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J. Math. Anal. Appl. 286 (2003) 359–362
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Note

Norm equality for a basic elementary operator

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Received 28 July 2002
Submitted by R. Curto

Abstract

Let $\mathcal{L}(H)$ be the algebra of bounded linear operators on a Hilbert space H. For $A, B \in \mathcal{L}(H)$, define the elementary operator $M_{A,B}$ by $M_{A,B}(X) = AXB$ ($X \in \mathcal{L}(H)$). We give necessary and sufficient conditions for any pair of operators A and B to satisfy the equation $\|I + M_{A,B}\| = 1 + \|A\|\|B\|$, where I is the identity operator on H.

Keywords: Norm; Numerical range; Elementary operators

Let H be a complex Hilbert space and let $\mathcal{L}(H)$ be the Banach algebra of all bounded linear operators on H. For $A, B \in \mathcal{L}(H)$, let L_A (respectively, R_B) denote the left (respectively, right) multiplication by A (respectively, B). The basic elementary operator (two-sided multiplication) $M_{A,B}$ induced by the operators A and B is defined by $M_{A,B} = L_A R_B$. An elementary operator on $\mathcal{L}(H)$ is a finite sum $R = \sum_{i=1}^{n} M_{A_i,B_i}$ of basic ones. A familiar example of elementary operators is the generalized derivation $\delta_{A,B}$ defined by $\delta_{A,B} = L_A - R_B$.

Many facts about the relation between the spectrum of R and spectrums of the coefficients A_i and B_i are known. This is not the case with the relation between the operator norm R and norms of A_i and B_i. Apparently, the only elementary operators on a Hilbert space for which the norm is computed are the basic ones and generalized derivations [10]. We refer to [2,4–11] for an intensive study of norms of elementary operators.

Let $A, B \in \mathcal{L}(H)$ and let I denote the identity operator on H. It is well known and easy to prove that $\|M_{A,B}\| = \|A\|\|B\|$. Thus we always have $\|I + M_{A,B}\| \leqslant 1 + \|A\|\|B\|$.

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0022-247X/$ – see front matter
doi:10.1016/S0022-247X(02)00657-1

In this note we shall give necessary and sufficient conditions for any pair of operators A and B to satisfy the equation $\|I + M_{A,B}\| = 1 + \|A\|\|B\|$.

In order to state our results in detail, we first recall some notation and results from the literature. Let $T \in \mathcal{L}(H)$. Following [10], the maximal numerical range of T is defined by

$$W_0(T) = \Big\{\lambda \in \mathbb{C}\colon \text{ there exists } \{x_n\} \subseteq H,\ \|x_n\| = 1 \text{ such that } \lim_n \langle Tx_n, x_n\rangle = \lambda \text{ and } \lim_n \|Tx_n\| = \|T\|\Big\},$$

and its normalized maximal numerical range is given by

$$W_N(T) = \begin{cases} W_0(T/\|T\|) & \text{if } T \neq 0, \\ 0 & \text{if } T = 0. \end{cases}$$

The set $W_0(T)$ is nonempty, closed, convex, and contained in the closure of the numerical range, see [10].

For $A \in \mathcal{L}(H)$, let $\sigma(A)$ and $\sigma_{ap}(A)$ denote, respectively, the spectrum and approximate point spectrum of A.

The next theorem is our main result.

Theorem 1. *For $A, B \in \mathcal{L}(H)$ the following are equivalent*:

(1) $\|I + M_{A,B}\| = 1 + \|A\|\|B\|$,
(2) $W_N(A^*) \cap W_N(B) \neq \emptyset$.

Proof. (1) $\Rightarrow$ (2) Suppose that $\|I + M_{A,B}\| = 1 + \|A\|\|B\|$. Then we can find two sequences $\{X_n\} \subseteq \mathcal{L}(H)$ and $\{x_n\} \subseteq H$ with $\|X_n\| = \|x_n\| = 1$ for each n such that

$$\lim_n \|X_n x_n + AX_nBx_n\| = 1 + \|A\|\|B\|.$$

Since

$$\|X_n x_n + AX_nBx_n\| \leqslant \|X_n x_n\| + \|AX_nBx_n\| \leqslant 1 + \|A\|\|B\|,$$

it follows that

$$\lim_n \|AX_nBx_n\| = \|A\|\|B\|.$$

On the other hand, we have for each n,

$$\|X_n x_n + AX_nBx_n\|^2 = \|X_n x_n\|^2 + \|AX_nBx_n\|^2 + 2\,\mathrm{Re}\langle X_n x_n, AX_nBx_n\rangle.$$

Consequently, we derive that

$$\lim_n \langle X_n x_n, AX_nBx_n\rangle = \|A\|\|B\|.$$

Thus $\lim_n \|A^*X_n x_n\| = \|A\|$ and $\lim_n \|X_nBx_n\| = \|B\|$ because $|\langle X_n x_n, AX_nBx_n\rangle| \leqslant \|A^*X_n x_n\|\|X_nBx_n\|$. For each $n \geqslant 1$, we have

$$\|\delta_{A^*,-B}\| \geqslant \|A^*X_n + X_nB\| \geqslant \|A^*X_n x_n + X_nBx_n\|.$$

Since $\lim_n \|A^*X_n x_n + X_nBx_n\| = \|A\| + \|B\|$ and $\|\delta_{A^*,-B}\| \leqslant \|A\| + \|B\|$, we conclude that $\|\delta_{A^*,-B}\| = \|A\| + \|B\|$. Thus, it follows from [10, Theorem 7] that $W_N(A^*) \cap W_N(B) \neq \emptyset$.

(2) $\Rightarrow$ (1) Let $\mu \in W_N(A^*) \cap W_N(B)$. Then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in H such that $\|x_n\| = \|y_n\| = 1$, $\lim_n \|A^*x_n\| = \|A\|$, $\lim_n \|By_n\| = \|B\|$, $\lim_n \langle A^*x_n, x_n\rangle = \mu\|A\|$, and $\lim_n \langle By_n, y_n\rangle = \mu\|B\|$. Set $A^*x_n = \alpha_n x_n + \beta_n u_n$, where $\alpha_n, \beta_n \in \mathbb{C}$, $u_n \in H$ with $\|u_n\| = 1$ and $\langle x_n, u_n\rangle = 0$. We may choose u_n so that $\langle A^*x_n, u_n\rangle = \beta_n \geqslant 0$ for all n. Set also $By_n = \gamma_n y_n + \delta_n v_n$, where $\gamma_n, \delta_n \in \mathbb{C}$, $\|v_n\| = 1$, $\langle y_n, v_n\rangle = 0$ and $\langle By_n, v_n\rangle = \delta_n \geqslant 0$.

Define a sequence $\{X_n\}_n \subseteq \mathcal{L}(H)$ by

$$X_n = \langle \cdot, y_n\rangle x_n + \langle \cdot, v_n\rangle u_n.$$

Then clearly $\|X_n\| = 1$ for all n, and we have

$$\langle X_n y_n, AX_nBy_n\rangle = \langle A^*y_n, \gamma_n y_n + \delta_n u_n\rangle = \alpha_n\gamma_n + \beta_n\delta_n.$$

By the definitions of the sequences $\{x_n\}$ and $\{y_n\}$, we derive that $\lim_n |\alpha_n|^2 + \beta_n^2 = \|A\|^2$ and $\lim_n |\alpha_n| = |\mu|\|A\|$. Thus, $\lim_n \beta_n = \sqrt{1-|\mu|^2}\|A\|$. In a similar way we obtain $\lim_n \delta_n = \sqrt{1-|\mu|^2}\|B\|$. Hence,

$$\begin{aligned}\lim_n \langle X_n y_n, AX_nBy_n\rangle &= \lim_n \alpha_n\gamma_n + \beta_n\delta_n \\ &= |\mu|^2\|A\|\|B\| + \big(1-|\mu|^2\big)\|A\|\|B\| = \|A\|\|B\|.\end{aligned}$$

From this we conclude that $\lim_n \|AX_nBy_n\| = \|A\|\|B\|$. Now, we have for each $n \geqslant 1$,

$$1 + \|A\|\|B\| \geqslant \|I + M_{A,B}\| \geqslant \|X_n + AX_nB\| \geqslant \|X_n y_n + AX_nBy_n\|.$$

Therefore,

$$\lim_n \|X_n y_n + AX_nBy_n\| = 1 + \|A\|\|B\| \leqslant \|I + M_{A,B}\| \leqslant 1 + \|A\|\|B\|.$$

Consequently,

$$\|I + M_{A,B}\| = 1 + \|A\|\|B\|. \qquad \square$$

Remark 2. (i) Let $A, B \in \mathcal{L}(H)$. It follows from Theorem 1, [10, Theorem 1], and [10, Theorem 8] that $0 \in W_0(A)$ if and only if $\|I - M_{A^*,A}\| = 1 + \|A\|^2$ if and only if $\|\delta_{A,A}\| = 2\|A\|$.

(ii) Also we conclude from Theorem 1 and [10] that the following are equivalent:

(1) $\|I + M_{A,B}\| = 1 + \|A\|\|B\|$,
(2) $\|\delta_{A^*,-B}\| = \|A\| + \|B\|$,
(3) $\|A\| + \|B\| \leqslant \|A - \lambda\| + \|B - \lambda\|$ for all $\lambda \in \mathbb{C}$.

An immediate consequence of Theorem 1 is the following

Corollary 3. *If $A \in \mathcal{L}(H)$, then $\|I + M_{A,A^*}\| = 1 + \|A\|^2$.*

Another consequence of Theorem 1 is the following result proved in [1,3].

Corollary 4. *If $A \in \mathcal{L}(H)$, then $\|I + A\| = 1 + \|A\|$ if and only if $\|A\| \in \sigma_{ap}(A)$.*

Proof. If $B = I$ in Theorem 1, then we see that $\|I + A\| = 1 + \|A\|$ if and only if $1 \in W_N(A^*)$. This is equivalent to the existence of a unit sequence $\{x_n\}_n$ in H such that $\lim_n \langle Ax_n, x_n\rangle = \|A\|$ and $\lim_n \|Ax_n\| = \|A\|$. From this we conclude that $\lim_n \|Ax_n - \|A\|x_n\| = 0$, that is, $\|A\| \in \sigma_{ap}(A)$. □

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