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ESSENTIAL NUMERICAL RANGE OF ELEMENTARY OPERATORS

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ABSTRACT. Let $A = (A_1, ..., A_p)$ and $B = (B_1, ..., B_p)$ denote two p-tuples of operators with $A_i \in \mathcal{L}(H)$ and $B_i \in \mathcal{L}(K)$. Let $R_{2,A,B}$ denote the elementary operators defined on the Hilbert-Schmidt class $\mathcal{C}^2(H, K)$ by $R_{2,A,B}(X) = A_1XB_1 + ... + A_pXB_p$. We show that

$$co\,[(W_e(A) \circ W(B)) \cup (W(A) \circ W_e(B))] \subseteq V_e(R_{2,A,B}).$$

Here $V_e(.)$ is the essential numerical range, $W(.)$ is the joint numerical range and $W_e(.)$ is the joint essential numerical range.

1. Introduction

Let $\mathcal{L}(H)$ denote the algebra of all bounded linear operators on a separable infinite-dimensional Hilbert space H. Let $A = (A_1, ..., A_p)$ and $B = (B_1, ..., B_p)$ denote two p-tuples of operators with $A_i \in \mathcal{L}(H)$ and $B_i \in \mathcal{L}(K)$. Let $R_{A,B} : \mathcal{L}(H) \longrightarrow \mathcal{L}(H)$ denote the elementary operators defined by

$$R_{A,B}(X) = A_1XB_1 + ... + A_pXB_p.$$

The class of Hilbert-Schmidt operators from a Hilbert space H to a Hilbert space K will be denoted by $\mathcal{C}^2(H, K)$ and, of course, $\mathcal{C}^2(H) = \mathcal{C}^2(H, H)$; see [8]. Recall that $\mathcal{C}^2(H, K)$ is a Hilbert space and that $A_iXB_i \in \mathcal{C}^2(H, K)$ for every $A_i \in \mathcal{L}(H), X \in \mathcal{C}^2(H, K)$ and $B_i \in \mathcal{L}(K)$. So the elementary operator $R_{A,B}$ is a bounded endomorphism of $\mathcal{C}^2(H, K)$. We denote by $R_{2,A,B}$ the restriction of $R_{A,B}$ to $\mathcal{C}^2(H, K)$.

If $\mathcal{A}$ is a Banach algebra with unit e, the algebraic numerical range of an arbitrary element $a \in \mathcal{A}$ is defined by

$$V(a) = \{f(a); f \in \mathcal{A}', \|f\| = f(e) = 1\}.$$

Here, of course, $\mathcal{A}'$ denotes the space of all continuous linear functionals on $\mathcal{A}$. Recall that $V(a)$ is a compact convex set.

For $T \in \mathcal{L}(H)$, the numerical range of T is defined as

$$W(T) = \{\langle Tx, x\rangle : \ \ \|x\| = 1\}.$$

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The essential numerical range, $V_e(T)$, is (by definition) the numerical range of the coset $T+K(H)$ in the Calkin algebra $\mathcal{L}(H)/K(H)$ where $K(H)$ is the ideal of all compact operators on H; see [1, 2] and [9].

It is known that $V_e(T) \subseteq W(T)^-$, the closure of $W(T)$.

For a p-tuple $A=(A_1,...,A_p)$ of operators on a Hilbert space H we define:

- i) the joint numerical range of A by
$$W(A)=\{(\langle A_1x,x\rangle,...,\langle A_px,x\rangle); \quad x\in H, \|x\|=1\};$$
- ii) the joint essential numerical range of A by
$\lambda \in W_e(A)$ if $\lambda=(\lambda_1,...,\lambda_p)\in C^p$ and there exists an orthonormal sequence (x_n) in H such that $\lambda_i=\mathrm{Lim}\,\langle A_ix_n,x_n\rangle\,,\ \ i=1,...,p.$

To simplify the statements, we shall use the following notation: for $\alpha,\beta\in C^n$, we let $\alpha\circ\beta=\sum_{i=1}^p \alpha_i\beta_i$ and for $K,L\subseteq C^n$,
$$K\circ L=\{\alpha\circ\beta,\ \ \alpha\in K,\beta\in L\}.$$
For vectors $x,y\in H$, the notation $x\otimes y$ will refer to the operator in $L(H)$ defined by $x\otimes y(z)=\langle z,y\rangle .x$.

In the past, elementary operators and their restrictions to norm ideals in $L(H)$ have been studied by many authors. Up to now, their spectra and their essential spectra have been characterized; see [4, 5, 3]. In [7], B. Magajna has determined the essential numerical range of the restriction of a generalized derivation to the class of Hilbert-Schmidt.

In this paper, we give some results about the essential numerical range of the restriction of an elementary operator to the class of Hilbert-Schmidt. More precisely, we prove that
$$co\left[(W_e(A)\circ W(B))\cup(W(A)\circ W_e(B))\right]\subseteq V_e(R_{2,A,B}),$$
and we give some consequences of this inclusion.

2. The essential numerical range

We need the following characterization of the essential numerical range, obtained by Fillmore, Stampfli, and Williams in [9].

Lemma 2.1. *Let $T\in\mathcal{L}(H)$. Each of the following conditions is necessary and sufficient in order that $\lambda\in V_e(T)$:*

(1) *$\langle Tx_n,x_n\rangle \longrightarrow \lambda$ for some sequence (x_n) of unit vectors such that $x_n \rightharpoonup 0$ weakly.*

(2) *$\langle Te_n,e_n\rangle \longrightarrow \lambda$ for some orthonormal sequence (e_n).*

The main result of this paper is the following.

Theorem 2.2. *Let H,K be two separable Hilbert spaces and $A=(A_1,...,A_p)$, $B=(B_1,...,B_p)$ two p-tuples with $A_i\in\mathcal{L}(H)$ and $B_i\in\mathcal{L}(K)$ for $i=1,...,p$. Then*
$$co\left[(W_e(A)\circ W(B))\cup(W(A)\circ W_e(B))\right]\subseteq V_e(R_{2,A,B}).$$

Proof. Let $\lambda\in W_e(A)$. There exists an orthonormal sequence (x_n) in H such that
$$\lambda_i=\mathrm{Lim}\,\langle A_ix_n,x_n\rangle \ \text{ for each } i=1,...,p.$$
Let $\mu\in W(B)$. There exists a unit vector y in K such that $\mu_i=\langle B_iy,y\rangle$. It is easily verified that $(x_n\otimes y)$ is an orthonormal sequence in $C^2(H,K)$ and
$$\langle A_i(x_n\otimes y)B_i,x_n\otimes y\rangle = tr(A_i(x_n\otimes y)B_i(y\otimes x_n))=\langle A_ix_n,x_n\rangle .\langle B_iy,y\rangle\,.$$

Hence,

$$\langle R_{2,A,B}(x_n \otimes y), x_n \otimes y\rangle = \sum_{i=1}^{p} \langle A_i x_n, x_n\rangle . \langle B_i y, y\rangle .$$

That is, $\lambda \circ \mu \in V_e(R_{2,A,B})$. □

The essential numerical range of the restriction of a generalized derivation to the class of Hilbert-Schmidt has been computed in [7], by B. Magajna. He has shown that

$$V_e(\delta_{2,A,B}) = co[(V_e(A) - W(B)^-) \cup (W(A)^- - V_e(B))].$$

Here we give only the easiest inclusion.

Corollary 2.3. *For $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(K)$,*

$$co[(V_e(A) - W(B)^-) \cup (W(A)^- - V_e(B))] \subseteq V_e(\delta_{2,A,B}).$$

If, in addition, $V_e(A) = W(A)^-$ or $V_e(B) = W(B)^-$, then we have equality.

Corollary 2.4. *For $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(K)$,*

$$co[(V_e(A).W(B)^-) \cup (W(A)^-.V_e(B))] \subseteq V_e(M_{2,A,B}),$$
$$V_e(L_{2,A}) = W(A)^- \quad and \quad V_e(R_{2,B}) = W(B)^-.$$

Proof. We have $W(A)^- \subseteq V_e(L_{2,A}) \subseteq W(L_{2,A})^- = W(A)^-$. □

3. Nonnegative operators and the essential numerical range

Lemma 3.1. *Let A be a nonnegative, selfadjoint operator and $AB = BA$. Then*

$$V_e(AB) \subseteq V_e(A)V_e(B). \tag{1}$$

Proof. Let $\lambda \in V_e(AB)$. There exists a sequence (x_n) of unit vectors in H such that $x_n \rightharpoonup 0$ weakly and

$$\lambda = \mathrm{Lim}\,\langle AB(x_n), x_n\rangle .$$

Let $y_n = A^{\frac{1}{2}}x_n$. If $y_{n_k} = 0$ for some subsequence, then 0 is in both sides of (1). If not and by passing to a subsequence if necessary, we can assume that $y_n \neq 0 \ \ \forall n$. Put $z_n = \frac{y_n}{\|y_n\|}$. Then (z_n) is a sequence of unit vectors with $z_n \rightharpoonup 0$ weakly and

$$\lambda = \mathrm{Lim}\,\langle Bz_n, z_n\rangle . \langle Ax_n, x_n\rangle .$$

But $\mathrm{Lim}\,\langle Bz_n, z_n\rangle \in V_e(B)$. So $\lambda \in V_e(A)V_e(B)$. □

Corollary 3.2. *Let $A \in \mathcal{L}(H)$ be a nonnegative, selfadjoint operator and $B \in \mathcal{L}(K)$. Then*

$$V_e(M_{2,A,B}) \subseteq W(A)^- W(B)^-.$$

Proof. Recall that $L_{2,A}R_{2,B} = R_{2,B}L_{2,A}$, $V_e(L_{2,A}) = W(A)^-$ and $V_e(R_{2,B}) = W(B)^-$. The rest is from Lemma 3.1. □

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