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On the essential numerical range

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Abstract. We introduce and study the essential numerical range for Banach space operators. This generalizes the corresponding well-known concept for Hilbert space operators.

1. Introduction

Let A be a complex normed algebra with unit. Denote by A^* the set of all bounded linear functionals on A. The *algebraic numerical range of an element* $a \in A$ is defined by

$$V(a, A) = \{f(a) : f \in A^*, f(1) = 1 = \|f\|\}.$$

It is well-known that $V(a, A)$ is a compact convex subset of the complex plane and $\sigma(a) \subset V(a, A)$ (see [BD1]). Moreover,

$$V(a, A) = \bigcap_{\mu \in \mathbb{C}} \{\lambda : |\lambda - \mu| \le \|a - \mu\|\}$$

and

$$\exp(-1) \cdot \|a\| \le \max\{|\lambda| : \lambda \in V(a, A)\} \le \|a\|.$$

Let $B(X)$ be the Banach algebra of all bounded linear operators acting on a complex Banach space X. For $T \in B(X)$ the *spatial numerical range* is defined by

$$W(T) = \Big\{\langle Tx, x^*\rangle : x \in X, x^* \in X^*, \|x\| = 1 = \|x^*\| = \langle x, x^*\rangle\Big\}.$$

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If X is a Hilbert space and $T \in B(X)$ then the above definition assumes a simpler form

$$W(T) = \Big\{\langle Tx, x\rangle : x \in X, \|x\| = 1\Big\}.$$

The algebraic and spatial numerical ranges of an operator are closely connected. For Hilbert space operators the set $W(T)$ is convex by the classical Toeplitz–Hausdorff theorem; moreover,

$$V(T, B(X)) = \overline{W(T)}.$$

For Banach space operators this is no longer true. By [BD1], Theorem 9.4, we have only

$$V(T, B(X)) = \overline{\text{conv}}\, W(T), \tag{1}$$

where $\overline{\text{conv}}$ denotes the closed convex hull.

An essential version of the numerical range has also been studied.

Denote by $\mathcal{K}(X)$ the ideal of all compact operators acting on a complex Banach space X, and let π be the canonical projection from $B(X)$ onto the Calkin algebra $B(X)/\mathcal{K}(X)$. Denote further by $\|\cdot\|_e$ the essential norm $\|T\|_e = \inf\{\|T+K\| : K \in \mathcal{K}(X)\}$.

Definition 1. [BD2] Let X be an infinite-dimensional Banach space and $T \in B(X)$. The *essential numerical range* $V_e(T)$ of T is defined by

$$V_e(T) = V\Big(\pi(T), B(X)/\mathcal{K}(X), \|\cdot\|_e\Big).$$

We summarize the basic properties of the essential numerical range in the following theorem:

Theorem 2. [BD2] *Let X be an infinite-dimensional Banach space and $T \in B(X)$. Then*:

(i) *$V_e(T)$ is a nonempty compact convex set and $\sigma_e(T) \subset V_e(T)$*;

(ii) *$V_e(T) = \{0\}$ if and only if $T \in \mathcal{K}(X)$*;

(iii) *$V_e(T) = \bigcap\Big\{V(T+K, B(X)) : K \in \mathcal{K}(X)\Big\}$*;

(iv) *$V_e(T) = \Big\{f(T) : f \in B(X)^*, f(I) = 1 = \|f\|, f(\mathcal{K}(X)) = \{0\}\Big\}$*;

(v) $\exp(-1)\cdot\|T\|_e \le \max\{|\lambda| : \lambda \in V_e(T)\} \le \|T\|_e$.

Here $\sigma_e(T)$ denotes the essential spectrum of T, $\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}$.

The essential spatial numerical range (defined by property (ii) of the following theorem) was studied for Hilbert space operators.

Theorem 3. ([FSW]) *Let H be an infinite-dimensional Hilbert space and $T \in B(H)$. Let $\lambda \in \mathbb{C}$. The following properties are equivalent*:

(i) $\lambda \in V_e(T)$;

(ii) *there exists a sequence* $(x_n) \subset H$ *such that* $\|x_n\| = 1$ *for all* n, $x_n \to 0$ *weakly, and* $\lim_{n\to\infty}\langle Tx_n, x_n\rangle = \lambda$;

(iii) *there exists an orthonormal sequence* $(u_n) \subset H$ *such that*

$$\lim_{n\to\infty}\langle Tu_n, u_n\rangle = \lambda.$$

Thus the algebraic and spatial essential numerical ranges for Hilbert space operator coincide.

The aim of this paper is to study the essential version of spatial numerical range for Banach space operators. We use the following natural definition:

Definition 4. Let X be an infinite-dimensional Banach space and $T \in B(X)$. Denote by $W_e(T)$ the set of all complex numbers λ with the property that there are nets $(u_\alpha) \subset X$, $(u_\alpha^*) \subset X^*$ such that

$$\|u_\alpha\| = \|u_\alpha^*\| = \langle u_\alpha, u_\alpha^*\rangle = 1$$

for all α, $u_\alpha \to 0$ weakly and $\langle Tu_\alpha, u_\alpha^*\rangle \to \lambda$.

In analogy to (1) we prove that the algebraic essential numerical range is equal to the convex hull of the set $W_e(T)$; however, instead of the essential norm in the Calkin algebra it is necessary to consider another measure of non-compactness. This is probably the reason why the essential spatial numerical ranges for Banach space operators have not been studied before.

Moreover, for reflexive Banach spaces it is sufficient to consider only sequences (instead of nets) in the definition of $W_e(T)$. The same is true for a more general class of Asplund spaces (i.e., Banach spaces with the property that each separable subspace has a separable dual).

2. Spatial essential numerical range

Let X be an infinite-dimensional complex Banach space and $T \in B(X)$. In this section we study the basic properties of the spatial essential numerical range.

Proposition 5. *$W_e(T)$ is a closed subset of $\mathbb{C}$.*

Proof. Let $\lambda \in W_e(T)^-$ and let $\lambda_n \in W_e(T)$, $\lambda_n \to \lambda$. Let $\varepsilon > 0$ and let $F = \{v_1^*, \dots, v_k^*\}$ be a finite subset of X^*. Find an n such that $|\lambda_n - \lambda| < \varepsilon/2$. Since $\lambda_n \in W_e(T)$, there are elements $u_{\varepsilon,F} \in X$ and $u_{\varepsilon,F}^* \in X^*$ such that $\|u_{\varepsilon,F}\| = 1 = \|u_{\varepsilon,F}^*\| = \langle u_{\varepsilon,F}, u_{\varepsilon,F}^*\rangle$, $|\langle Tu_{\varepsilon,F}, u_{\varepsilon,F}^*\rangle - \lambda_n| < \varepsilon/2$ and $|\langle u_{\varepsilon,F}, v_j^*\rangle| < \varepsilon \quad (j = 1, \dots, k)$. Thus

$$|\langle Tu_{\varepsilon,F}, u_{\varepsilon,F}^*\rangle - \lambda| \le |\langle Tu_{\varepsilon,F}, u_{\varepsilon,F}^*\rangle - \lambda_n| + |\lambda_n - \lambda| < \varepsilon.$$

Consider the nets $(u_{\varepsilon,F})$, $(u_{\varepsilon,F}^*)$ ordered by the relation $(\varepsilon, F) \le (\varepsilon', F')$ if and only if $\varepsilon' \le \varepsilon$ and $F' \supset F$. It is easy to see that $u_{\varepsilon,F} \to 0$ weakly and $\langle Tu_{\varepsilon,F}, u_{\varepsilon,F}^*\rangle \to \lambda$.

Hence $\lambda \in W_e(T)$ and $W_e(T)$ is closed. ■

Proposition 6. *Let $T \in B(X)$ and $\lambda \in \mathbb{C}$. The following properties are equivalent:*

(i) $\lambda \in W_e(T)$;

(ii) *for every subspace $M \subset X$ of finite codimension and each $\varepsilon > 0$ there are $x \in M$ and $x^* \in X^*$ such that $\|x\| = \|x^*\| = 1 = \langle x, x^*\rangle$ and $|\langle Tx, x^*\rangle - \lambda| \le \varepsilon$.*

Proof. (ii)⇒(i): For $M \subset X$ with codim $M < \infty$ and $\varepsilon > 0$ choose $x_{M,\varepsilon} \in M$ and $x_{M,\varepsilon}^* \in X^*$ such that $\|x_{M,\varepsilon}\| = \|x_{M,\varepsilon}^*\| = 1 = \langle x_{M,\varepsilon}, x_{M,\varepsilon}^*\rangle$ and $|\langle Tx_{M,\varepsilon}, x_{M,\varepsilon}^*\rangle - \lambda| \le \varepsilon$. Consider the nets $(x_{M,\varepsilon})$ and $(x_{M,\varepsilon}^*)$ ordered by the relation $(M, \varepsilon) \le (M', \varepsilon')$ if and only if $M' \subset M$ and $\varepsilon' \le \varepsilon$. Clearly $x_{M,\varepsilon} \to 0$ weakly and $\langle Tx_{M,\varepsilon}, x_{M,\varepsilon}^*\rangle \to \lambda$. Thus $\lambda \in W_e(T)$.

(i)⇒(ii): Conversely, let $\lambda \in W_e(T)$ and let $(u_\alpha) \subset X$ and $(u_\alpha^*) \subset X^*$ be the nets with the properties required in the definition of $W_e(T)$. Let $M \subset X$ be a subspace of finite codimension and let $\varepsilon > 0$. We may assume that $M \ne X$ since otherwise the statement is obvious. Let $L = M \cap T^{-1}M$. Then L is a subspace of X of finite codimension. Let $k = \operatorname{codim} L$.

Consider the finite-dimensional space X/L. By the Auerbach lemma there are elements $\tilde{w}_1, \dots, \tilde{w}_k \in X/L$ and

$$v_1^*, \dots, v_k^* \in (X/L)^* = L^\perp \subset X^*$$

such that $\|\tilde{w}_i\|_{X/L} = 1 = \|v_j^*\|$ and $\langle \tilde{w}_i, v_j^*\rangle = \delta_{i,j}$ (the Kronecker symbol) for all $i, j = 1, \dots, k$. Choose $\delta > 0$ small enough ($\delta \le \min\{(4k)^{-1}, \frac{\varepsilon^2}{40^2\|T\|^2 k}\}$). Choose $v_i \in X$ such that $v_i + L = \tilde{w}_i$ and $\|v_i\| < 1 + \delta$ for all $i = 1, \dots, k$. Thus $\langle v_i, v_j^*\rangle = \delta_{i,j} \quad (i, j = 1, \dots, k)$.

Moreover, $L = \bigcap_{j=1}^k \operatorname{Ker} v_j^*$. Find β such that $|\lambda - \langle Tu_\beta, u_\beta^* \rangle| < \varepsilon/2$ and $|\langle u_\beta, v_j^* \rangle| \le \delta$ for $j = 1, \dots, k$. Let $y = u_\beta - \sum_{j=1}^k \langle u_\beta, v_j^* \rangle v_j$. Then $\langle y, v_j^* \rangle = 0$ for $j = 1, \dots, k$, and so $y \in L$. Further

$$\|y - u_\beta\| \le \sum_{j=1}^k |\langle u_\beta, v_j^* \rangle| \cdot \|v_j\| < k\delta(1+\delta) \le 2k\delta.$$

Let $y_1 = \frac{y}{\|y\|}$ and $y_1^* = \frac{u_\beta^*|_M}{\|u_\beta^*|_M\|}$. We have $y_1 \in L \subset M$, $y_1^* \in M^*$, $\|y_1\| = 1 = \|y_1^*\|$ and

$$\|y_1 - y\| = |\, 1 - \|y\| \,| = |\, \|u_\beta\| - \|y\| \,| \le \|u_\beta - y\| \le 2k\delta.$$

Thus

$$\|y_1 - u_\beta\| \le \|y_1 - y\| + \|y - u_\beta\| \le 4k\delta$$

and

$$|\langle y_1, u_\beta^* \rangle - 1| = |\langle y_1 - u_\beta, u_\beta^* \rangle| \le \|y_1 - u_\beta\| \le 4k\delta.$$

Further $\|u_\beta^*|_M\| \le \|u_\beta^*\| = 1$ and

$$\|u_\beta^*|_M\| \ge |\langle y_1, u_\beta^* \rangle| \ge |\langle u_\beta, u_\beta^* \rangle| - |\langle u_\beta - y_1, u_\beta^* \rangle| \ge 1 - \|u_\beta - y_1\| \ge 1 - 4k\delta.$$

We have

$$\|y_1^* - u_\beta^*|_M\| = \Big|\, 1 - \|u_\beta^*|_M\| \,\Big| \le 4k\delta,$$

and so

$$|\langle y_1, y_1^* \rangle - 1| \le |\langle y_1, u_\beta^* \rangle - 1| + |\langle y_1, y_1^* - u_\beta^*|_M \rangle| \le 4k\delta + \|y_1^* - u_\beta^*|_M\| \le 8k\delta.$$

By the Bishop–Phelps–Bollobás Theorem, there are $x \in M$ and $z^* \in M^*$ such that $\|x\| = \|z^*\| = 1 = \langle x, z^* \rangle$, $\|x - y_1\| \le \sqrt{32k\delta} \le 6\sqrt{k\delta}$ and $\|z^* - y_1^*\| \le \sqrt{32k\delta} \le 6\sqrt{k\delta}$. We have

$$\|z^* - u_\beta^*|_M\| \le \|z^* - y_1^*\| + \|y_1^* - u_\beta^*|_M\| \le 6\sqrt{k\delta} + 4k\delta \le 10\sqrt{k\delta}.$$

Let x^* be an extension of z^* to X such that $\|x^*\| = \|z^*\| = 1$. Then $\|x\| = \|x^*\| = 1 = \langle x, x^* \rangle$. Since $Ty_1 \in M$, we have

$$\begin{aligned}
&|\langle Tx, x^* \rangle - \lambda| \\
&\qquad \le |\langle Tx - Ty_1, x^* \rangle| + |\langle Ty_1, x^* \rangle - \lambda| \\
&\qquad \le \|T\| \cdot \|x - y_1\| + |\langle Ty_1, z^* \rangle - \lambda| \\
&\qquad \le 6\|T\|\sqrt{k\delta} + |\langle Tu_\beta, u_\beta^* \rangle - \lambda| + |\langle T(y_1 - u_\beta), u_\beta^* \rangle| + |\langle Ty_1, z^* - u_\beta^*|_M \rangle| \\
&\qquad \le 6\|T\|\sqrt{k\delta} + \varepsilon/2 + 4\|T\|k\delta + 10\|T\|\sqrt{k\delta} \\
&\qquad \le \varepsilon/2 + 20\|T\|\sqrt{k\delta} \le \varepsilon.
\end{aligned}$$

■

We now introduce another seminorm in $B(X)$ (see e.g. [LS], or [M2], Sec. 23). For $T \in B(X)$ let

$$\|T\|_\mu = \inf\Big\{\|T|_M\| : M \subset X \text{ a subspace of finite codimension}\Big\}.$$

It is well-known that $\|\cdot\|_\mu$ is a "measure of non-compactness", i.e., $\|T\|_\mu = 0$ if and only if T is compact. Moreover, $\|\cdot\|_\mu$ is an algebra seminorm, i.e.,

$$\|T+S\|_\mu \le \|T\|_\mu + \|S\|_\mu, \quad \|TS\|_\mu \le \|T\|_\mu \cdot \|S\|_\mu, \quad \|\alpha T\|_\mu = |\alpha| \cdot \|T\|_\mu$$

for all $T, S \in B(X)$ and $\alpha \in \mathbb{C}$. Thus $\|\cdot\|_\mu$ defines an algebra norm on the Calkin algebra $B(X)/\mathcal{K}(X)$.

For $T \in B(X)$ we define a new essential numerical range by

$$V_\mu(T) = V\Big(T, B(X)/\mathcal{K}(X), \|\cdot\|_\mu\Big).$$

Thus $V_\mu(T)$ is the set of all complex numbers λ such that there is a functional $\widetilde{\Phi} \in (B(X)/\mathcal{K}(X), \|\cdot\|_\mu)^*$ satisfying $\|\widetilde{\Phi}\| = 1 = \widetilde{\Phi}(I+\mathcal{K}(X))$ and $\widetilde{\Phi}(T+\mathcal{K}(X)) = \lambda$. Equivalently, there is a functional $\Phi \in B(X)^*$ such that $\Phi(\mathcal{K}(X)) = 0$, $\Phi(I) = 1$, $\Phi(T) = \lambda$ and $|\Phi(S)| \le \|S\|_\mu$ for all $S \in B(X)$.

In particular, $V_\mu(T)$ is a closed convex set and

$$\exp(-1) \cdot \|T\|_\mu \le \max\{|\lambda| : \lambda \in V_\mu(T)\} \le \|T\|_\mu.$$

Remarks 7. (i) If H is a Hilbert space then $\|T\|_\mu$ coincides with the essential norm $\|T\|_e$, and so

$$V_\mu(T) = V_e(T).$$

(ii) In general $\|T\|_\mu \le \|T\|_e$, and so

$$V_\mu(T) \subset V_e(T).$$

However, in general the norms $\|\cdot\|_e$ and $\|\cdot\|_\mu$ are not equivalent, see [AT]. Thus the above inclusion can be proper (by Theorem 2 (v) and the corresponding property of $V_\mu(T)$).

(iii) Although in general $V_\mu(T) \ne V_e(T)$, for any Banach space X and $T \in B(X)$ we have $V_\mu(T) = \{0\}$ if and only if $V_e(T) = \{0\}$; this is true if and only if T is compact.

Theorem 8. *Let $T \in B(X)$. Then $V_\mu(T) = \operatorname{conv} W_e(T)$.*

Proof. Let $\lambda \in W_e(T)$. Let (u_α), (u_α^*) be the nets with the required properties.

For $\beta, \gamma \in \mathbb{C}$ define $\Phi(\beta I + \gamma T) = \beta + \gamma\lambda$. We show that

$$|\Phi(\beta I + \gamma T)| \leq \|\beta I + \gamma T\|_\mu$$

for all $\beta, \gamma \in \mathbb{C}$. Write for short $S = \beta I + \gamma T$.

Let $M \subset X$ be a subspace of finite codimension and $\varepsilon > 0$. Then there are $x \in M$ and $x^* \in X^*$ such that $\|x\| = \|x^*\| = 1 = \langle x, x^* \rangle$ and $|\langle Tx, x^* \rangle - \lambda| < \varepsilon$. Thus

$$\|S|_M\| \geq \|Sx\| \geq |\langle Sx, x^* \rangle| = |\beta + \gamma \langle Tx, x^* \rangle| \geq |\beta + \gamma\lambda| - |\gamma\varepsilon|.$$

Since $\varepsilon > 0$ was arbitrary, we have $\|S|_M\| \geq |\beta + \gamma\lambda|$, and so $\|S\|_\mu \geq |\beta + \gamma\lambda|$. Thus $\|S\|_\mu \geq |\Phi(S)|$ for each $S \in \bigvee\{I, T\}$. By the Hahn–Banach Theorem, it is possible to extend Φ to a functional (denoted also by Φ) on $B(X)$ such that $|\Phi(R)| \leq \|R\|_\mu$ for all $R \in B(X)$. In particular, $\Phi(K) = 0$ for each compact operator $K \in \mathcal{K}(X)$. By definition, $\lambda = \Phi(T) \in V_\mu(T)$. Since $V_\mu(T)$ is a convex set, we have $\operatorname{conv} W_e(T) \subset V_\mu(T)$.

To show the opposite inclusion, we first prove the following lemma.

Lemma 9. *Let $S \in B(X)$ and $0 \in V_\mu(S)$. Then there exists $\eta \in W_e(S)$ with* $\operatorname{Re} \eta \geq 0$.

Proof. By definition, there exists a functional $\Phi \in B(X)^*$ such that $\Phi(I) = 1$, $\Phi(S) = 0$ and $|\Phi(R)| \leq \|R\|_\mu$ for all $R \in B(X)$.

Let $\varepsilon > 0$ and $M \subset X$ be a subspace of finite codimension. Let $a > 0$ be sufficiently large. We have $|\Phi(aI + S))| = a \leq \|aI + S\|_\mu$. Then there exists $x \in M, \|x\| = 1$ such that $\|(aI + S)x\| > a - \varepsilon$. By the Hahn–Banach Theorem, there is an $x^* \in X^*$ with $\|x^*\| = 1$ and

$$\langle ax + Sx, x^* \rangle = \|ax + Sx\| > a - \varepsilon.$$

We have

$$\operatorname{Re} \langle Sx, x^* \rangle = \operatorname{Re} \langle ax + Sx, x^* \rangle - \operatorname{Re} \langle ax, x^* \rangle > a - \varepsilon - |\langle ax, x^* \rangle| \geq -\varepsilon,$$

and

$$\operatorname{Re} \langle ax, x^* \rangle = \operatorname{Re} \langle ax + Sx, x^* \rangle - \operatorname{Re} \langle Sx, x^* \rangle > a - \varepsilon - \|S\|.$$

Thus $\operatorname{Re}\langle x, x^*\rangle > 1 - \frac{\varepsilon+\|S\|}{a}$. By choosing a sufficiently large, we can find $x \in M$ and $x^* \in X^*$ such that $\|x\| = 1 = \|x^*\|$ and $|\langle x, x^*\rangle - 1| < \varepsilon^2/4$.

By the Bishop–Phelps–Bollobás Theorem, there exist $y \in X$ and $y^* \in X^*$ such that

$$\|y\| = 1 = \|y^*\| = \langle y, y^*\rangle, \quad \|y - x\| < \varepsilon \quad \text{and} \quad \|y^* - x^*\| < \varepsilon.$$

Thus $\operatorname{dist}\{y, M\} \le \|y - x\| < \varepsilon$ and

$$\begin{aligned}\operatorname{Re}\langle Sy, y^*\rangle &= \operatorname{Re}\langle Sx, x^*\rangle + \operatorname{Re}\langle S(y-x), x^*\rangle + \operatorname{Re}\langle Sy, y^* - x^*\rangle| \\ &> -\varepsilon - 2\varepsilon\|S\|.\end{aligned}$$

Therefore we have nets $(y_{M,\varepsilon}) \subset X, (y^*_{M,\varepsilon}) \subset X^*$ (indexed by subspaces $M \subset X$ of finite codimension and by $\varepsilon > 0$) such that

$$\|y_{M,\varepsilon}\| = 1 = \|y^*_{M,\varepsilon}\| = \langle y_{M,\varepsilon}, y^*_{M,\varepsilon}\rangle, \quad \operatorname{dist}\{y_{M,\varepsilon}, M\} < \varepsilon$$

and $\operatorname{Re}\langle Sy_{M,\varepsilon}, y^*_{M,\varepsilon}\rangle > -\varepsilon - 2\varepsilon\|S\|$ for all M and $\varepsilon > 0$.

We show that the net $(y_{M,\varepsilon})$ converges weakly to 0. Let $u^* \in X^*$, $\|u^*\| = 1$, let $\tau > 0$ and $L = \operatorname{Ker}\, u^*$. For every $(M, \varepsilon) \ge (L, \tau)$ (i.e., $M \subset L, \varepsilon \le \tau$) we have $\operatorname{dist}\{y_{M,\varepsilon}, L\} < \varepsilon \le \tau$, and so

$$|\langle y_{M,\varepsilon}, u^*\rangle| \le \operatorname{dist}\{y_{M,\varepsilon}, L\} < \tau.$$

Hence the net $(y_{M,\varepsilon})$ converges weakly to 0.

Set $\eta_{M,\varepsilon} = \langle Sy_{M,\varepsilon}, y^*_{M,\varepsilon}\rangle$. We have $\operatorname{Re}\,\eta_{M,\varepsilon} > -\varepsilon - 2\varepsilon\|S\|$. Find a subnet (y_β) of $y_{M,\varepsilon}$ such that the corresponding numbers (η_β) are converging to some number η. Clearly (y_β) converges weakly to 0, $\operatorname{Re}\,\eta \ge 0$ and $\eta \in W_e(S)$.

Continuation of the proof of Theorem 8. Since $W_e(T)$ is closed, its convex hull $\operatorname{conv}\, W_e(T)$ is also closed. Suppose on the contrary that there is a $\lambda \in V_\mu(T) \setminus \operatorname{conv}\, W_e(T)$. Then there are $\mu \in \mathbb{C}, |\mu| = 1$ and $q \in \mathbb{R}$ such that $\operatorname{Re}(\mu\lambda) > q$ and

$$\operatorname{conv}\, W_e(T) \subset \{z \in \mathbb{C} : \operatorname{Re}(\mu z) < q\}.$$

Let $S = \mu T - \mu\lambda I$. Then $0 \in V_\mu(S)$ and

$$W_e(S) = \{\mu z - \mu\lambda : z \in W_e(T)\} \subset \{z \in \mathbb{C} : \operatorname{Re}\, z < 0\},$$

a contradiction with Lemma 9.

This finishes the proof of Theorem 8. ■

Proposition 10. *Let $T \in B(X)$. Then $\sigma_e(T) \subset W_e(T)$.*

Proof. Let $\lambda \in \sigma_e(T)$, i.e., $T-\lambda$ is not Fredholm. We distinguish two cases.

Suppose first that $T-\lambda$ is not upper semi-Fredholm. Let $\varepsilon > 0$ and let $M \subset X$ be a subspace of finite codimension.

Then there exists $x \in M$ with $\|x\| = 1$ and $\|(T-\lambda)x\| < \varepsilon$. Find $x^* \in X^*$ such that $\|x^*\| = 1 = \langle x, x^* \rangle$. Then

$$|\langle Tx, x^* \rangle - \lambda| = |\langle (T-\lambda)x, x^* \rangle| \le \|(T-\lambda)x\| < \varepsilon.$$

By Proposition 6, $\lambda \in W_e(T)$.

Suppose now that $T-\lambda$ is upper semi-Fredholm but not Fredholm. Then $\dim \operatorname{Ker}(T^*-\lambda) = \infty$.

Let $u_1^*, \dots, u_n^* \in X^*$ be of norm one and let $\varepsilon > 0$. Denote by F' the subspace of X^* spanned by the elements $u_1^*, \dots, u_n^*$. Since $\dim \operatorname{Ker}(T^*-\lambda) = \infty$ and $\dim F' < \infty$, there exists $v^* \in \operatorname{Ker}(T^*-\lambda)$ such that $\|v^*\| = 1 = \operatorname{dist}\{v^*, F'\}$, see [K, p. 199]. Therefore there exists a functional $v^{**} \in F'^{\perp} \subset X^{**}$ such that $\|v^{**}\| = 1 = \langle v^*, v^{**} \rangle$.

By the local reflexivity principle, see e.g. [FHH], Theorem 9.15, there exists $v \in X$ such that $v \in {}^{\perp}F'$, $\langle v, v^* \rangle = 1$ and $\|v\| \le 1 + \varepsilon$. Thus $\langle \frac{v}{\|v\|}, v^* \rangle = \frac{1}{\|v\|} \ge \frac{1}{1+\varepsilon} \ge 1 - \varepsilon$.

By the Bishop–Phelps–Bollobás Theorem, there exist $w \in X$, $w^* \in X^*$ such that $\|w\| = \|w^*\| = 1 = \langle w, w^* \rangle$, $\left\| w - \frac{v}{\|v\|} \right\| < 2\sqrt{\varepsilon}$ and $\|w^* - v^*\| \le 2\sqrt{\varepsilon}$. Thus

$$\begin{aligned} |\langle Tw, w^* \rangle - \lambda| &\le \left| \left\langle T\Big(w - \frac{v}{\|v\|}\Big), w^* \right\rangle \right| + \left| \left\langle \frac{Tv}{\|v\|}, w^* - v^* \right\rangle \right| + \left| \left\langle \frac{Tv}{\|v\|}, v^* \right\rangle - \lambda \right| \\ &\le 2\|T\|\sqrt{\varepsilon} + 2\|T\|\sqrt{\varepsilon} + \left| \left\langle \frac{v}{\|v\|}, \lambda v^* \right\rangle - \lambda \right| \\ &\le 4\|T\|\sqrt{\varepsilon} + |\lambda| \cdot \left| \frac{1}{\|v\|} - 1 \right| \le 4\|T\|\sqrt{\varepsilon} + \varepsilon|\lambda|. \end{aligned}$$

Further,

$$|\langle w, u_i^* \rangle| = |\langle w - v, u_i^* \rangle| \le \|w - v\| \le \left\| w - \frac{v}{\|v\|} \right\| + \left\| \frac{v}{\|v\|} - v \right\| < 2\sqrt{\varepsilon} + \varepsilon$$

for all $i = 1, \dots, n$.

In this way we obtain nets $(w_\alpha) \subset X$ and $(w_\alpha^*) \subset X^*$ indexed by finite subsets of X^* and positive numbers ε which satisfy all the conditions of the definition of $W_e(T)$.

Hence $\sigma_e(T) \subset W_e(T)$.

■

3. Reflexive and Asplund spaces

For operators on reflexive Banach spaces (and more generally, on Asplund spaces) it is sufficient to consider only sequences instead of nets in the definition of $W_e(T)$. Thus the situation is more similar to the Hilbert space case.

Recall that a sequence (x_n) in a Banach space X is called *basic* if every vector $u \in \bigvee\{x_n : n \in \mathbb{N}\}$ can be uniquely expressed as a sum $u = \sum_{n=1}^{\infty} \alpha_n x_n$ for some complex coefficients α_n.

By a classical result of Banach, see [FHH], Prop. 6.13, a sequence $(x_n) \subset X$ consisting of nonzero vectors is basic if and only if there is a constant $k > 0$ such that for all $r, m \in \mathbb{N}$, $r < m$ and complex numbers $\alpha_1, \dots, \alpha_m$ we have

$$\Big\|\sum_{i=1}^{r} \alpha_i x_i\Big\| \le k \cdot \Big\|\sum_{i=1}^{m} \alpha_i x_i\Big\|.$$

Theorem 11. *Let X be a reflexive Banach space and $T \in B(X)$. Let $\lambda \in \mathbb{C}$. Then the following conditions are equivalent*:

(i) $\lambda \in W_e(T)$;

(ii) *there exist sequences $(x_n) \subset X$, $(x_n^*) \subset X^*$ such that*

$$\|x_n\| = \|x_n^*\| = \langle x_n, x_n^* \rangle = 1$$

for all n, $x_n \to 0$ weakly and $\langle Tx_n, x_n^* \rangle \to \lambda$;

(iii) *there exists a basic sequence $(x_n) \subset X$ and a sequence $(x_n^*) \subset X^*$ such that*

$$\|x_n\| = \|x_n^*\| = \langle x_n, x_n^* \rangle = 1$$

for all n and $\langle Tx_n, x_n^* \rangle \to \lambda$.

Proof. (ii)⇒(i): Clear.

(iii)⇒(ii): By the James Theorem, see e.g. [FHH], Th. 6.11, every basic sequence in a reflexive Banach space is weakly converging to 0 (in fact this property characterizes reflexive Banach spaces). This implies (ii).

(i)⇒(iii): This follows from the following (more general) lemma.

■

Lemma 12. *Let X be a Banach space, let $T \in B(X)$, $\lambda \in W_e(T)$ and let $0 < \varepsilon_n < 1$, $\varepsilon_n \to 0$. Then there exist sequences $(x_n) \subset X$ and $(x_n^*) \subset X^*$ such that $\langle Tx_n, x_n^* \rangle \to \lambda$, $\|x_n\| = 1 = \|x_n^*\| = \langle x_n, x_n^* \rangle \quad (n \in \mathbb{N})$, and for all $m, r \in \mathbb{N}$, $r < m$ and complex numbers $\alpha_1, \dots, \alpha_m$ we have*

$$\Big\|\sum_{i=1}^{r} \alpha_i x_i\Big\| \le (1-\varepsilon_r)^{-1} \Big\|\sum_{i=1}^{m} \alpha_i x_i\Big\|.$$

In particular, the sequence (x_n) is basic.

Proof. We construct the sequences $(x_n) \subset X$, $(x_n^*) \subset X^*$ and subspaces $X = L_0 \supset L_1 \supset \cdots$ of finite codimension inductively. Choose $x_1 \in X$ and $x_1^* \in X^*$ such that $\|x_1\| = 1 = \|x_1^*\| = \langle x_1, x_1^* \rangle$ and $|\langle Tx_1, x_1^* \rangle - \lambda| < 1$.

Let $k \ge 1$ and suppose that the vectors $x_1, \dots, x_k \in X$, functionals $x_1^*, \dots, x_k^* \in X^*$ and subspaces $L_0, \dots, L_{k-1} \subset X$ have already been constructed. Set $F_k = \bigvee\{x_1, \dots, x_k\}$. By [M1], Lemma 1, there exists a subspace $M_k \subset X$ of finite codimension such that $\|f + m\| \ge (1-\varepsilon_k)\max\{\|f\|, \|m\|/2\}$ for all $f \in F_k$ and $m \in M_k$.

Set $L_{k+1} = L_k \cap M_k$. Then codim $L_{k+1} < \infty$ and $L_{k+1} \subset L_k$. By Proposition 6, there exist $x_{k+1} \in L_{k+1}$ and $x_{k+1}^* \in X^*$ such that $\|x_{k+1}\| = \|x_{k+1}^*\| = 1 = \langle x_{k+1}, x_{k+1}^* \rangle$ and $|\langle Tx_{k+1}, x_{k+1}^* \rangle - \lambda| < (k+1)^{-1}$.

Let (x_n), (x_n^*) be the sequences constructed in the above described way. Clearly $\lim\langle Tx_n, x_n^* \rangle = \lambda$. Let $r, m \in \mathbb{N}$, $r < m$ and let $\alpha_1, \dots, \alpha_m \in \mathbb{C}$. Since $\sum_{i=1}^{r} \alpha_i x_i \in F_r$ and $\sum_{i=r+1}^{m} \alpha_i x_i \in M_r$, we have

$$\Big\|\sum_{i=1}^{r} \alpha_i x_i\Big\| \le (1-\varepsilon_r)^{-1} \Big\|\sum_{i=1}^{m} \alpha_i x_i\Big\|.$$

Since $\sup_k (1-\varepsilon_k)^{-1} < \infty$, the sequence (x_n) is basic. ∎

Results similar to those in Theorem 11 are true also for a more general class of Asplund spaces. Recall that a Banach space X is called *Asplund* if each separable subspace of X has a separable dual.

Let (x_n) be a basic sequence in a Banach space X. Let $X_0 = \bigvee_n x_n$. The sequence (x_n) is called *shrinking* if the dual functionals $y_n^* \in X_0^*$ defined by $\langle x_m, y_n^* \rangle = \delta_{m,n} \quad (m, n = 1, 2, \dots)$ generate X_0^*.

Theorem 13. *Let X be an Asplund space, $T \in B(X)$, $\lambda \in \mathbb{C}$. Then the following conditions are equivalent*:

(i) $\lambda \in W_e(T)$;

(ii) *there exist sequences $(x_n) \subset X$, $(x_n^*) \subset X^*$ such that*

$$\|x_n\| = \|x_n^*\| = \langle x_n, x_n^* \rangle = 1$$

for all n, $x_n \to 0$ weakly and $\langle Tx_n, x_n^* \rangle \to \lambda$;

(iii) *there exists a shrinking basic sequence $(x_n) \subset X$ and a sequence $(x_n^*) \subset X^*$ such that*

$$\|x_n\| = \|x_n^*\| = \langle x_n, x_n^* \rangle = 1$$

for all n and $\langle Tx_n, x_n^* \rangle \to \lambda$.

Proof. (ii)⇒(i): Clear.

(iii)⇒(ii): Any shrinking basic sequence is weakly converging to zero.

(i)⇒(iii): Set $\varepsilon_k = 1/k$. By Lemma 12, there exists sequences $(x_n) \subset X$ and $(x_n^*) \subset X^*$ with the properties described there. Let $X_0 = \bigvee_n x_n$. Then X_0^* is separable. By [FHH], Proposition 8.34 and Theorem 8.19, (x_n) is a shrinking basic sequence.

■

4. Sequential numerical range

For a Banach space X and $T \in B(X)$ we can define the *sequential essential numerical range* $W_\omega(T)$ as the set of all complex numbers λ with the property that there are sequences $(x_n) \subset X$, $(x_n^*) \subset X^*$ such that

$$\|x_n\| = \|x_n^*\| = \langle x_n, x_n^* \rangle = 1$$

for all $n \in N$, $x_n \to 0$ weakly and $\langle Tx_n, x_n^* \rangle \to \lambda$.

It is clear that $W_\omega(T) \subset W_e(T)$. For operators on Asplund spaces we have $W_\omega(T) = W_e(T)$ by the previous theorem.

On the other hand, in general $W_\omega(T) \neq W_e(T)$. To see this, let $T \in B(X)$ be a completely continuous operator (i.e., $\|Tx_n\| \to 0$ for every sequence $(x_n) \subset X$ weakly converging to 0) which is not compact. Then $W_w(T) \subset \{0\}$ and $\sup\{|z| : z \in W_e(T)\} = \sup\{|z| : z \in V_\mu(T)\} \geq \exp(-1) \cdot \|T\|_\mu > 0$.

Corollary 14. *Let X be an Asplund space. An operator $T \in B(X)$ is compact if and only if it is completely continuous.*

Example 15. Let $X = \ell_1$. Then every sequence $(x_n) \subset X$ weakly converging to 0 is also strongly converging, i.e., $\|x_n\| \to 0$. Consequently, $W_\omega(T) = \emptyset$ for every operator $T \in B(X)$.

Example 16. Let H be a separable infinite-dimensional Hilbert space and let $X = \bigoplus_{n=1}^\infty H$ (the ℓ_1-direct sum). Let $T = \bigoplus_{n=1}^\infty (1 - n^{-1}) I_H$. Then $1 - n^{-1} \in W_\omega(T)$ for each n and $1 \notin W_\omega(T)$. Consequently, $W_\omega(T)$ is not closed.

Proof. It is easy to see that $1 - n^{-1} \in W_\omega(T)$ for each n (consider an orthonormal sequence in the n-th copy of H).

We show that $1 \notin W_\omega(T)$. Suppose on the contrary that there are sequences $(x_n) \subset X$, $(x_n^*) \subset X^*$ such that $x_n \to 0$ weakly, $\|x_n\| = 1 = \|x_n^*\| = \langle x_n, x_n^* \rangle$ for all n, and $\langle T x_n, x_n^* \rangle \to 1$. In particular, $\|T x_n\| \to 1$.

For $n \in \mathbb{N}$, let $x_n = \bigoplus_{j=1}^\infty h_j^{(n)}$ with $h_j^{(n)} \in H$, $\sum_{j=1}^\infty \|h_j^{(n)}\| = 1$. Let

$$a_n = \max \Big\{ k : \sum_{j=1}^{k-1} \|h_j^{(n)}\| \le \frac{1}{5} \Big\}$$

and

$$b_n = \min \Big\{ k : \sum_{j=k+1}^{\infty} \|h_j^{(n)}\| \le \frac{1}{5} \Big\}.$$

Since $\|T x_n\| \to 1$, we have $a_n \to \infty$. By passing to a subsequence if necessary, we may assume without loss of generality that $a_1 < b_1 < a_2 < b_2 < \cdots$.

For $n, j \in \mathbb{N}$ let $g_j^{(n)} = \frac{h_j^{(n)}}{\|h_j^{(n)}\|}$ if $h_j^{(n)} \neq 0$ and $g_j^{(n)} = 0$ otherwise. Consider the functional $x^* = \bigoplus_{n=1}^{\infty} \bigoplus_{j=a_n}^{b_n} g_j^{(n)} \in X^*$. Then

$$\|x^*\| = \sup \Big\{ \|g_j^{(n)}\| : n \in \mathbb{N}, a_n \le j \le b_n \Big\} \le 1.$$

For each n we have

$$\begin{aligned} |\langle x_n, x^* \rangle| &\ge \Big| \Big\langle \sum_{j=a_n}^{b_n} h_j^{(n)}, g_j^{(n)} \Big\rangle \Big| - \Big| \Big\langle \sum_{j=1}^{a_n-1} h_j^{(n)}, g_j^{(n)} \Big\rangle \Big| - \Big| \Big\langle \sum_{j=b_n+1}^{\infty} h_j^{(n)}, g_j^{(n)} \Big\rangle \Big| \\ &\ge \sum_{j=a_n}^{b_n} \|h_j^{(n)}\| - \sum_{j=1}^{a_n-1} \|h_j^{(n)}\| - \sum_{j=b_n+1}^{\infty} \|h_j^{(n)}\| \ge \frac{3}{5} - \frac{1}{5} - \frac{1}{5} = \frac{1}{5}, \end{aligned}$$

a contradiction with the assumption that $x_n \to 0$ weakly. Hence $1 \notin W_\omega(T)$ and $W_\omega(T)$ is not closed.

■

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