a finite-dimensional Hilbert space. The Riesz Representation Theorem allows us to associate a unique unit vector to each functional in \( E_1^* \). Then we represent \( \tau : [0, 1] \times E_1 \times E_1 \to [0, 1] \times E_1 \times E_1 \) given by \( \tau(x, u, v) = \tau(x, u_1, v_1) \) with \( u, v, u_1, v_1 \) corresponding to \( \varphi, \psi, \varphi_1, \psi_1 \) respectively.

LEMMA 3.1. If \( E \) is a finite-dimensional real Hilbert space and \( T \) is a surjective isometry on \( C^1([0, 1], E) \) then \( T \) maps constant functions to constant functions.
Proof. We assume that there exists a constant function \( f \in C^1([0, 1], E) \) with \( f(t) = a \), a vector in \( E \), such that \( Tf \) is not constant. This means there exists \( x_0 \in [0, 1] \) such that \( (Tf)'(x_0) \neq 0 \). We choose a vector \( v_0 \) in \( E \) orthogonal to \( (Tf)'(x_0) \), i.e., \( \langle (Tf)'(x_0), v_0 \rangle = 0 \). We set \( \tau(x_0, u, v) = (x_1, u_1, v_1) \). Then
\[
\langle (Tf)(x_0), u \rangle + \langle (Tf)'(x_0), v \rangle = \langle a, u_1 \rangle.
\]
Lemma 2.2 implies that
\[
\langle (Tf)(x_0), u \rangle + \langle (Tf)'(x_0), v \rangle = \langle a, u_1 \rangle.
\]
Therefore \( \langle (Tf)(x_0), u \rangle = \langle a, u_1 \rangle \) and \( \langle (Tf)'(x_0), u \rangle = 0 \) for every \( v \). This contradicts our initial assumption that \( (Tf)'(x_0) \neq 0_E \), and completes the proof.

For a fixed \( x \in [0, 1] \) and \( v \in E \), we define \( \tau(x, v) : E_1 \to E_1 \) by
\[
\tau(x, v)(u) = v_1 \quad \text{provided that} \quad \tau(x, u, v) = (x_1, u_1, v_1).
\]

**Lemma 3.2.** If \( E \) is a finite-dimensional real Hilbert space then, for fixed \( x \in [0, 1] \) and \( v \in E \), \( \tau(x, v) \) is constant.

**Proof.** We follow the steps in the proof of Lemma 2.2 with the following modification. We consider functions of the form \( f(t) = (t-x_1) a \) with \( a \) a unit vector in \( E \), and set \( F(u) = v_1 \) with \( u \) and \( v_1 \) associated with the functions \( \varphi \) and \( \psi_1 \), respectively. A similar strategy to that followed in Lemma 2.2 allows us to conclude that either \( F \) is constant or \( (Tf)' \) is zero. If \( (Tf)' \) is zero, then \( Tf \) is constant. Lemma 3.1 and the surjectivity of \( T \) imply that \( f \) must be constant. This contradiction completes the proof.

**Lemma 3.3.** If \( E \) is a finite-dimensional real Hilbert space, \( x \) and \( x_1 \) are such that \( \tau(x, u, v) = (x_1, u_1, v_1) \), and \( f \in C^1([0, 1], E) \), then \( f(x_1) = 0 \) implies that \( (Tf)(x_1) = 0_E \).

**Proof.** Equation (2.1) reduces to
\[
\langle (Tf)(x), u \rangle + \langle (Tf)'(x), v \rangle = \langle f'(x_1), v_1 \rangle.
\]
Now considering \( u_0 \in E_1 \), Lemmas 2.1 and 3.2 imply that
\[
\langle (Tf)(x), u \rangle + \langle (Tf)'(x), v \rangle = \langle f'(x_1), v_1 \rangle.
\]
Therefore \( \langle (Tf)(x), u - u_0 \rangle = 0 \). Since \( u_0 \) is chosen arbitrarily in \( E_1 \) we conclude that \( Tf(x) = 0_E \).

**Lemma 3.4.** If \( E \) is a finite-dimensional real Hilbert space and \( T \) is a surjective isometry on \( C^1([0, 1], E) \), then there exists a surjective isometry \( U \) on \( E \) and a homeomorphism \( \sigma \) on the interval \( [0, 1] \) such that
\[
T(f)(t) = U(f(\sigma(t))
\]
for every \( f \in C^1([0, 1], E) \).

**Proof.** We define \( U(v) = T(\delta)(0) \) with \( \delta \) representing the constant function equal to \( v \). Since \( T \) is a surjective isometry, \( U \) is also a surjective isometry on \( E \). Given \( f \in C^1([0, 1], E) \) and \( x_1 \in [0, 1] \) we denote by \( f_1 \) the function given by \( f_1(t) = f(t) - f(x_1) \). Lemma 3.3 implies that \( T(f_1)(x) = 0_E \). Therefore
\[
T(f)(x) = U(f(\sigma(x))).
\]
We set \( \sigma(x) = x_1 \); Lemmas 1.4 and 2.1 imply that \( \sigma \) is a homeomorphism.

**Theorem 3.5.** If \( E \) is a finite-dimensional real Hilbert space, then \( T \) is a surjective isometry on \( C^1([0, 1], E) \) if and only if there exists a surjective isometry on \( E \) such that for every \( f \),
\[
T(f)(x) = U(f(\sigma(x)))
\]
with \( \sigma = \text{Id} \) or \( \sigma = 1 - \text{Id} \).

**Proof.** It is clear that a composition operator of the form described in the theorem is a surjective isometry on \( C^1([0, 1], E) \). Conversely, if \( T \) is a surjective isometry then Lemma 3.4 asserts the existence of a surjective isometry \( U \) on \( E \) and a homeomorphism \( \sigma \) on the interval \([0, 1] \) such that
\[
T(f)(t) = U(f(\sigma(t)))
\]
for every \( f \in C^1([0, 1], E) \). In particular, if \( f(x) = xa \) with \( a \) an arbitrary vector in \( E \), then \( T(f)(x) = \sigma(x)U(a) \). This implies that \( \sigma \) is continuously differentiable. Similar considerations applied to \( T^{-1} \) imply that \( \sigma^{-1} \) is also continuously differentiable. Therefore \( \sigma \) is a diffeomorphism of \([0, 1] \). Since
\[
\|T(f)\| = \max_{x_0} \|Tf(x)\|_E + \|Tf'(x)\|_E \quad \text{and} \quad T(f)(x) = U(f(\sigma(x))) \quad \text{with} \quad U \quad \text{an isometry on} \quad E\text{, we have}
\]
\[
\|Tf\|_1 = \max \{ \|Uf(\sigma(x))\|_E + \|Uf'(\sigma(x))\|_E | \sigma'(x) \} = \max \{ \|f(\sigma(x))\|_E + \|f'(\sigma(x))\|_E | \sigma'(x) \}
\]
and
\[
\|f\|_1 = \max \{ \|f(x)\|_E + \|f'(x)\|_E \} = \|f(x_0)\|_E + \|f'(x_0)\|_E
\]
for some \( x_0 \in [0, 1] \). Therefore \( |\sigma'(x_0)| \leq 1 \). On the other hand, \( T^{-1}(f)(x) = U^{-1}f(\sigma^{-1}(x)) \) and
\[
\tau \|T^{-1}f\|_1 = \max \{ \|U^{-1}f(\sigma^{-1}(x))\|_E + \|U^{-1}f'(\sigma^{-1}(x))\|_E | \sigma^{-1}(x) \} = \max \{ \|f(\sigma^{-1}(x))\|_E + \|f'(\sigma^{-1}(x))\|_E | \sigma^{-1}(x) \}
\]
Therefore \( |\sigma'(x_0)| = 1/|\sigma'(0)| \leq 1 \) and so \( |\sigma'(x_0)| \geq 1 \). To conclude that \( |\sigma'(x)| = 1 \) for all \( x \), we need to show that for every \( x \in [0, 1] \) there exists \( f \) such that \( \|f\|_1 = \|f(x)\|_E + \|f'(x)\|_E \) and \( f'(x) \neq 0 \). We consider
\( f_x(t) = \lambda_x(t)a \) with \( a \) a unit vector in \( E \) and \( \lambda_x \) given as in (1.1)

\[
\lambda_x(t) = \begin{cases} 
-\frac{1}{2}(x^2 - t^2) + (x-1)(x-t) & \text{for } 0 \leq t \leq x, \\
-\frac{1}{2}(t^2 - x^2) + (x+1)(t-x) & \text{for } x \leq t \leq 1.
\end{cases}
\]

Hence \(|\sigma'| = 1\) and so \( \sigma = \text{Id} \) or \( 1 - \text{Id} \).

**Remark 3.6.** If the range space \( E \) is an infinite-dimensional separable Hilbert space then there are nonsurjective isometries. Let \( \{e_n\}_{n \in \mathbb{N}} \) be an orthonormal basis and \( U \) be the operator defined by \( U(e_n) = e_{2n} \). The isometry \( T : C^1([0,1],E) \to C^1([0,1],E) \) given by \( T(f)(x) = U(f(x)) \) is not surjective. It is not clear, whenever \( E \) is finite-dimensional, if there are isometries on \( C^1([0,1],E) \) which are not surjective.

Theorem 3.5 was stated for range spaces that are finite-dimensional Hilbert spaces over the reals, and we now extend our characterization to finite-dimensional Hilbert spaces over the complex numbers.

**Corollary 3.7.** If \( E \) is a finite-dimensional complex Hilbert space, then \( T \) is a surjective isometry on \( C^1([0,1],E) \) if and only if there exists a surjective isometry \( U \) on \( E \) such that, for every \( f \),

\[
T(f)(x) = U(f(\sigma(x)))
\]

with \( \sigma = \text{Id} \) or \( \sigma(x) = 1 - \text{Id} \).

**Proof.** The space \( E \) is equipped with an inner product over \( C \), denoted by \((\cdot,\cdot)\). This inner product induces a norm on \( E \), denoted by \( \| \cdot \| \), and the norm \( \| \cdot \|_1 \) is defined on the space \( C^1([0,1],E) \). We define another inner product \((\cdot,\cdot)\) on \( E \) by

\[
(u,v) = \text{Re} (u,v).
\]

The space \( E \) with multiplication only by real scalars and equipped with this real inner product \((\cdot,\cdot)\), is a Hilbert space, denoted by \( \tilde{E} \). The induced norm is denoted by \( \| \cdot \| \) and

\[
\|f\|_1 = \sup_{x \in [0,1]} \{ \|f(x)\| + \|f'(x)\| \}
\]

is the corresponding norm on \( C^1([0,1],[\tilde{E}]) \). The identity map \( (E, \| \cdot \|) \to (E, \| \cdot \|) \) is real linear. Furthermore, given \( u \in E \) we have

\[
\|u\|^2 = (u,u) = \text{Re}(u,u) = \|u\|^2.
\]

Consequently, \((\tilde{E}, \| \cdot \|)\) and \((E, \| \cdot \|)\) are linearly isometric as real Banach spaces. If \( T \) is a surjective isometry on \( C^1([0,1], \tilde{E}) \), then \( \tilde{T} \), on \( C^1([0,1],\tilde{E}) \), given by \( \tilde{T}(f) = T(f) \) is also a surjective isometry. In fact,

\[
\|\tilde{T}f\|_1 = \sup_{t \in [0,1]} \{ \|\tilde{T}f(t)\| + \|\tilde{T}f'(t)\| \} = \sup_{t \in [0,1]} \{ \|f(t)\| + \|f'(t)\| \} = \|f\|_1.
\]

Theorem 3.5 now asserts that there exists a real isometry \( U \) on \( \tilde{E} \) and \( \sigma = \text{Id} \) or \( 1 - \text{Id} \) so that \( \tilde{T}(f)(t) = U(f(\sigma(t))) \). Then it follows that \( T(f)(t) = U(f(\sigma(t))) \). It also follows that \( U \) is a complex linear isometry by considering constant functions. This concludes the proof.

**4. Generalized bi-circular projections on \( C^{(1)}([0,1],E) \).** In this section we give a characterization of all generalized bi-circular projections on \( C^{(1)}([0,1],E) \) with \( E \) a finite-dimensional complex Hilbert space. We start by reviewing the definition of generalized bi-circular projection.

**Definition 4.1.** A bounded linear projection \( P \) on \( C^{(1)}([0,1],E) \) is said to be a generalized bi-circular projection if and only if there exists a modulus \( 1 \) complex number \( \lambda \), different from 1, so that \( P + \lambda \text{Id} - P \) is an isometry \( T \) on \( C^{(1)}([0,1],E) \).

The isometry \( T \) must satisfy the following operator quadratic equation:

\[
T^2 - (1 + \lambda)T + \lambda \text{Id} = 0.
\]

Since \( T \) is a surjective isometry, Theorem 3.5 implies the existence of a surjective isometry \( U \) on \( E \) such that

\[
U^2f(x) - (1 + \lambda)U(f(\sigma(x))) + \lambda f(x) = 0.
\]

Therefore if \( \lambda = -1 \) then \( U^2 = \text{Id} \) and \( P \) is the average of the identity with an isometric reflection \( R(f)(x) = U(f(\sigma(x))) \). If \( \lambda \neq -1 \), then \( \sigma(x) = x \) for every \( x \in [0,1] \) and \( U^2 - (1 + \lambda)U + \lambda \text{Id} = 0 \). Hence

\[
P(f) = \frac{U - \lambda \text{Id}}{1 - \lambda}f(x).
\]

We summarize the previous considerations in the following proposition.

**Proposition 4.2.** Let \( E \) be a finite-dimensional complex Hilbert space. Then \( P \) is a generalized bi-circular projection on \( C^{(1)}([0,1],E) \) if and only if there exists a generalized bi-circular projection \( P_E \) on \( E \) so that \( Pf(x) = P_E(f(x)) \).

**Remark 4.3.** We wish to thank the referee for several helpful suggestions that resulted in a substantial improvement of this paper. The referee also suggested that the proof of our main result could be shortened by using results by Jarosz and Pathak in [9].

**References**


Weighted variable $L^p$ integral inequalities for the maximal operator on non-increasing functions

by

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Abstract. Let $B_p$ be the Ariño–Muckenhoupt weight class which controls the weighted $L^p$-norm inequalities for the Hardy operator on non-increasing functions. We replace the constant $p$ by a function $p(x)$ and examine the associated $L^{p(x)}$-norm inequalities of the Hardy operator.

1. Introduction. The weights $w : \mathbb{R}_+ \to \mathbb{R}_+$ for which the Hardy operator

$$Hf(x) = \frac{1}{x} \int_0^x f(t) \, dt$$

on non-negative non-increasing functions $f$ (we write simply $f \uparrow$) is bounded:

$$\int_0^\infty Hf(x)^p w(x) \, dx \leq c_p \int_0^\infty f(x)^p w(x) \, dx, \quad 1 \leq p < \infty,$$

have been characterized by Ariño and Muckenhoupt [1] by the condition

$$w \in B_p : \int_0^\infty \left( \frac{r}{x} \right)^p w(x) \, dx \leq c_p \int_0^\infty w(x) \, dx. \quad (2)$$

A different proof of (1)$\Rightarrow$(2) was given by me in [7] where it is also apparent that in the implication (2)$\Rightarrow$(1) the constant $c_p$ can be taken to be $(c+1)^p$. For (1)$\Rightarrow$(2) one uses the test function $f = \chi_{[0,r]}$ and (2) follows with $c = c_p$. We also note that for $f \uparrow$, $Hf(x)$ equals $Mf(x)$, the Hardy–Littlewood maximal function.

In the past few years a great deal of attention has been paid to the problem of the boundedness of $M$ on variable $L^p$-spaces. If $p : \mathbb{R}^n \to [1, \infty)$ and $w : \mathbb{R}^n \to \mathbb{R}_+$, let $L^{p(x)}(w)$ be the collection of all functions $f : \mathbb{R}^n \to \mathbb{R}$

2000 Mathematics Subject Classification: Primary 42B25.
Key words and phrases: weights, Hardy operator, variable $L^p$.

DOI: 10.4064/sm192-1-5 [51] © Instytut Matematyczny PAN, 2009