Asymptotics of generalized Hadwiger numbers

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Abstract

We give asymptotic estimates for the number of non-overlapping homothetic copies of some centrally symmetric oval B which have a common point with a 2-dimensional domain F having rectifiable boundary, extending previous work of the L.Fejes-Toth, K.Borockzy Jr., D.G.Larman, S.Sezgin, C.Zong and the authors. The asymptotics compute the length of the boundary ∂F in the Minkowski metric determined by B. The core of the proof consists of a method for sliding convex beads along curves with positive reach in the Minkowski plane. We also prove that level sets are rectifiable subsets, extending a theorem of Erdös, Oleksiv and Pesin for the Euclidean space to the Minkowski space.

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1 Introduction

For closed topological disks $F, B \subseteq \mathbb{R}^d$, we denote by $N_\lambda(F, B) \in \mathbb{Z}_+$ the following generalized Hadwiger number. Let $A_{F,B,\lambda}$ denote the family of all sets, homothetic to B in the ratio λ, which have only boundary points in common with F. Then $N_\lambda(F, B)$ is the greatest integer k such that $A_{F,B,\lambda}$ contains k sets with pairwise disjoint interiors. In particular, $N_1(F, F)$ is the Hadwiger number of F and $N_\lambda(F, F)$ the generalized Hadwiger number considered first by Fejes Toth for polytopes in ([8, 9]) and further in [2]. Extensive bibliography and results concerning this topic can be found in [3]. The main concern of this note is to find asymptotic estimates for $N_\lambda(F, B)$ as λ approaches 0, in terms of geometric invariants of F and B, as it was done for $F = B$ in [2], and to seek for the higher order terms.

Roughly speaking, counting the number of homothetic copies of B packed along the surface of a d-dimensional body F amounts to compute the $(d-1)$-area of its boundary, up to a certain density factor depending only on B. The density factor is especially simple when dimension $d = 2$.

A bounded convex centrally symmetric domain B determines a Banach structure on $\mathbb{R}^n$ and thus a metric, usually called the Minkowski metric associated to B (see [17, 21]). In particular it makes sense to consider the length of curves with respect to the Minkowski metric.

The main result of this paper states the convergence of the number of homothetic copies times the homothety factor to half of the Minkowski length of ∂F, in the case of planar domains F having rectifiable boundary. In order to achieve this we need first a regularity result concerning level sets that we are able to prove in full generality in the first section. This is a generalization of a theorem due to Erdös, Oleksiv and Pesin for the Euclidean space to the Minkowski space. The core of the paper is the second section which is devoted to the proof of the main result stated above. We first prove it for curves of positive reach (following Federer [7]) and then deduce the general case from this. The remaining sections contain partial results concerning the higher order terms for special cases (convex and positive reach domains) and an extension of the main result in higher dimensions for domains with convex and smooth boundary.

2 Level sets

Through out this section B will denote a centrally symmetric compact convex domain in $\mathbb{R}^n$. Any such B determines a norm $\| \|_B$ by $\|x-y\|_B = \|x-y\|/\|o-z\|$, where $\|\|$ is the Euclidean norm, o is the center of B and z is a point on the boundary ∂B of B such that the half-lines $|oz$ and $|xy$ are parallel. When equipped with this norm, $\mathbb{R}^n$ becomes a Banach space whose unit disk is isometric to B. We also denote by d_B the distance in the $\| \|_B$ norm, called also the Minkowski metric structure on $\mathbb{R}^n$ associated to B. We set xy, respectively $|xy$, $|xy|$ for the line, respectively half-line and segment determined by the points x and y. As it is well-known in Minkowski geometry segments are geodesics but when B is not strictly convex one might have also other geodesic segments than the usual segments.

The goal of this section is to generalize the Erdös theorem about the Lipschitz regularity of level sets from the Euclidean space to an arbitrary Minkowski space (see [6]). We will make use of it only for $n = 2$ in the next section but we think that the general result is also of independent interest (see also [11, 12]).

The theorem for the Euclidean space was stated and the beautiful ideas of the proof were sketched by Erdös in [6]; forty years later the full details were worked out by Oleksiv and Pesin in [18].

Theorem 1. *If the set M is bounded and r is large enough then the level set $M_r = \{x \in \mathbb{R}^n; d_B(x, M) = r\}$ is a Lipschitz hypersurface in the Minkowski space. Furthermore, for arbitrary $r > 0$ the level set M_r is the union of finitely many Lipschitz hypersurfaces and in particular it is a $(n-1)$-rectifiable subset of $\mathbb{R}^n$.*

Proof. Our proof extends the one given by Erdös [6] and Oleksiv and Pesin in [18]. Let r_0 such that $M \subset B(c, r_0)$, where $B(c, r_0)$ denotes the metric ball of radius r_0 centered at c. Consider first r large enough in terms of r_0.

Lemma 2.1. *Let $B(x, r)$ be such that $B(x, r) \cap B(c, r_0) \neq \emptyset$ and $B(x, r) \setminus B(c, r_0) \neq \emptyset$. Set γ for the angle under which we can see $B(c, r_0)$ from x. Then, for any $\varepsilon > 0$ there exists some $r_1(\varepsilon, r_0)$ which depends only B, r_0 and ε such that, for any $r \geq r_1(\varepsilon, r_0)$ we have $\gamma < \varepsilon$.*

Proof. If $r_{\max}$ (respectively $r_{\min}$) denotes the maximum (respectively minimum) Euclidean radius of B, then

$$\sin\frac{\gamma}{2} \leq \frac{r_0 r_{\max}}{r_1 r_{\min}} \tag{1}$$

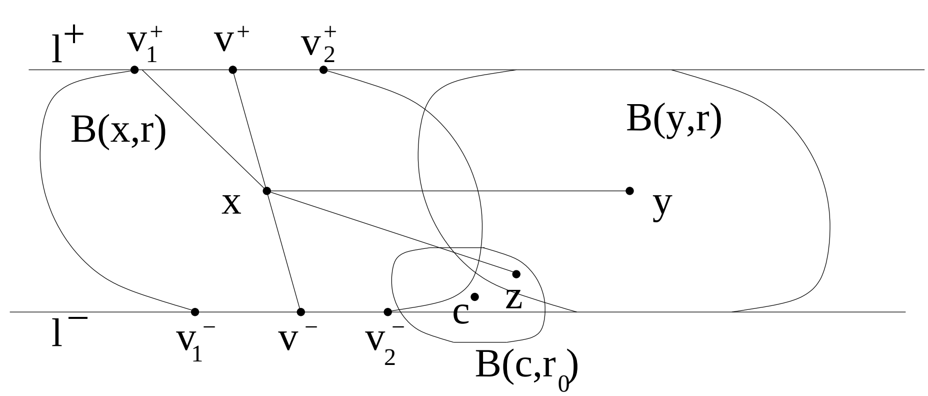
□

Lemma 2.2. *There exists some $\alpha(B) < \pi$ such that $\widehat{yxz} \leq \alpha$, for any $z \in B(y, r) \setminus \mathrm{int}B(x, r)$.*

Proof. The problem is essentially two-dimensional as we can cut the two metric balls by a 2-plane containing the line xy and the point z. Suppose henceforth B is planar and consider support lines l^+ and l^- parallel to xy.

Since $l^+ \cap \partial B(x, r)$ is convex it is a segment $|v_1^+ v_2^+|$, possibly degenerate to one point. We choose v_1^+ to be the farthest from $l^+ \cap B(y, r)$ among v_1^+ and v_2^+. By symmetry $l^- \cap \partial B(x, r)$ is a parallel segment $|v_1^- v_2^-|$, with $v_1^+ v_1^-$ parallel to $v_2^+ v_2^-$. Let v^+ (and v^-) be the midpoint of $|v_1^+ v_2^+|$ (respectively of $|v_1^- v_2^-|$). Observe that $x \in |v^+ v^-|$. We assume that v_2^+ and v_2^- lie in the half-plane determined by $v^+ v^-$ and containing y.

We claim then that $\widehat{yxz} \leq \max(\widehat{yxv_1^+}, \widehat{yxv_1^-})$. This amounts to prove that any $z \in B(y, r) \setminus B(x, r)$ should lie in the half-plane determined by the line $v_1^+ v_1^-$ and containing y, as in the picture below.



Suppose the contrary, namely that there exists $z \in B(y,r) \setminus B(x,r)$ in the opposite half-plane. Let T be the translation in the direction $|yx$ of length $|yx|$. We have $T(B(y,r)) = B(x,r)$ and $z \in B(y,r)$, hence $T(z) \in B(x,r)$. The half-line $|T(z)z$ intersects the segment $|v^+v^-|$ in a point $w \in B(x,r)$.

Suppose first that B is strictly convex. Then both w and $T(z)$ belong to $B(x,r)$ while the point $z \notin \text{int}B(x,r)$. This contradicts the strict convexity of $B(x,r)$, since $T(z) \neq w$.

The direction v^+v^- is called the dual d^* of $d = xy$ with respect to B (also called the B-orthogonal, as introduced by Birkhoff).

It suffices now to remark that for given B the quantity $\sup_\pi \sup_d \max(\angle(d,d^*), \angle(d,-d^*))$, the supremum being taken over all planes π and all directions d, is bounded from above by some $\alpha < \pi$. In fact the space of parameters is a compact (a Grassmannian product the sphere) and that this angle cannot be π unless the planar slice degenerates.

Let us assume now that B is not strictly convex. Then the argument above shows that w, $T(z)$ belong to $B(x,r)$ while the point $z \notin \text{int}B(x,r)$. Therefore $z \in \partial B(x,r)$ and hence $w, z, T(z) \in \partial B(x,r)$. Thus w belongs to one of the two support lines l^+ or l^-. By symmetry it suffices to consider the case when $w = v^+$. Since $T(\partial B(y,r) \cap l^+) \subset \partial B(x,r) \cap l^+$ it follows that $\partial B(y,r) \cap l^+$ is the segment $|T^{-1}(v_1^+)T^{-1}(v_2^+)|$. Thus z belongs to the half-plane determined by $T^{-1}(v_1^+)$ and $T^{-1}(v_1^-)$, which is contained into the one determined by $v_1^+v_1^-$ and containing y.

The compactness argument above extends to the non strict convex B. □

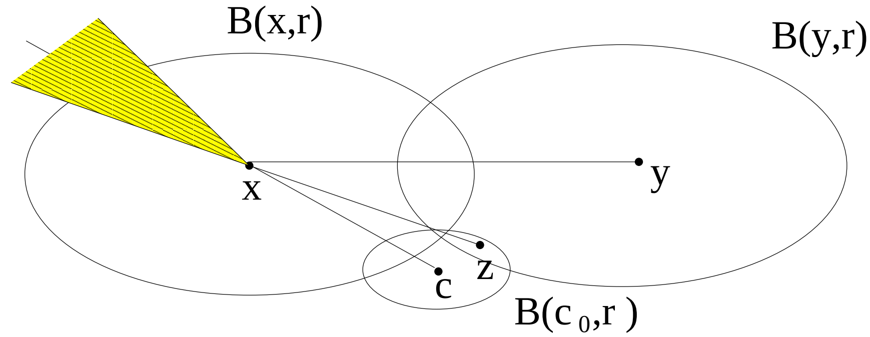
Remark 1. We have $\widehat{yxz} \leq \max(\widehat{yxv^+}, \widehat{yxv^-})$ if $z \in B(y,r) \setminus B(x,r)$. The proof is similar. The upper bound is valid for the closure of $B(y,r) \setminus B(x,r)$ as well. Therefore it holds also for $B(y,r) \setminus \text{int}(B(x,r))$ provided that B is strictly convex, but not in general, see for instance the case when B is a rectangle and xy is parallel to one side.

If β is an angle smaller than $\frac{\pi}{2}$ we set $K(x,\beta,|cx)$ for the cone with vertex x of total angle 2β, of axis $|cx$ and going outward c.

Lemma 2.3. *Let us choose ε such that $\alpha(B) + \varepsilon < \pi$. Then for any point $x \in M_r$, with $r \geq r_1(\varepsilon, r_0)$ we have $K(x, \pi - \alpha(B) - \varepsilon, |cx) \cap M_r = \{x\}$.*

Proof. If $x \in M_r$ then $\text{int}(B(x,r)) \cap M = \emptyset$. Moreover, $M \subset B(c,r_0)$ and so $M \subset B(c,r_0) \setminus \text{int}(B(x,r))$. Let now $y \in M_r$, $y \neq x$. Thus $B(y,r) \cap (B(c,r_0) \setminus \text{int}(B(x,r)) \neq \emptyset$. Let then z be a point from this set.

Then $z \in B(y,r) \setminus \text{int}B(x,r)$ so that by lemma 2.2 $\angle(yxz) \leq \alpha(B)$. Further lemma 2.1 shows that $|\angle(czx)| \leq \varepsilon$, provided that $r \geq r_1(\varepsilon, r_0)$. Thus the angle made between the half-lines $|cx$ and $|xy$ is at least $\pi - \alpha - \varepsilon$, as can be seen in the figure.



In particular y cannot belong to the cone $K(x, \pi - \alpha(B) - \varepsilon, |cx)$. This proves the lemma. □

Proof of the theorem. Set $\beta = \pi - \alpha(B) - \varepsilon$ and let $r \geq r_1(\varepsilon, r_0)$.

First take any $x \in M_r$ and let $U = M_r \cap K(c, \beta/2, |cx)$. If $u \in U$ then $K(u, \beta/2, |cx) \subset K(u, \beta, |cu)$ and hence

$$K(u, \beta/2, |cx) \cap M_r \subset K(u, \beta, |cu) \cap M_r = \emptyset \tag{2}$$

This means that for each $u \in U$ the cone with angle β and axis parallel to the fixed half-line $|cx$ contains no other points of U. Therefore U is the graph of a function of $n-1$ variables satisfying a Lipschitz condition with constant equal to $\frac{1}{\tan\beta}$.

Let consider now the case when r is arbitrary positive. Choose then s such that $r_1(\varepsilon, s) < r$. Split M into a finite number of sets M_j such that each M_j has diameter at most s. It follows that $M_r \subset \cup_j M_{j_r}$. Since each M_{j_r} is locally Lipschitz it follows that M is locally the union of finitely many Lipschitz hypersurfaces. ☐

Corollary 1. *If $M \subset \mathbb{R}^2$ then for almost all r the level set M_r is a 1-dimensional Lipschitz manifold i.e. the union of disjoint simple closed Lipschitz curves.*

Proof. In fact Ferry proved (see [10]) that for almost all r the level set M_r is a 1-manifold. ☐

Remark 2. Lipschitz curves are precisely those curves which are rectifiable. Notice also that the rectifiability does not depend on the particular Minkowski metric, as already observed by Gólab ([13, 14]).

Remark 3. Stachó ([19]) proved that level sets M_r in the Minkowski space are rectifiable in the sense of Minkowski for all but countably many r generalizing earlier results of Szökefalvi-Nagy for planar sets.

3 Planar domains: approaching the perimeter

Unless explicitly stated otherwise, throughout this section, B will denote a centrally symmetric plane oval, where by oval we mean a compact convex domain with non-empty interior.

We assume henceforth that ∂F is a rectifiable curve, namely it is the image of a Lipschitz map from a bounded interval into the plane. Set $p_B(\partial F)$ for the length of ∂F in the norm $\| \, \|_B$.

Our main result generalizes theorem 1 from ([2]), where we considered the case $F = B$ and thus F was convex.

Theorem 2. *For any symmetric oval B and topological disk F with rectifiable boundary in the plane, we have*

$$p_B(\partial F) = 2 \lim_{\lambda \to 0} \lambda N_\lambda(F, B) \tag{3}$$

Remark 4. The guiding principle of this paper is that we can construct some outer packing measure for sets in the Minkowski space which is similar to the packing measure defined by Tricot (see [22]) but uses only *equal* homothetic copies of B which are packed *outside* and hang on the respective set. These constraints make it much more rigid than the measures constructed by means of the Caratheodory method (see [7]). On the other hand it is related to the Minkowski content and the associated curvature measures.

For a fractal set F consider those s for which $\lim_{\lambda\to 0}(2\lambda)^s N_\lambda(F, B)$ is finite non-zero. If this set consists in a singleton, then call it the Hadwiger dimension of F and the above limit the Hadwiger s-measure of F. This measure is actually supported on the "frontier" ∂F of F. Although it is not, in general, a bona-fide measure but only a pre-measure, there is a standard procedure for converting it into a measure. Explicit computations for De Rham curves show that these make sense for a large number of fractal curves. One might expect such measures be Lipschitz functions on the space of measurable curves endowed with the Hausdorff metric.

3.1 Curves of positive reach

Federer introduced in [7] subsets of positive reach in Riemannian manifolds. His definition extends immediately to Finsler manifolds and in particular to Minkowski spaces, as follows:

Definition 1. *The closed subset $A \subset R^n$ has positive reach if it admits a neighborhood U such that for all $p \in U$ there exists a unique point $\pi(p) \in A$ which is the closest point of A to p i.e. such that $d_B(p, \pi(p)) = d_B(p, A)$.*

It is clear that convex sets and sets with boundary of class $\mathcal{C}^2$ have positive reach in the Euclidean space. A classical theorem of Motzkin characterized convex sets as those sets of positive reach in any Minkowski space whose unit disk B is strictly convex and smooth (see [23], Theorem 7.8, p.94). Moreover, Bangert characterized completely in [1] the sets of positive reach in Riemannian manifolds, as the sub-level sets of functions f, admitting local charts $(U, \varphi_U : U \to \mathbb{R}^n)$ and $\mathcal{C}^\infty$ functions h_U such that $(f + h_U) \circ \varphi_U^{-1}$ are convex functions. Another characterization was recently obtained by Lytchak ([16]), as follows. Subsets A of positive reach in Riemann manifolds are those which are locally convex with respect to some Lipschitz continuous Riemann metric on the manifold, and equivalently those for which the inner metric d^A induced on A by the Riemann distance verifies the inequality

$$d^A(x,y) \le d(x,y)(1 + Cd(x,y)^2) \tag{4}$$

for any $x, y \in A$ with $d(x,y) \le \rho$, for some constants $C, \rho > 0$. Federer proved in [7] that Lipschitz manifolds of positive reach are $\mathcal{C}^{1,1}$ manifolds. This was further showed to hold true more generally for topological manifolds of positive reach (see [16]).

On the other hand the sets of positive reach might depend on the specific Minkowski metric on $\mathbb{R}^n$. For instance if B is a square in $\mathbb{R}^2$ then any other rectangle F having an edge parallel to one of B has not positive reach. In fact a point in a neighborhood of that edge has infinitely many closest points.

Remark 5. It seems that sets of positive reach are the same for a Riemannian metric on $\mathbb{R}^n$ and the Minkowski metric d_B associated to a strictly convex smooth B (see also [23] for the extension of the Motzkin theorem to Minkowski spaces).

We will prove now the main theorem for sets of positive reach:

Proposition 1. *If ∂F is a Lipschitz curve of positive reach with respect to the Minkowski metric d_B then $\lim_{\lambda\to 0} 2\lambda N_\lambda(F,B) = p_B(\partial F)$.*

Proof. We start by reviewing a number of notations and concepts. Let $A_{<\varepsilon}$ (respectively $A_{\le\varepsilon}$ and A_ε) denote the set of points at distance less than (respectively less or equal than, or equal to) ε from A, in the metric d_B.

Recall from [7] the following definition:

Definition 2. *The reach $r(A)$ of the set A is defined to be the larger ε (possibly ∞) such that each point x of the open neighborhood $A_{<\varepsilon}$ has a unique $\pi(x) \in A$ realizing the distance from x to A.*

Assume from now that F is a planar domain such that ∂F is a Lipschitz curve which has positive reach. We will consider henceforth only those values of $\lambda > 0$ for which $2\lambda < r(\partial F)$.

Definition 3. *Elements of $A_{F,B,\lambda}$ are called* beads *(or λ-beads if one wants to specify the value of λ) and a configuration of λ-beads with disjoint interiors is called a* λ-necklace. *The necklace is said to be* complete *(respectively* almost complete*) if all (respectively all but one) pairs of consecutive beads have a common point. A necklace is* maximal *if it contains $N_\lambda(F,B)$ beads.*

The main step in proving the proposition is to establish first:

Proposition 2. *If ∂F is Lipschitz and has positive reach then there exist maximal almost complete λ-necklaces for any $\lambda < \frac{1}{2}r(\partial F)$.*

Consider now a maximal almost complete necklace and $P(\lambda)$ be the associated polygon whose vertices are the centers of the beads. Let a and c denote the pair of consecutive vertices of $P(\lambda)$ realizing the maximal distance among consecutive vertices. These are the centers of those beads A and C of the necklace which might not touch each other. The distance between the beads A and C is called the *gap* of the almost complete necklace.

Proposition 3. *A maximal almost complete λ-necklace of the simple closed curve ∂F whose reach is greater than 2λ has gap smaller than 3λ. Consequently the perimeter $p_B(P(\lambda))$ of $P(\lambda)$ satisfies the following inequalities:*

$$0 \le p_B(P(\lambda)) - 2\lambda N_\lambda(F,B) < 3\lambda \tag{5}$$

Observe that the set $(\partial F)_\lambda$ has two components, namely the one contained in the interior of F and that exterior to F. We set $\partial^+ F_\lambda = (\partial F)_\lambda \cap (\mathbb{R}^2 \setminus F)$. Moreover, it is easy to see that $\partial^+ F_\lambda = \partial(F_{\le\lambda})$.

Proposition 4. *Suppose that F is a planar domain whose boundary ∂F is rectifiable (without assuming that the reach is positive). Then for any $\lambda > 0$ we have:*

$$p_B(\partial^+ F_\lambda) \leq p_B(\partial F) + \lambda p_B(\partial B) \tag{6}$$

Proof of Proposition 1 assuming Propositions 2, 3 and 4. Recall now that $P(\lambda)$ is a polygon with N_λ vertices inscribed in $\partial^+ F_\lambda$. Each pair of consecutive vertices of the polygon determines an oriented arc of $\partial^+ F_\lambda$. Furthermore, each edge corresponds to a pair of consecutive beads and thus the arcs associated to different edges of $P(\lambda)$ do not overlap. We will show later also that $\partial^+ F_\lambda$ is connected. These imply that the perimeter of $P(\lambda)$ is bounded from above by the length of $\partial^+ F_\lambda$. Therefore we have the inequalities:

$$p_B(P(\lambda)) \leq p_B(\partial^+ F_\lambda) \leq p_B(\partial F) + \lambda p_B(\partial B) \tag{7}$$

Let λ goes to 0. We derive that:

$$\lim_{\lambda \to 0} p_B(P(\lambda)) \leq p_B(\partial F) \tag{8}$$

On the other hand recall that $P(\lambda)$ converges to ∂F since the distance between consecutive vertices is bounded by 2λ. Using the fact that the Lebesgue-Minkowski length is lower semi-continuous (see [5]) we find that:

$$\lim_{\lambda \to 0} \inf p_B(P(\lambda)) \geq p_B(\partial F) \tag{9}$$

The two inequalities above imply that $\lim_{\lambda \to 0} p_B(P(\lambda))$ exists and is equal to $p_B(\partial F)$. In particular

$$\lim_{\lambda \to 0} 2\lambda N_\lambda(F, B) = \lim_{\lambda \to 0} p_B(P(\lambda)) = p_B(\partial F) \tag{10}$$

and Proposition 1 is proved. □

Remark 6. One can also consider packings with disjoint homothetic copies of B lying in F and having a common point with the complement $\mathbb{R}^2 - \text{int}(F)$. Then a similar asymptotic result holds true.

3.2 Proof of Proposition 2

Consider a maximal necklace and join consecutive centers of beads by segments to obtain a polygon $P(\lambda)$. We want to slide the beads along ∂F so that all but at one pairs of consecutive beads have a common boundary point. Observe that $P(\lambda)$ is a polygon with $N_\lambda = N_\lambda(F, B)$ vertices inscribed in $\partial^+ F_\lambda$.

Let $\pi : \partial^+ F_\lambda \to \partial F$ be the map that associates to the point x the closest point $\pi(x) \in \partial F$. Since $\lambda < r(\partial F)$ the map π is well-defined and continuous.

Lemma 3.1. *The projection map $\pi : \partial^+ F_\lambda \to \partial F$ is surjective.*

Proof. Assume the contrary, namely that π would not be surjective. Continuous maps between compact Hausdorff spaces are closed so that π is closed. Moreover each connected component of $\partial^+ F_\lambda$ is sent by π into a closed connected subset of ∂F.

If some image component consists of one point then $\partial^+ F_\lambda$ is a metric circle centered at that point and thus ∂F has a point component, which is a contradiction.

Give these boundary curves the clockwise orientation. The orientation induces a cyclic ordering on each component. Moreover, this cyclic order restricts to a linear order on any proper subset, in particular on small neighborhoods of a point. When talking about left (or right) position with respect to some point we actually consider points which are smaller (or greater) than the respective point with respect to the linear order defined in a neighborhood of that point.

Let assume that some image component is a proper arc within ∂F. This arc has the right boundary point $\pi(s)$ and there is no other point in the image sitting to the right of $\pi(s)$, in a small neighborhood of $\pi(s)$. Lt s' be maximal such that $\pi(t) = \pi(s)$ for all t in the right of s in the interval from s to s'. As we saw above this is a proper subset of $\partial^+ F_\lambda$.

Choose then some $t \in \partial^+ F_\lambda$ which is nearby s' and slightly to the right of s'. Therefore, we have $\pi(s) \neq \pi(t)$. By hypothesis $\pi(t) \in \partial F$ should sit slightly to the left and closed-by to $\pi(s)$, by the continuity of the map π.

There are several possibilities:

1. the segments $|s\pi(s)|$ and $|t\pi(t)|$ intersect in a point u (see case 1. in the figure below).

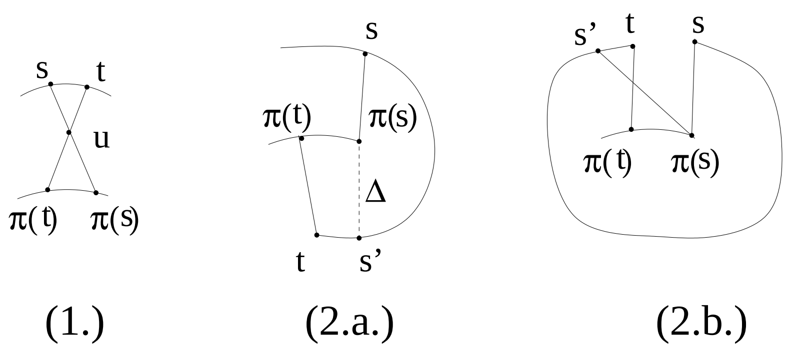
   If $d_B(s,u) < d_B(t,u)$ then $d_B(s,\pi(t)) \leq d_B(s,u) + d_B(u,\pi(t)) < d_B(t,u) + d_B(u,\pi(t)) = \lambda$ and thus $d_B(s,\partial F) < \lambda$ contradicting the fact that $s \in \partial^+ F_\lambda$.

   If $d_B(s,u) > d_B(t,u)$ then $d_B(u,\pi(s)) < d_B(u,\pi(t)$ and hence $d_B(t,\pi(s)) \leq d_B(t,u) + d_B(u,\pi(s)) < d_B(t,u) + d_B(u,\pi(t) = \lambda$, leading to a contradiction again.

   Suppose now that $d_B(s,u) = d_B(t,u)$. The previous argument shows that $d_B(s,\pi(t)) \leq \lambda$. In order to avoid the contradiction above the inequality cannot be strict, so that $d_B(s,\pi(t)) = \lambda = d_B(s,\partial F)$. This means that there are two points on ∂F realizing the distance to s. This contradicts the fact that the reach of ∂F was supposed to be larger than λ.

2. The segments $|s\pi(s)|$ and $|t\pi(t)|$ have empty intersection.

   (a) Moreover, the segments $|s'\pi(s)|$ and $|t\pi(t)|$ are disjoint (see the case 2.a. on the figure below).



   In this situation we observe that the arc of metric circle ss', the arc of ∂F going clock-wisely from $\pi(t)$ to $\pi(s)$ and the segments $|s\pi(s)|$ and $|t\pi(t)|$ bound a domain Δ in the plane. The arc of ∂F which is complementary to the clockwise arc $\pi(t)\pi(s)$ joins $\pi(s)$ and $\pi(t)$ and thus it has to cut at least once more the boundary of the domain Δ. However this curve cannot intersect:

   i. neither the arc $\pi(t)\pi(s)$, since ∂F is a simple curve;
   ii. nor the segments $|s\pi(s)|$ and $|t\pi(t)|$, since it would imply that there exist points in ∂F at distance smaller than λ on $\partial^+ F_\lambda$.
   iii. nor the arc of metric circle $ss' \subset \partial^+ F_\lambda$, since the distance between $\partial^+ F_\lambda$ and ∂F is $\lambda > 0$.

   Thus each alternative above leads to a contradiction.

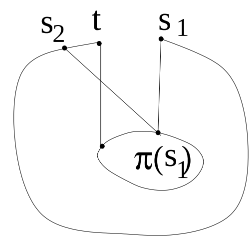
   (b) The segments $|s'\pi(s)|$ and $|t\pi(t)|$ are disjoint (case 2.b. in the figure above).

   Here we conclude as in the first case by using s' in the place of s and get a contradiction again.

Therefore our assumption was false so that the image component is all of ∂F. Notice that we actually proved that π is open. □

Lemma 3.2. *The fibers of the projection map $\pi : \partial^+ F_\lambda \to \partial F$ are either points or connected arcs. In particular $\partial^+ F_\lambda$ is connected.*

Proof. Let $\pi(s_1) = \pi(s_2)$ for two distinct points s_1 and s_2 and assume that π is not constant on the clockwise arc $s_1 s_2$. Pick up some v in the arc $s_1 s_2$. According to the proof of the previous lemma we cannot have $\pi(v)$ sitting to the left of $\pi(s_1)$, for v near s_1. Thus $\pi(v)$ sits in the right of $\pi_(s_1)$. Moreover, if w lies between v and s_2 the same argument shows that $\pi(w)$ sits in the right of $\pi(v)$. Consequently the image by π of the arc $s_1 s_2$ covers completely ∂F and the situation is that from the figure below.



Take now any t in the complementary arc s_2s_1. If $\pi(t) \neq \pi(s_1)$ then $|t\pi(t)|$ intersects either $|s_1\pi(s_1)|$ or else $|s_2\pi(s_2)|$, leading to a contradiction as in the proof of the previous lemma. The lemma follows. □

We will need to have informations about the rectifiability of the set $\partial^+ F_\lambda$, as follows:

Lemma 3.3. *If $0 < \lambda < r(\partial F)$ then $\partial^+ F_\lambda$ is a Lipschitz curve and in particular a $\mathcal{C}^{1,1}$ simple closed curve.*

Proof. Since λ is smaller than the reach $r(\partial F)$ it follows that $\partial^+ F_\lambda$ has also positive reach. The proof from [10] shows that $\partial^+ F_\lambda$ is a 1-manifold. Thus, by Theorem 1 the set $\partial^+ F_\lambda$ is a Lipschitz 1-manifold. Lemma 3.2 shows that $\partial^+ F_\lambda$ is connected and thus it is a simple closed curve. □

Therefore the curve $\partial^+ F_\lambda$ is rectifiable. Recall that $\partial^+ F_\lambda$ has an orientation, say the clockwise one. Consider a maximal λ-necklace $\mathcal{B}$ and suppose that there exists a pair of consecutive beads which do not touch each other. There is induced a cyclic order on the beads of any λ-necklace: the beads B_1, B_2 and B_3 are cyclically ordered if the three corresponding points on which the B_i touch ∂F are cyclically ordered. As $\lambda < r(\partial F)$ each λ-bead intersects ∂F in a unique point and thus the definition makes sense.

Consider two consecutive beads which do not touch each other. If $x \in \partial F$ let l_x be some support line for ∂F at x and B_x (depending also on l_x) the translate of λB which admits l_x as support line at x. We assume that going from x to the center of B_x we go locally outward F. We call B_x the virtual λ-bead attached at x. Actually the virtual bead might intersect ∂F and thus be not a bead.

The consecutive beads are B_p and B_q for $p, q \in \partial F$. We want to slide B_q in counterclockwise direction among the virtual beads B_x, where x is going from q to p along ∂F until B_x touches B_p. Let $\mathcal{B}_x$ be the virtual necklace obtained from the necklace $\mathcal{B}$ by replacing the bead B_q by the virtual bead B_x.

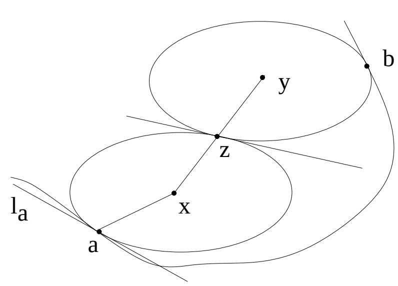
If all virtual necklaces $\mathcal{B}_x$ are genuine necklaces then we obtained another maximal necklace in which the pair of consecutive beads are now touching each other. We continue this procedure while possible. Eventually we stop either when the necklace was transformed into an almost complete one, or else the sliding procedure cannot be performed anymore.

Let then assume we have two consecutive beads which cannot get closer by sliding. Let then a be the first point on the curve segment from q to p (running counter-clockwisely) where the sliding procedure gets stalked. We have then two possibilities:

1. B_a touches ∂F in one more point.
2. B_a touches another bead B_b from the necklace $\mathcal{B}$.

In the first situation the center of B_a is at distance λ from ∂F and the distance is realized twice. Thus $r(\partial F) \leq \lambda$, contradicting our choice of λ.

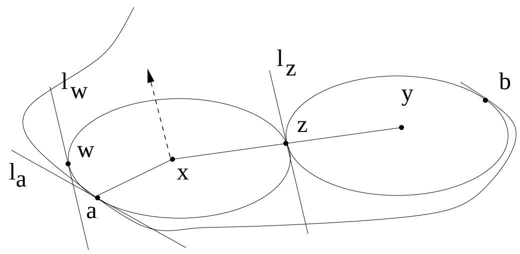
The analysis of the second alternative is slightly more delicate. Let z be the midpoint of the segment $|xy|$ joining the centers of the two beads B_a and B_b respectively.



Let l_z be a support line at z, common to both B_a and B_b.

Lemma 3.4. *Either l_a and l_z are parallel or else they intersect in the half-plane determined by xy and containing the germ of the arc of ∂F issued from a which goes toward p.*

Proof. Assume the contrary and let then l_w be a support line to B_a which is parallel to l_z and touches ∂B_a into the point $w \in \partial B_a$. The cyclic order on ∂B_a is then z, a and w. Consider the arc of ∂F issued from a. Since the reach of ∂F is larger than λ we have w and all points of $B_a \setminus \{a\}$ are contained in $\mathbb{R}^2 - F$. Thus there is some ε-neighborhood of w which is still contained in the open set $\mathbb{R}^2 - F$. This implies that we can translate slightly B_a along l_z within the strip determined by l_z and l_w such that it does not intersect ∂F anymore.



The translated B_a will remain disjoint from $\mathrm{int}(B_b)$ because the later lies in the other half-plane determined by l_z. Pushing it further towards F along l_a we find that the sliding can be pursued beyond a, contradicting our choice for a. This proves the claim. □

Lemma 3.5. *For any $t \in |xy|$ we have $d_B(t, \partial F) \leq 2\lambda$.*

Proof. The segment $|xy|$ is covered by $B_a \cup B_b$ and the triangle inequality shows that $\min(d_B(t,a), d_B(t,b)) \leq 2\lambda$, which implies the claim. □

Consider now $\partial^+ F_{2\lambda}$. By lemmas 3.1 and 3.3 the projection $\pi : \partial^+ F_{2\lambda} \to \partial F$ is a surjection. Let us choose some $w \in \partial^+ F_{2\lambda}$ such that $\pi(w) = a$. Set x' for the midpoint of the segment $|aw|$.

Lemma 3.6. *The metric ball $B(x', \lambda)$ is a λ-bead.*

Proof. As $a \in B(x', \lambda) \cap \partial F$ it suffices to show that $B(x', \lambda) \subset \mathbb{R}^2 \setminus \mathrm{int}(F)$. Suppose the contrary and let $p \in \mathrm{int}(B(x', \lambda)) \cap \mathrm{int}(F)$. There exists then some $p' \in |px| \setminus \{p\}$ with $p' \in B(x', \lambda) \cap \partial F$. The diameter of $B(x', \lambda)$ is 2λ and so $d_B(w, p) \leq 2\lambda$, but p' lies on the segment $|pw|$ so that $d_B(p', w) < 2\lambda$. This implies that $d_B(w, \partial F) < 2\lambda$ which is a contradiction. This establishes the lemma. □

The diameter of a λ-bead is obviously 2λ. We say that points u and v are *opposite* points in the bead if they realize the diameter of the bead. If B is strictly convex the each boundary point has a unique opposite point. This is not anymore true in general. Given a point on the boundary of a rectangle any point on the opposite side is an opposite of the former one.

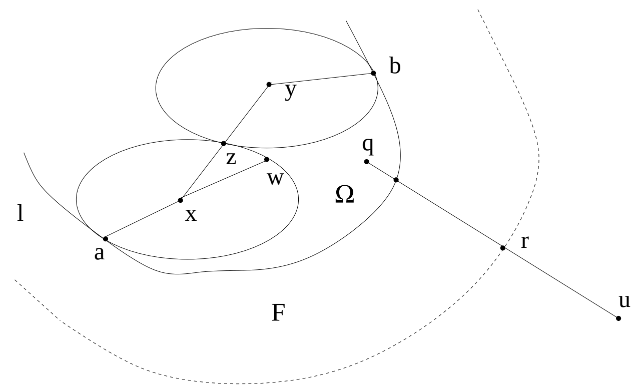
Lemma 3.7. *There exists some point w which is opposite to a in B_a such that $w \in \partial^+ F_{2\lambda}$.*

Proof. Let us assume first that ∂F is smooth at a, or equivalently that it has unique support line at a. As both $B(x', \lambda)$ and B_a are λ-beads which have the same support line l_a (since it is unique) it follows that they coincide. In other terms w is one of the points opposite to a in B_a.

Consider now the general case when ∂F is not necessarily smooth at a. Let l_a^+ and l_a^- denote the extreme positions of the support lines to ∂F at a. Thus l_a belongs to the cone determined by l_a^+ and l_a^-. Recall that ∂F was supposed to be Lipschitz and thus by the Rademacher theorem it is almost everywhere differentiable. There exists then a sequence of points $p_j^\pm \in \partial F$ converging to a such that ∂F is smooth at p_j^+ and p_j^- and the tangent lines at p_j^+ (respectively p_j^-) converge to l_a^+ (respectively to l_a^-).

Let $w_j^\pm$ be points on $\partial^+ F_{2\lambda}$ such that $\pi(w_j^\pm) = p_j^\pm$. It follows that w_j^+ (respectively w_j^-) converge towards a point w^+ (respectively w^-) which lies on the boundary of a λ-bead $B(x^+, \lambda)$ (respectively $B(x^-, \lambda)$ having the support line l_a^+ (respectively l_a^-). Further $\pi(w^+) = \pi(w^-) = a$. The proof of Lemma 3.2 shows that the arc of the metric circle centered at a and of radius 2λ which joins w^+ to w^- is also contained in $\partial^+ F_{2\lambda}$. The point w which is opposite to a in the λ-bead B_a is contained in this arc and thus it belongs to $\partial^+ F_{2\lambda}$. □

End of the proof of Proposition 2. The clockwise arc ab of ∂F and the union of segments $|ax| \cup |xy| \cup |yb|$ which is disjoint from ∂F bound together a simply connected domain Ω_0 in the plane. Then $\Omega = \Omega_0 \setminus \mathrm{int}(B_a) \cup \mathrm{int}(B_b)$ is also a topological disk, possibly with an arc attached to it (if $B_a \cap B_b$ is an arc) since it is obtained from Ω_0 by deleting out two small disks touching the boundary and having connected intersection.



According to lemma 3.4 w belongs either to Ω (for instance when B is strictly convex) or else to $B_a \cap B_b$ (when the support line l_a meets B_a along a segment).

On the other hand the curve $\partial^+ F_{2\lambda}$ contains both the point $w \in \Omega$ and points outside Ω. In fact the arc ab is contained in ∂F which bounds the domain F. Pick up a point q of Ω and r in the arc ab such that the half-line $|vr$ does not meet $|ax| \cup |xy| \cup |yb|$. Then $|vr$ intersects the domain F and thus at least once the clockwise arc ba. Let r be such a point. Then there exists $u \in \partial^+ F_{2\lambda}$ for which $\pi(u) = r$. It is clear that $u \notin \Omega$. Otherwise, by Jordan curve theorem the segment $|ru|$ should intersect once more the clockwise arc ab and this would contradict the fact that $d_B(u, r) = d_B(u, \partial F)$.

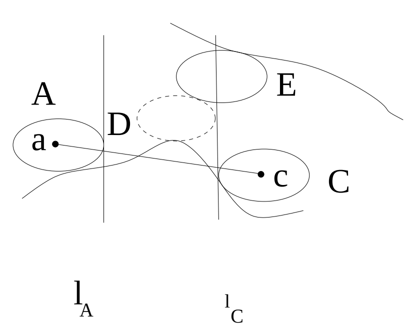
Therefore the curve $\partial^+ F_{2\lambda}$ has to exit the domain Ω and there are two possibilities:

1. either $\partial^+ F_{2\lambda}$ meets $\mathrm{int}(B_a) \cup \mathrm{int}(B_b)$. This will furnish points of $\partial^+ F_{2\lambda}$ at distance less than 2λ from either a or b and thus from ∂F and hence it leads to a contradiction.

2. or else $\partial^+ F_{2\lambda}$ meets $\partial B_a \cup \partial B_b$. In this case any point from $(\partial B_a \cup \partial B_b) \cap \partial^+ F_{2\lambda}$ is at distance 2λ both from a and from b. In particular the distance 2λ is not uniquely realized and this contradicts the choice of $2\lambda < r(\partial F)$.

3.3 Proof of Proposition 3

Assume that $d_B(a, c) \geq 5\lambda$. Let $d \in |ac|$ be the midpoint of $|ac|$ and D denote the translate of A centered at d. The triangle inequality shows that $A \cap D = C \cap D = \emptyset$. The segment $|ac|$ intersects once each one of A and C. Consider the support lines l_A and l_C at these points. Since A and C are obtained by a translation one from the other, we can choose the support lines to be parallel. The convexity of D implies that D is contained in the strip S determined by the parallel lines l_A and l_C.



If $D \cap \partial F$ is empty, then we translate it within S until it touches first ∂F. If D intersects non-trivially the interior of F on one side of the segment $|ac|$, then we translate it in the opposite direction until the contact between D and F is along boundary points. We keep the notation D for the translated oval. However, by the maximality of our almost complete λ-necklace, we cannot add D to our beads to make a necklace. Thus D has to intersect either once more ∂F, or else another bead E from the necklace.

Let consider the first situation. We deflate gradually the bead D by a homothety of ratio going from 1 to 0 by keeping its boundary contact with ∂F until we reach a position where all contact points between D and F are boundary points in ∂F. This implies that the reach of ∂F is less than λ which contradicts our choice of λ.

When the second alternative holds true we make use of the following:

Lemma 3.8. *If two λ-beads intersect each other and there exist λ-beads between them (both in the clockwise and the counterclockwise directions) then the reach of ∂F is at most 2λ.*

Proof. If D and E are the two beads which intersect non-trivially at z let d and e be the points where they touch ∂F. One can choose one of the arcs de or ed of ∂F such that together with $|dz| \cup |ze|$ bound a simply connected bounded domain Ω_0 which is disjoint from F. There exists at least one other λ-bead say $G \subset \Omega_0$. Then we can find as above a point $w \in \partial G$ which lies in $\partial^+ F_{2\lambda}$. Therefore $\partial^+ F_{2\lambda}$ contains points from Ω_0. It is not hard to see that the argument given at the end of the proof of Proposition 2 shows that $\partial^+ F_{2\lambda}$ has also points from outside Ω_0. However $\partial^+ F_{2\lambda}$ is connected and disjoint from ∂F and hence it has to cross $D \cup E$. But then we will find that either there are points on $\partial^+ F_{2\lambda}$ of distance 2λ from both d and e (contradicting the fact that the reach was larger than 2λ) or else we find point at distance strictly less than 2λ from either d or e, which contradicts the definition of $\partial^+ F_{2\lambda}$. □

In our case both A and C are λ-bead disjoint from D both in clockwise and counterclockwise directions. Thus if D intersects another bead E, different from A and C, of the necklace then the reach of ∂F will be smaller than 2λ. This contradiction shows that we can add D to our necklace and the Proposition 3 is proved.

3.4 Proof of Proposition 4

We will prove first the Proposition 4 in the case when ∂F is a polygon Q. Denote by Q_λ the set of points lying outside Q and having distance λ to Q (or, this is the same, to ∂Q). Let us define a (not necessarily simple) curve W_λ as follows. To each edge e of Q there is associated a parallel segment e_λ which is the translation of e in outward (with respect to Q) direction dual to e.

Recall the definition of the dual to a given direction. Assume for the moment that ∂B is strictly convex. If d is a line then let d_+ and d_- be support lines to ∂B which are parallel to d; by the strict convexity assumption each lines d_+, d_- intersects ∂B into one point p_+, p_- respectively. Then the dual of d is the line p_+p_- (which passes through the origin). If ∂B is not strict convex then it might still happen that each support line parallel to d has one intersection point with ∂B, in which case the definition of the dual is the same as above. Otherwise $d_+ \cap \partial B$ has at least two points and thus, by convexity, it should be a segment z_+t_+. In a similar way $d_- \cap \partial B$ is the a segment z_-t_- which is the symmetric of z_+t_+ with respect to the center of B. Thus $z_+z_-t_-t_+$ is a parallelogram having two sides parallel to d. The direction of the other two sides is the dual of d.

It is immediate then that $d_B(e, e_\lambda) = \lambda$.

For each vertex v of Q where the edges e and f meet together we will associate an arc v_λ of the circle $\lambda\partial B$ of radius λ. Let n_e and n_f be the length λ vectors whose directions are dual to e and f respectively and are pointing outward Q. Let v_λ be the arc of λB corresponding to the trajectory drawn by n_e when rotated to arrive in position n_f while pointing outward of Q.

Let us order cyclically the edges $e_1, e_2, \ldots, e_n$ of Q clockwisely and the vertices v_j (which is common to e_j and e_{j+1}). Let also A_j (respectively B_j) denote the left (respectively right) endpoint of ${e_j}_\lambda$. Set α_j for the interior angle (with respect to Q) between e_j and e_{j+1}. Observe that the configuration around two consecutive edges is one of the following type:

1. if $\alpha_j \leq \pi$ then ${e_j}_\lambda$ and ${e_{j+1}}_\lambda$ are disjoint and joined by the arc ${v_j}_\lambda$ is which is locally outside Q;

2. if $\pi < \alpha_j < 2\pi$ then $e_{j\lambda}$ and $e_{j+1\lambda}$ intersect at some point C_j.

Let us define $e_{j\lambda}^*$ to be the segment whose left endpoint is C_{j-1}, if $\alpha_{j-1} > \pi$ and A_j elsewhere while the right endpoint is C_j, if $\alpha_j > \pi$, and B_j otherwise. Let also $v_{j\lambda}^*$ be empty when $\alpha_j > \pi$ and the arc $v_{j\lambda}$ otherwise.

Set W_λ for the union of edges $e_{j\lambda}^*$ and of arcs $v_{j\lambda}^*$. Notice that W_λ might have (global) self-intersections.

Observe that $Q_\lambda \subset W_\lambda$. Notice that the inclusion might be proper.

We claim now that:

Lemma 3.9. *The length of W_λ verifies*

$$p_B(W_\lambda) \leq p_B(Q) + \lambda p_B(\partial B) \tag{11}$$

Proof. The arcs $v_{j\lambda}$ are naturally oriented, using the orientation of ∂Q. Moreover, its orientation is positive if $\alpha_j \leq \pi$ and negative otherwise. Since $\sum_{j=1}^n \alpha_j = (n-2)\pi$ we have $\sum_{j=1}^n (\pi - \alpha_j) = 2\pi$, which means that the algebraic sum of the arcs $v_{j\lambda}$ is once the circumference of $\lambda\partial B$. Thus

$$\sum_{j=1}^n \sigma(v_j) p_B(v_{j\lambda}) = \lambda p_B(\partial B) \tag{12}$$

where $\sigma_j \in \{-1, 1\}$ is the sign giving the orientation of $v_{j\lambda}$. It follows that

$$\lambda p_B(\partial B) + p_B(Q) = \sum_{j=1}^n \sigma(v_j) p_B(v_{j\lambda}) + \sum_{j=1}^n p_B(e_j) \tag{13}$$

Now, $\sigma(j) = -1$ if and only if C_j is defined (i.e. the angle $\alpha_j > \pi$). Thus

$$\begin{aligned} \sum_{j=1}^n \sigma(v_j) p_B(v_{j\lambda}) + \sum_{j=1}^n p_B(e_j) &= \sum_{j=1; \alpha_j \leq \pi}^n (p_B(v_{j\lambda}) + p_B(e_j)) + \\ &+ \sum_{j=1; \alpha_j > \pi}^n (|A_{j+1}C_j|_B + |C_jB_j|_B - p_B(v_{j\lambda}) + p_B(e_{j\lambda}^*)) = \\ &= p_B(W_\lambda) + \sum_{j=1; \alpha_j > \pi}^n (|A_{j+1}C_j|_B + |C_jB_j|_B - p_B(v_{j\lambda}) \geq \\ &\geq p_B(W_\lambda) \end{aligned} \tag{14}$$

The last inequality follows from

$$|A_{j+1}C_j|_B + |C_jB_j|_B \geq p_B(v_{j\lambda}) \tag{15}$$

In fact, it is proved in ([21], p.121), see also or the elementary proof from ([17], 3.4., p.111-113), that a convex curve is shorter than any other curve surrounding it. Moreover the direction B_jC_j is dual to e_j and thus it is tangent to a copy of λB translated at v_j; in a similar way $A_{j+1}C_j$ is dual to e_{j+1} and thus tangent to the same copy of λB. In other words the arc $v_{j\lambda}$ determined by A_{j+1} and B_j is surrounded by the union $|A_{j+1}C_j| \cup |C_jB_j|$ of two support segments. The convexity of ∂B implies the inequality above, and in particular our claim. □

Remark 7. One can use the signed measures defined by Stachó in [20] for computing the length of $\partial^+ F_\lambda$ and to obtain, as a corollary, the result of Proposition 4. Our proof for planar rectifiable curves has the advantage to be completely elementary.

End of the proof of proposition 4. Let now ∂F be an arbitrary rectifiable simple curve. It is known that there exists a sequence of polygons Q_n inscribed in ∂F such that $\lim_n p_B(Q_n) = p_B(\partial F)$. Here p_B denotes the Jordan (equivalently Lebesgue) length of the respective curve, in the Minkowski metric.

Therefore $Q_{n\lambda}$ is a sequence of rectifiable curves which converge to $\partial^+ F_\lambda$. By theorem 1 $\partial^+ F_\lambda$ is the union of finitely many Lipschitz 1-manifolds and thus the Lebesgue length of $\partial^+ F_\lambda$ makes sense. By the lower semi-continuity of the Lebesgue length (see e.g. [5]) it follows that

$$\liminf_n p_B(Q_{n\lambda}) \geq p_B(\partial^+ F_\lambda) \tag{16}$$

However we proved above that for simple polygonal lines Q_n we have:

$$p_B(Q_{n\lambda}) \leq p_B(Q_n) + \lambda p_B(\partial B) \tag{17}$$

Passing to the limit $n \to \infty$ we obtain

$$p_B(\partial^+ F_\lambda) \leq \liminf_n p_B(Q_n) + \lambda p_B(\partial B) = p_B(\partial F) + \lambda p_B(\partial B) \tag{18}$$

Therefore Proposition 4 follows.

3.5 Curves of zero reach

Consider now an arbitrary simple closed Lipschitz curve ∂F in the plane. When sliding λ-beads for achieving almost completeness of necklaces we might get stalked because we encounter points of ∂F with reach smaller than λ. Let us introduce the following definitions.

Definition 4. *The clockwise arc ab of ∂F is a λ-corner if there exists a λ-bead B_a which touches ∂F at a and b and such that there is no λ-bead B_x for x in the interior of the arc ab (except possibly for B_a).*

Definition 5. *The clockwise arcs aa' and $b'b$ of ∂F form a* long *λ-gallery if there exist two* disjoint *λ-beads B_a and $B_{a'}$ with $\{a, b\} \subset B_a \cap \partial F$ and $\{a', b'\} \subset B_{a'} \cap \partial F$ such that:*

1. *there is no λ-bead touching the arcs aa' or bb';*
2. *at least one complementary arc among $a'b'$ and ba admits a λ-bead which is disjoint from B_a and $B_{a'}$.*

Definition 6. *The clockwise arcs aa' and $b'b$ of ∂F form a* short *λ-gallery if there exist two λ-beads B_a and $B_{a'}$ with non-empty intersection, $\{a, b\} \subset B_a \cap \partial F$ and $\{a', b'\} \subset B_{a'} \cap \partial F$ such that:*

1. *any λ-bead touching $aa' \cup b'b$ should intersect the boundary beads $B_a \cup B_{a'}$;*
2. *there is no 2λ-bead touching the arcs $aa' \cup b'b$;*

Observe that λ-corners do not really make problems in sliding λ-beads, because we can jump from a to b keeping the same bead and we can continue the sliding from there on.

Set Z_λ for the set of points that belong to some λ-gallery (long or short).

Lemma 3.10. *For each $\lambda > 0$ the number of maximal λ-galleries is finite.*

Proof. Assume that we have infinitely many λ-galleries. They have to be disjoint, except possibly for their boundary points. Thus the length of their arcs converges to zero. Moreover, the associated pairs of arcs of ∂F converge towards a pair of two points at distance 2λ. Thus all but finitely many galleries are short galleries. The lengths of intermediary arcs (those joining consecutive gallery arcs in the sequence) should have their length going to zero since their total length is finite.

Consider now the union of two consecutive galleries in the sequence together with the intermediary arcs between them. We claim that if we are deep enough in the sequence then this union will also be a gallery, thus contradicting the maximality. Assume the contrary, namely that the union is not a short gallery. Then one should find either a λ-bead touching one intermediary arc which is disjoint from the boundary beads, or else a 2λ-bead.

In the first case the intermediary arc joins two points x, y of intersecting λ-beads and surrounds a disjoint λ-bead. Let z be a common point for the two boundary beads. Then the union of $|xz| \cup |zy|$ with the intermediary arc forms a closed curve surrounding the boundary of a λ-bead. In particular its length is larger than or equal to $\lambda p_B(\partial B)$. Since $d_B(x, z), d_B(z, y) \leq 2\lambda$ it follows that the length of the intermediary arc is at least $\lambda(p_B(\partial B) - 4) \geq 2\lambda$. However intermediary arcs should have length going to zero, so this is a contradiction.

The second alternative tells us that there exists a 2λ-bead touching the intermediary arc. Let the arcs be xy and $x'y'$. Then we claim that the union of the arcs xx' (in the boundary of the bead), $x'y'$, $y'y$ (in the boundary of the bead) and yx is a closed curve surrounding the convex 2λ-bead. Therefore their total length is at least $2\lambda p_B(\partial B)$.

In fact suppose that the 2λ-bead of center w intersects the arc xx'. Observe that w is not contained in the interior of the λ-bead because otherwise the 2λ-bead would contain it and thus there will be no place for the arc of ∂F. Further we find that the distance function $d_B(z, w)$ for z in the arc xx' will have points where it takes values smaller than 2λ. As $d_B(x, w), d_B(x', w) \geq 2\lambda$ it follows that the distance function will have at least two local maxima. However since B is convex the distance function to a point cannot have several local maxima unless when B is not strictly convex and there is a segment of maxima. This proves the claim.

1. 2.

However the sum of the lengths of the arcs xx' and $y'y$ is smaller than $\frac{4\lambda}{3}p_B(\partial B)$ if we are far enough in the sequence. Indeed the two boundary λ-beads intersect each other and their centers become closer and closer as we approach the limit bead. Then the perimeter of the union of the two convex λ-beads converge to the perimeter of one bead. In particular, at some point it becomes smaller than $\frac{4\lambda}{3}p_B(\partial B)$.

This implies that the length of the arcs xy and $x'y'$ is at least $\frac{2\lambda}{3}p_B(\partial B) \geq 4\lambda$. This is in contradiction with the fact that intermediary arcs should converge to points. □

Lemma 3.11. *For any $\lambda > 0$ we have*

$$p_B(\partial F \setminus Z_{2\lambda} \cup Z_\lambda) \leq \lim_{\delta \to 0} 2\delta N_\delta(\partial F) \tag{19}$$

Proof. Since there are finitely many maximal 2λ-galleries consider δ be small enough such that two δ-beads which touch a maximal 2λ-gallery at each end point should be disjoint.

Let then choose a δ-necklace $\mathcal{N}_j$ for each connected component A_j of $\partial F \setminus Z_{2\lambda}$. We claim that the union of necklaces $\cup_j \mathcal{N}$ is a necklace on ∂F.

No bead exterior to a 2λ-gallery can intersect the arcs of that gallery. Extreme positions of δ-beads are contained in boundary beads and thus only boundary points of the gallery can be touched by the necklace.

Two component necklaces are separated by a gallery. We chose δ such that the last δ-bead of one necklace is disjoint from the first δ-bead of the next component.

Remark now that δ-necklaces with $\delta < \lambda$ are otherwise disjoint. In fact suppose that two beads from different necklaces (or one bead from a necklace and an arc A_j) intersect each other. Going far enough to one side of the arcs we should find large enough beads and hence 2λ-beads, since beads lie in $\mathbb{R}^2 \setminus F$. Going to the other side, if we can find a 2λ-beads then the two arcs contain a 2λ-gallery, contradicting our assumptions. Otherwise the remaining part forms a 2λ-corner and in particular the arcs belong to the same component. Then the beads should be disjoint, since they are beads of the same necklace. The same proof works for the bead intersecting an arc.

Let then $N_\delta(A_j)$ be the maximal cardinal of a δ-necklace in $\mathbb{R}^2 - F$ such that all beads touch the arc A_j. We set (by abuse of notation) $N_\delta(\partial F \setminus Z_{2\lambda} \cup Z_\lambda) = \sum_j N_\delta(A_j)$.

Summing up we proved above that

$$N_\delta(\partial F \setminus Z_{2\lambda} \cup Z_\lambda) \leq N_\delta(\partial F) \tag{20}$$

Recall now that each arc A_j is a Lipschitz curve of reach at least 2λ. The proof of Proposition 1 can be carried over not only for simple closed curves but also for simple Lipschitz arcs of positive reach without essential modifications, with a slightly different upper bound in Proposition 4.

Thus the result holds true for each one of the arcs A_j. As we have finitely many such arcs A_j we obtain

$$\lim_{\delta\to 0} 2\delta N_\delta(\partial F \setminus Z_{2\lambda} \cup Z_\lambda) = \sum_j p_B(A_j) = p_B(\partial F \setminus Z_\lambda) \tag{21}$$

The inequality above implies the one from the statement. □

Lemma 3.12. *We have* $\lim_{\lambda\to 0} p_B(Z_\lambda) = 0$.

Proof. For each λ-gallery there is some $\mu(\lambda)$ such that its points are not contained in any μ-gallery. Assume the contrary. Then there exists a sequence of $\lambda_j \to 0$ of nested λ_j-galleries. Their intersection point is an interior point of these arcs and thus it yields a point where the curve ∂F has a self-intersection, which is a contradiction. Thus the claim follows.

Since the number of λ-galleries is finite there is a sequence $\lambda_j \to 0$ such that λ_j-galleries are pairwise disjoint. Thus

$$\sum_j p_B(Z_{\lambda_j}) \le p_B(\partial F) \tag{22}$$

and hence $\lim_j p_B(Z_{\lambda_j}) = 0$.

Further any Z_λ is contained into some the union of Z_{λ_j} for some $j > j(\lambda)$. The result follows. □

Lemma 3.13. *We have* $\lim_{\lambda\to 0} 2\lambda N_\lambda(F, B) \le p_B(\partial F)$.

Proof. Let $P(\lambda)$ be the polygon associated to a maximal λ-necklace on ∂F. Then

$$2\lambda N_\lambda(F, B) \le p_B(P(\lambda)) \tag{23}$$

For all λ the set $\partial^+ F_\lambda$ is the union of finitely many Lipschitz curves and $P(\lambda)$ is a polygon inscribed in $\partial^+ F_\lambda$. However, it might happen that $\partial^+ F_\lambda$ has several components, possibly infinitely many.

Recall that we defined in the proof of Proposition 4 the intermediary curve $W_\lambda = W_\lambda(Q)$ which is associated to a polygon Q. We can define $W_\lambda(F)$ as the Hausdorff limit of $W_\lambda(Q_n)$ where Q_n is approximating ∂F. Or else we can choose Q which approximates closed enough to ∂F so that the vertices of $P(\lambda)$ belong to $W_\lambda(Q)$.

Moreover, $W_\lambda(Q)$ is now a closed curve, which might have self-intersections. The polygon $P(\lambda)$ is inscribed in $W_\lambda(Q)$ and we can associate disjoint arcs to different edges, since edges are associated to consecutive beads. Therefore we have:

$$p_B(P(\lambda)) \le p_B(W_\lambda(Q)) \tag{24}$$

Then the proof of Proposition 4 actually shows that

$$p_B(W_\lambda(Q)) \le p_B(Q) + \lambda p_B(\partial B) \le p_B(\partial F) + \lambda p_B(\partial B) \tag{25}$$

The inequalities above imply that

$$2\lambda N_\lambda(\partial F) \le p_B(\partial F) + \lambda p_B(\partial B) \tag{26}$$

and taking the limit when λ goes to zero yields the claim. □

End of the proof of Theorem 1. By Lemma 3.11 the limit is at least $p_B(\partial F \setminus Z_\lambda)$, for any λ. Using Lemma 3.12 this lower bounds converges to $p_B(\partial F)$ when λ goes to zero. Then Lemma 3.13 concludes the proof.

4 Second order estimates

The aim of this section is to understand better the rate of convergence in Theorem 2. First, we have the very general upper bound below:

Proposition 5. *For any planar simply connected domain F with Lipschitz boundary we have*

$$2\lambda N_\lambda(F,B) \leq p_B(\partial F) + \lambda p_B(\partial B) \tag{27}$$

Proof. This is an immediate consequence of the proof of Lemma 3.13. □

When F is convex, we can obtain more effective estimates of the rate of convergence for the lower bound:

Proposition 6. *For any symmetric oval B and convex disk F in the plane, the following inequalities hold true:*

$$p_B(F) - 2\lambda \leq 2\lambda N_\lambda(F,B) \leq p_B(F) + \lambda p_B(\partial B) \tag{28}$$

Proof. By approximating the convex curve ∂F by convex polygons we deduce the following extension of the classical tube formula to Minkowski spaces:

$$p_B(\partial^+ F_\lambda) = p_B(F) + \lambda p_B(\partial B) \tag{29}$$

Notice that for non-convex F we have only an inequality above.

Let $B_1,\ldots,B_N$ be a maximal necklace with beads which are translates of λB and $o_1,o_2,\ldots,o_N$ be their respective centers, considered in a cyclic order around F. Since $B_i \cap F$ contains at least one boundary point it follows that $o_i \in F_\lambda$ and $B_i \subset F_{2\lambda}$.

Since B and F are convex it follows that F_λ is convex. Therefore the polygon $P = o_1 o_2 \cdots o_N$ is convex since its vertices belong to $\partial^+ F_\lambda$ and, moreover, $P \subset F_\lambda$.

It is not true in general that P contains F, and we have to modify P.

If the necklace is incomplete, we can fill in the space left by adjoining an additional translate B^*_{N+1} homothetic to B in the ratio $\lambda\mu$, with $\mu < 1$, which has a common point with each one of F, B_1 and B_N. Set o_{N+1} for its center.

Now, we claim that the polygon $P^* = o_1 o_2 \cdots o_{N+1}$ contains F. In fact, $d_B(o_i,o_{i+1}) \leq 2$ since B_i and B_{i+1} have a common point, which is at unit distance from the centers. But their interiors have empty intersection thus $d_B(o_i,o_{i+1}) = 2\lambda$ and the segment $|o_i o_{i+1}|$ contains one intersection point from $\partial B_i \cap \partial B_{i+1}$. Furthermore, the same argument shows that $d_B(o_N,o_{N+1}) = d_B(o_1,o_{N+1}) = (1+\mu)\lambda$ and each segment $|o_N o_{N+1}|$ and $|o_{N+1} o_1|$ contains one boundary point from the corresponding boundaries intersections. Thus the boundary of P^* is contained in $\cup_{i=1}^{N+1} B_i \cup B^*_{N+1}$, the later being disjoint from the interior of F. This proves our claim. Remark that P^* is not necessarily convex.

We know that $P \subset F_\lambda$ and $d_B(o_i,o_{i+1}) \geq 2$ (for $i = 1,2,\ldots,N$) because B_i and B_{i+1} have no common interior points. Since a convex curve surrounded by another curve is shorter than the containing one we obtain:

$$2\lambda N \leq p_B(P) \leq p_B(\partial^+ F_\lambda) = p_B(\partial F) + \lambda p_B(\partial B) \tag{30}$$

Next, recall that $F \subset P^*$ and $d_B(o_i,o_{i+1}) \leq 2$, where $i = 1,2,\ldots,N+1$, since consecutive beads have at least one common point. This implies that:

$$p_B(\partial F) \leq p_B(P^*) < 2\lambda(N+1) \tag{31}$$

The two inequalities above prove the Proposition 6. □

Consider a more general case when F is not necessary convex. We assume that F is *regular*, namely that its boundary is the union of finitely many arcs with the property that each arc is either convex or concave. The endpoints of these maximal arcs are called vertices of ∂F. This is the case, for instance, when ∂F is a piecewise analytic curve. Moreover we will suppose that F has positive reach. This is the case for instance when F admits a support line through each vertex of ∂F, which leaves a neighborhood of the vertex in F on one side of the half-plane.

The estimates for the rate of convergence will not be anymore sharp. By hypothesis, ∂F can be decomposed into finitely many arcs A_i, $i = \overline{1,m}$, which we call pieces, so that each piece is either convex or concave.

Proposition 7. *Consider a symmetric oval B and a regular topological disk F of positive reach having $c(F)$ convex pieces and $d(F)$ concave pieces. Then the following inequalities hold:*

$$p_B(F) - 2\lambda(2c(F) + p_B(\partial B)d(F) + 3d(F)) \leq 2\lambda N_\lambda(F, B) \leq p_B(F) + 2\lambda p_B(\partial B) \tag{32}$$

for $2\lambda < r(\partial F)$.

Proof. If $\mathcal{N}$ is a maximal necklace on F denote by $\mathcal{N}|_{A_j}$ its trace on the arc A_j, i.e. one considers only those beads that touch A_j. Consider also maximal necklaces M_{A_j} on each arc A_j, consisting of only those beads sitting outside F which have common points to A_j. Consider now the union of the maximal necklaces M_{A_j}. Beads of M_{A_j} cannot intersect ∂F since the reach is larger than λ. Moreover beads from different necklaces cannot intersect (according to Lemma 3.8) unless the beads are consecutive beads i.e. one is the last bead on A_j and the other is the first bead on the next (in clockwise direction) arc A_{j+1}.

Therefore if we drop the last bead from each M_{A_j} and take their union we obtain a necklace on ∂F. This implies that:

$$\sum_{i=1}^{m} N_\lambda(A_i, B) - N_\lambda(F, B) \leq c(F) + d(F) \tag{33}$$

We analyze convex arcs in the same manner as we did for ovals in the previous Proposition. Since the arc A_j has positive reach we can slide all beads to the left side. Add one more smaller bead in the right side which touches the arc at its endpoint, if possible. The centers of the beads form a polygonal line P^* with at most $N_\lambda(A_j, B) + 1$ beads. Join its endpoints to the endpoints of the arc A_j by two segments of length no larger than λ. This polygonal line surrounds the convex arc A_j and thus its length is greater than $p_B(A_j)$. Therefore, for each convex arc A_j we have:

$$2\lambda(N_\lambda(A_j, B) + 1) \geq p_B(A_j) \tag{34}$$

The next step is to derive similar estimates from below for a concave arc A_s. Since the arc has positive reach we can slide all beads to its left side. If there is more space left to the right let us continue the arc A_s by adding a short arc on its right side along a limit support line at the right endpoint so that we can add one more bead to our necklace which touches the completed arc A_s^* at its endpoint. We can choose this line so that the reach of A_s^* is the same as that of A_s.

Let the beads have centers o_i, $i = \overline{1, N+1}$, where $N = N_\lambda(A_s, B)$, the last one being the center of the additional bead. Then $d_B(o_i, o_{i+1}) = 2\lambda$ and $d_B(o_i, A) = \lambda$, as in the convex case. The point is that the function $d_B(x, A)$ is not anymore convex, as it was for convex arcs. However, for any point $x \in |o_i o_{i+1}|$ we have $d_B(x, A) \leq \min(d_B(x, o_i) + d_B(o_i, A), d_B(x, o_{i+1}) + d(o_{i+1}, A)) \leq 2\lambda$.

If P^* denotes the polygonal line $o_1 o_2 \cdots o_{N+1}$ then $P^* \subset A^*_{s2\lambda}$. Moreover the points which are opposite to the contacts between the beads and A_s^* belong to $A^*_{s2\lambda}$. Join in pairs the endpoints of P^* and those of $A^*_{s2\lambda}$ by two segments of length λ and denote their union with P^* by $\overline{P^*}$. The arc A_s was considered concave of positive reach and this means that for small enough $\lambda < r(\partial F)/2$ the boundary $\partial A^*_{s2\lambda}$ is still concave of positive reach. Looking from the opposite side $A^*_{s2\lambda}$ is a convex arc. Moreover $\overline{P^*}$ encloses (from the opposite side) this convex arc and thus $p_B(\overline{P^*}) \geq p_B(A^*_{s2\lambda}) \geq p_B(A_{s2\lambda})$.

The formula giving the perimeter for the parallel has a version for the inward deformation of convex arcs, or equivalently, for outward deformations of concave arcs, which reads as follows:

$$p_B(A_{s2\lambda}) = p_B(A_s) - 2\lambda p_B(X_{A_s}) \tag{35}$$

where $X_{A_s} \subset \partial B$ is the image of A_s by the Gauss map associated to B. As $X_{A_s} \subset \partial B$ we obtain

$$p_B(A_s) - 2\lambda p_B(\partial B) \leq p_B(P^*) + 2\lambda \leq 2\lambda N_\lambda(A_s, B) + 4\lambda \tag{36}$$

Summing up these inequalities we derive the inequality from the statement. □

Remark 8. The proofs above work for arbitrary convex B, not necessarily centrally symmetric. In this case, we could obtain:

$$p_{\overline{B}}(F) = 2 \lim_{\lambda \to 0} \lambda N_\lambda(F, B) \tag{37}$$

where $\overline{B} = \frac{1}{2}(B - B) = \{z \in \mathbb{R}^2;\ \text{there exist}\, x, y \in B,\ \text{such that}\, 2z = x - y\}$.

Consider the set $N(F, B)$ of all positive integers that appear as $N_\lambda(F, B)$ for some $\lambda \in (0, 1]$.

Corollary 2. *If F is regular and its boundary has positive reach then large enough consecutive terms in $N(F, B)$ are at most distance $11d(F) + 2c(F) + 4$ apart. When F is convex consecutive terms in $N(F, B)$ are at most distance 4 apart, if B is not a parallelogram and 5 otherwise.*

Proof. Let us consider F convex. Theorem 2 shows that

$$\frac{p_B(F)}{2\lambda} - 1 < N_\lambda(F, B) \le \frac{p_B(F)}{2\lambda} + \frac{p_B(\partial B)}{2} \tag{38}$$

Moreover one knows that $p_B(\partial B) \le 8$ (see [2] and references there) with equality only when B is a parallelogram. In particular any interval $(\alpha, \alpha + \frac{p_B(\partial B)}{2}]$ contains at least one element of $N(F, B)$. If $c < d$ are two consecutive elements of $N(F, B)$ this implies that $c \in (d - \frac{p_B(\partial B)}{2} - 1, d)$, and thus

$$d - c < \frac{p_B(\partial B)}{2} + 1 \le 5 \tag{39}$$

Since c, d are integers it follows that $d - c \le 4$, if B is not a parallelogram.

When F is arbitrary the inequality in theorem 3 shows that any interval of length $11d(F) + 2c(F) + 4$ contains some $N_\lambda(F, B)$. We conclude as above. □

Corollary 3. *Consecutive terms in $N(B, B) \subset \mathbb{Z}_+$ are at most distance 4 apart.*

Proof. If $F = B$ is a parallelogram then $N_\lambda(F, B) = 4\left[\frac{1}{\lambda}\right] + 4$ and thus $N(F, B) = 4(\mathbb{Z}_+ - \{0, 1\})$. □

Remark 9. If F is not convex then we can have gaps of larger size in $N(F, B)$. Take for instance F having the shape of a staircase with k stairs and B a square. As in the remark above we can compute $N_\lambda(F, B) = 4k\left[\frac{1}{\lambda}\right] + 4$ and thus there are gaps of size $4k$.

5 Higher dimensions

The previous results have generalizations to higher dimensions in terms of some Busemann-type areas defined by B. Theorem 2, when $F = B$, was extended in [4] and ([3], 9.10). The result involves the presence of an additional density factor which seems more complicated for $d > 2$.

For a convex body K in $\mathbb{R}^d$ one defines the translative packing density $\delta(K)$ to be the supremum of the densities of periodic packings by translates of K and set $\Delta(K) = \frac{1}{\delta(K)}\mathrm{vol}(K)$. Alternatively, $\Delta(K) = \inf_{T,n} \mathrm{vol}(T)/n$ over all tori T and integers n such that there exists a packing with n translates of K in T, where T is identified with a quotient of $\mathbb{R}^d$ by a lattice.

We consider from now on that B and F are convex and smooth.

Proposition 8. *We have for a convex smooth $F \subset \mathbb{R}^d$ and a centrally symmetric smooth domain $B \subset \mathbb{R}^d$ that*

$$\lim_{\lambda \to \infty} \lambda^{d-1} N_\lambda(F, B) = \int_{\partial F} \frac{1}{\Delta(B \cap T_x)} dx \tag{40}$$

where $x \in \partial F$ and T_x is a hyperplane through the center of B which is parallel to the tangent space at ∂F in x. Here $B \cap T_x \subset T_x$ is identified to a $(d-1)$-dimensional domain in $\mathbb{R}^{d-1}$.

Proof. The proof from ([3] 9.10) can be adapted to work in this more general situation as well. □

Although the present methods do not extend to general arbitrary domains with rectifiable boundary the previous proposition seem to generalize at least when the boundary has positive reach.

Remark 10. We have an obvious upper bound

$$N_\lambda(F,B) \le \lambda^{-d}\frac{\mathrm{vol}(F_{2\lambda}) - \mathrm{vol}(F)}{\mathrm{vol}(B)} = \lambda^{1-d}\mathrm{area}_B(\partial F) + o(\lambda^{1-d}) \tag{41}$$

which follows from the inclusion $\cup_{i=1}^N B_i \subset F_{2\lambda}$ with B_i having disjoint interiors and the Steiner formula (see [15]).

6 Remarks and conjectures

The structure of the sets $N(F,B)$ is largely unknown. One can prove that when F is convex and both F and B are smooth then $N(F,B)$ contains all integers from $N_1(F,B)$ on, at least in dimension 2. For general F we saw that we could have gaps. It would be interesting to know whether $N(B,B)$ contains all sufficiently large integers when $F = B$ is a convex domain and not a parallelohedron. It seems that Corollary 3 can be generalized to higher dimensions as follows:

Conjecture 1. *The largest distance between consecutive elements of $N(F,B)$, where F is convex is at most 2^d with equality when $F = B$ is a parallelohedron.*

Another natural problem is to understand the higher order terms in the asymptotic estimates. Or, it appears that second order terms from section 4 are actually oscillating according to the inequalities in proposition 6 as below:

Corollary 4. *For convex F we have*

$$-2 \le l_-(F,B) = \liminf_{\lambda\to 0} \frac{2\lambda N_\lambda(F,B) - p_B(F)}{\lambda} \le \limsup_{\lambda\to 0} \frac{2\lambda N_\lambda(F,B) - p_B(F)}{\lambda} = l_+(F,B) \le \frac{p_B(\partial B)}{2} \tag{42}$$

The exact meaning of $l_-(F,B)$ and $l_+(F,B)$ is not clear. Assume that $F = B$. We computed:

1. If B is a disk then $l_-(F,B) = -2$ and $l_+(F,B) = 0$;
2. If B is a square then $l_-(F,B) = 0$ and $l_+(F,B) = 4$;
3. If B is a regular hexagon then $l_-(F,B) = 0$ and $l_+(F,B) = 3$;
4. If B is a triangle then $l_-(F,B) = 0$ and $l_+(F,B) = 3$.

There are various other invariants related to the second order terms. Set

$$J_k = \{\lambda \in (0,1];\ \text{there exists a complete } \lambda-\text{necklace } B_1,\dots,B_k\},\ \ I_k = \{\lambda\in(0,1]; N_\Lambda(F,B) = k\} \tag{43}$$

so that $J_k \subset I_k$. Then I_k are disjoint connected intervals but we don't know whether this is equally true for J_k. It seems that J_k are singletons when B is a round disk.

Let $\{r\} = r - [r]$ denote the fractionary part of r.

Conjecture 2. *There exists some constant $c = c(B)$ such that the following limit exists*

$$\lim_{r\to\infty,\{r\}=\alpha} 2N_{c(B)/r}(F,B) - r = \varphi(\alpha) \tag{44}$$

where $\varphi : [0,1) \to [-2, p_B(\partial B)]$ is a right continuous function with finitely many singularities. If F and B are polygons then φ is linear on each one of its intervals of continuity.

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