[1] Compositio Mathematica 59 (1986) 265–321

BOUNDED REINHARDT DOMAINS IN BANACH SPACES

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Introduction

In [29] Thullen classified the bounded Reinhardt domains in $\mathbb{C}^2$ using the orbit of the origin under the action of the biholomorphic automorphisms as a method of classification. Using Thullen's results, J^* triple systems, and a result of Braun et al. [3], Vigué [35] classified the bounded homogeneous Reinhardt domains in a complex Banach space with a basis (as explained in §1 this is equivalent to characterizing those Banach spaces with an unconditional basis of unconditionality constant 1 in which the unit ball is symmetric).

In this paper we extend Vigué's work in two directions. First of all, we characterize those Banach spaces with an unconditional finite dimensional decomposition with unconditionality constant 1 whose unit balls are symmetric. This also provides an independent proof of Vigué's result. Secondly, we classify all bounded Reinhardt domains in a Banach space with basis in a manner similar to that given by Thullen [29] in $\mathbb{C}^2$ and extended to arbitrary finite dimensional domains by Sunada [28].

This article is organized as follows. In §1 we introduce the concepts that are used most frequently later. §2 contains a result classifying Reinhardt decompositions of finite dimensional bounded irreducible domains and uses Lie algebra techniques. The result is applied in §3 to characterize those Banach spaces which have both a one unconditional finite dimensional decomposition and a symmetric unit ball. We also give some consequences of this classification. In §4 we characterize the bounded Reinhardt domains in a Banach space with unconditional basis in terms of the orbit of the origin under the biholomorphic automorphisms of the domain. This result is applied in §5 to determine the orbit for the unit ball of certain Tsirelsohn spaces. Finally §6 contains a discussion of convexity of the domains considered in §4.

Since the concepts involved in this article are drawn from the theory of symmetric spaces and Banach space theory – two AREAS which have had little overlap until recently – we felt it suitable to make this paper as self-contained as possible and consequently have included the basic

§1. Background

This section contains the basic definitions from Banach space theory and the theory of symmetric spaces used in the sequel.

Our Banach space notation and terminology will follow that of [22]. In particular, B_E is the open unit ball of a Banach space E, $[A]$ denotes the closed linear span of the subset A of E, $\mathrm{co}(A)$ is the convex hull of A, and if $(e_n)_{n\in\mathbb{N}}$ is the unit vector basis of c_0 and $\{E_n\}_{n\in\mathbb{N}}$ are Banach spaces then

$$\left(\sum_n \oplus E_n\right)_{c_0}$$

is the Banach space of all sequences (x_n) with $x_n \in E_n$ and $\sum_n \|x_n\| e_n \in c_0$. We'll also write $E \approx F$ when E and F are isomorphic and $E \cong F$ when they are isometrically isomorphic.

Let E be a complex Banach space and let $\mathbb{F}$ be a complete orthogonal family of projections on E. A subset D of E is *circular* if $x \in D$ if and only if $\lambda x \in D$ for all $|\lambda| = 1$, and a circular domain containing the origin is *Reinhardt* – we shall always assume in this paper that Reinhardt domains contain the origin – (with respect to $\mathbb{F}$) *if* it is invariant under the transformations

$$x \to \lambda Px + (\mathrm{id} - P)x \quad \text{for all } P \in \mathbb{F},\ |\lambda| = 1.$$

We shall consider only countable families $\mathbb{F}$ of finite dimensional projections.

PROPOSITION 1.1: *If E contains a bounded Reinhardt domain D then the operators*

$$x \to \sum_{i=1}^{n} \lambda_i P_i x$$

are uniformly bounded for all $n \in \mathbb{N}$, $|\lambda_i| = 1$, and distinct $P_1, \cdots, P_n \in \mathbb{F}$.

PROOF: First observe that $\mathrm{co}(D)$ is also a Reinhardt domain. If $x \in \mathrm{co}(D)$, then

$$(\lambda_1 P_1 + (\mathrm{id} - P_1)) \circ \cdots \circ (\lambda_n P_n + (\mathrm{id} - P_n))(x)$$

$$= \sum^{n} \lambda_i P_i x + x - \sum^{n} P_i x \in \mathrm{co}(D) \quad \text{for all } |\lambda_i| = 1. \tag{1.1}$$

Choosing $\lambda_i = -1$ for all $1 \leqslant i \leqslant n$ in (1.1) results in

$$y = \sum_{i=1}^{n} P_i x \in \mathrm{co}(D).$$

Substituting y for x in (1.1) yields

$$\sum_{i=1}^{n} \lambda_i P_i x = \sum_{i=1}^{n} \lambda_i P_i y \in \mathrm{co}(D) \quad \text{for all } |\lambda_i| = 1. \tag{1.2}$$

Hence, if r and R are constants satisfying $rB_E \subseteq D \subseteq RB_E$, then $rB_E \subseteq \mathrm{co}(D) \subseteq RB_E$ and so

$$\left\| \sum_{i=1}^{n} \lambda_i P_i \right\| = \frac{1}{r} \sup \left\{ \left\| \sum_{i=1}^{n} \lambda_i P_i x \right\| \mid x \in rB_E \right\} \leqslant \frac{R}{r}$$

by (1.2). ∎

A sequence $\{E_n\}_{n \in \mathbb{N}}$ of finite dimensional subspaces of E is a *finite dimensional decomposition* (FDD) of E if every $x \in E$ can be uniquely decomposed as $x = \sum_n x_n$ with $x_n \in E_n$. An FDD is *unconditional* (a UFDD) if $\sum_n x_n$ converges unconditionally, or equivalently, if there is a constant $K \geqslant 1$ so that

$$\left\| \sum_n \lambda_n x_n \right\| \leqslant K \left\| \sum_n x_n \right\| \quad \text{for all } |\lambda_n| \leqslant 1. \tag{1.3}$$

The smallest constant K satisfying (1.3) is the *unconditionality constant*, and we'll say that $\{E_n\}_{n \in \mathbb{N}}$ is a K-UFDD if (1.3) holds. The interested reader may consult [22] and [7] for further information and recent developments on finite dimensional decompositions of Banach spaces.

Proposition 1.1 and (1.3) show that if E contains a bounded Reinhardt domain D then $\{PE \mid P \in \mathbb{F}\}$ is a UFDD with unconditionality constant no more than R/r. In particular, if B_E is Reinhardt then $\mathbb{F}$ must determine a 1-UFDD. If $\{E_n\}$ is a 1-UFDD for E we'll refer to $E = E_1 \oplus E_2 \oplus \cdots$ as a *Reinhardt decomposition* of E. When each E_n is 1-dimensional E has a basis (e_n) of elements $e_n \in E_n$ choosen so that $0 < \inf \| e_n \| \leqslant \sup \| e_n \| < \infty$. A basis (e_n) is normalized if $\| e_n \| = 1$ for all n, and a normalized basis is *K-unconditional* when the subspaces $E_n = [\{e_n\}]$ form a K-UFDD.

Our definition of a Reinhardt domain containing the origin is a natural generalization of the classical notion. Thus the obvious setting for studying these domains in infinite dimensional spaces are Banach

The theory we develop is isometric, not isomorphic, and so the results do not generally extend to spaces with a K-UFDD when $K > 1$. A partial exception to this rule occurs in §4, where the results hold modulo a renorming of the space which is given by a specific positive diagonal transformation.

A basis $(e_n)_{n\in\mathbb{N}}$ of E is said to be *symmetric* with *symmetric constant 1* if for each permutation π of $\mathbb{N}$ and each sequence $(\lambda_n)_n$ of complex numbers of modulus one, the map

$$\sum_n x_n e_n \to \sum_n \lambda_n x_n e_{\pi(n)}$$

is an isometry of E. Symmetric bases arise in §5 and should not be confused with the concepts of symmetric domain and symmetric Banach space introduced below. In §5 we also discuss a particular Banach space: Tsirelsohn's space. This rather remarkable space has already proved useful in infinite dimensional holomorphy [1]. We define the space in §5 but further details may be found in [6,8,11,17,22].

Let D be a domian in E. A mapping $f\colon D \to D$ is said to be *biholomorphic* if

(i) f is bijective,
(ii) f is holomorphic (or Fréchet differentiable),
(iii) f^{-1} is holomorphic.

If E is a finite dimensional space then (i) and (ii) imply (iii). This is not generally true for infinite dimensional E. As a general reference for infinite dimensional holomorphy we refer to Dineen [9] and Franzoni and Vesentini [12].

$G(D)$ will denote the group of biholomorphic automorphisms of D. When there is no fear of confusion we'll write G for $G(D)$. The stabilizer of the origin or the isotropy subgroup at the origin is the subgroup $K(D) = K$ of G consisting of all $g \in G$ with $g(0) = 0$. A theorem of Cartan [5] says that K is the group of linear automorphisms of D when D is a bounded circular domain. Both K and G are topological groups with the operator topology and composition of functions as the group operation (Vigué and Isidro [36]). When D is symmetric (see below) G is also a Lie group. In general it is unknown when G is Lie group, although G always has a topology finer than the operator topology in which it is a Lie group (see Vigué [31]).

D is said to be *symmetric* if for each $x \in D$ there is a symmetry $\sigma_x \in G$, i.e., a map satisfying $\sigma_x(x) = x$ and $\sigma_x'(x) = -\mathrm{id}$. D is *homogeneous* if for each x, $y \in D$ there is a map $g \in G$ with $g(x) = y$. A bounded symmetric domain is always homogeneous (Vigué [31]), and the

$\sigma_x = g^{-1} \circ (-\mathrm{id}) \circ g$, where $g \in G$ takes x to 0. These two concepts agree in the contexts considered in this article.

Let D now be the unit ball of a finite dimensional Banach space E. We'll say E is *symmetric* if D is symmetric. D, as well as E, is said to be *irreducible* if it is symmetric and is not isometrically isomorphic to the direct product of two nontrivial symmetric domains. This concept of irreducibility has several generalizations to infinite dimensional spaces (see Vigué [32,34]). A classical theorem of Cartan [4] states that any finite dimensional symmetric space is a direct product, unique up to permutation of terms, of irreducible spaces. We shall call this decomposition the *Cartan decomposition* of the space. Cartan [4] also classified the irreducible finite dimensional domains (see Theorem 2.4 for a description of these).

A *holomorphic vector field* on D is a differential operator $X = h\frac{\partial}{\partial x}$, where $h \in H(D, E)$, the holomorphic functions from D to E, and x denotes the variable in D. For $f \in H(D, E)$ we have $(Xf)(x) = f'(x)(h(x))$. The set of holomorphic vector fields on D forms a complex Lie algebra with the Lie bracket operation $[X, Y] = (Yh - Xk)\frac{\partial}{\partial x}$, where $Y = k\frac{\partial}{\partial x}$. We restrict ourselves to the *complete* vector fields on D, i.e., those which occur as the derivative at $t = 0$ of a one parameter subgroup

$$\phi\colon \mathbb{R} \to G(D)$$

$$t \to g_t.$$

The complete vector fields form a real Lie algebra, denoted by $g = g(D)$. By a slight abuse of notation we will consider $g \subseteq H(D, E)$ and omit the "$\frac{\partial}{\partial x}$" notation.

Several facts of fundamental importance to the present work were given in [3] and [21]. Every $X \in g(D)$ is a polynomial of degree $\leqslant 2$, and, writing

$$G(D).0 \equiv \{ g(0) \mid g \in G(D) \}$$

for the orbit of the origin by $G(D)$, there is a unique closed complex subspace F of E so that

$$D \cap F = G(D).0. \tag{1.4}$$

Moreover, to each $\xi \in F$ there corresponds a complete vector field $X_\xi \in g(D)$ given by

where Z is a certain trilinear map determined by the action of $g(D)$ on D. F is in fact characterized by this property, i.e.,

$$F = \{ X(0) \mid X \in g(D) \}. \tag{1.6}$$

The triple (E, F, Z) is called the *partial J^* triple system* associated with D. The map Z is called the *Jordan triple product* and has the following properties:

(i) Z: $F \times E \times E \to E$ is continuous, complex linear and symmetric in the latter two variables, and complex conjugate-linear in the first variable.

(ii) For all $\xi \in F$ the map

$$h\colon x \to Z(\xi, \xi, x)$$

is a hermitian operator, i.e., $\exp(ith) \in K(D)$ for all $t \in \mathbb{R}$.

(iii) Z satisfies the Jordan triple identity: for all $\xi, \eta, x \in F$ and $y, z \in E$

$$\begin{aligned} Z(\xi, \eta, Z(x, y, z)) = &-Z(Z(\eta, \xi, x), y, z) \\ &+Z(x, Z(\xi, \eta, y), z) \\ &+Z(x, y, Z(\xi, \eta, z)). \end{aligned}$$

When D is homogeneous $F = E$. In this case the pair (E, Z) is called a J^* triple system. Much work on (partial) J^* triple systems has been done by Braun et al. [3], Kaup [18,19,20], Kaup and Upmeier [21], and Vigué [31,32,34,35], to which we refer the reader for further information.

§2. Reinhardt decompositions of finite dimensional irreducible spaces

In this section we classify all Reinhardt decompositions of irreducible symmetric finite dimensional Banach spaces. In the next section we remove the irreducibility condition and apply our results to characterize those symmetric Banach spaces with a nontrivial Reinhardt decomposition. Our proof uses Lie algebraic techniques and proceeds by examining case by case the four classical domains and the two exceptional domains of Cartan's classification. For the benefit of the non-specialist we first sketch the general theory which we apply in this section. We refer to Drucker [10], Hegason [15], Humphreys [16], Loos [23], and Wolf [37] for

Let D denote the unit ball of a finite dimensional Banach space E and suppose D is an irreducible domain. By differentiating the one parameter subgroups at the origin, one obtains the Lie algebra of the Lie group. The elements of the Lie algebra are sometimes called the infinitesimal transformations of the Lie group. Let g and k denote the Lie algebras of G and K respectively. These Lie algebras are vector spaces over $\mathbb{R}$ (indeed $g \cap ig = 0$) and we let $g^{\mathbb{C}}$ and $k^{\mathbb{C}}$ denote their complexifications.

If τ denotes a maximal abelian subalgebra of k then τ is also a maximal abelian subalgebra of g and, furthermore, it is the Lie algebra of a certain subgroup of K which we denote by T. We let $\tau^{\mathbb{C}}$ denote the complexification of τ.

A *toral group* is a group which is homomorphic to t^n for some positive integer n where $t = \{z \in \mathbb{C} \mid |z| = 1\}$ has multiplication of complex numbers as its group operation. The group T is a maximal (with respect to dimension) toral subgroup of K. Any toral subgroup of K is contained in a maximal toral subgroup and any two maximal toral subgroups of K are conjugate under an element of K. A closed subgroup of K is a toral subgroup if and only if it is connected and abelian. Furthermore, a toral subgroup is maximal if and only if its Lie algebra is a maximal abelian Lie sub-algebra of k.

The symmetry at the origin in D induces a decomposition of g of the form $k + p$. The Lie algebra k contains a *central element* which induces a decomposition of $g^{\mathbb{C}}$ into $k^{\mathbb{C}} + p^+ + p^-$. The space p^+ may be identified with the original space E.

If $X \in g^{\mathbb{C}}$ then the mapping $Y \in g^{\mathbb{C}} \to [X, Y] \in g^{\mathbb{C}}$ is written $\mathrm{ad}(X)$. Let α be a linear functional on $\tau^{\mathbb{C}}$ and let g^α denote the linear subspace of $g^{\mathbb{C}}$ given by

$$g^\alpha = \{ X \in g^{\mathbb{C}} \mid \mathrm{ad}(H)(X) = \alpha(H)X \quad \text{for all } H \in \tau^{\mathbb{C}} \}.$$

If $\alpha \neq 0$ and $g^\alpha \neq 0$ then the linear functional α is called a *root* and g^α is called a *root space*. If α is a root then g^α is a one dimensional space and either $g^\alpha \subset k^{\mathbb{C}}$, $g^\alpha \subset p^+$, or $g^\alpha \subset p^-$. The roots corresponding to these cases are called the *compact*, *positive noncompact* and *negative noncompact roots*, respectively and denoted by Δ_c, Δ_p^+ and Δ_p^-, respectively. It can be shown that the mappings $\{\mathrm{ad}(H)\}_{H \in \tau^{\mathbb{C}}}$ are simultaneously diagonalizable. Hence

$$g^{\mathbb{C}} = \tau^{\mathbb{C}} + \sum_{\alpha \neq 0} g^\alpha \quad \text{and} \quad p^+ = \sum_{\alpha \in \Delta_p^+} g^\alpha.$$

A more concrete representation of k, p^+ and p^- can be given in terms

and Kaup and Upmeier [21]):

$$p^+ = \{h(0) \mid h \in g(D)\}$$

$$k = \{h'(0) \mid h \in g(D)\}$$

$$p^- = \{h''(0) \mid h \in g(D)\}$$

LEMMA 2.1: *Let* $T \in L(E, E)$. *If* $T \in \tau^{\mathbb{C}}$ *then* $\left[T\frac{\partial}{\partial z}, \xi\frac{\partial}{\partial z}\right] = T(\xi)\frac{\partial}{\partial z}$ *for all* $\xi \in E$. *If, furthermore,* T *is a projection then* $\alpha(T) = 0$ *or* 1 *for all* $\alpha \in \Delta_p^+$.

PROOF: $[T\frac{\partial}{\partial z}, \xi\frac{\partial}{\partial z}] = (T(\xi) - (\xi)'(T(z)))\frac{\partial}{\partial z} = T(\xi)\frac{\partial}{\partial z}$. If $\alpha \in \Delta_p^+$ and x_α is an eigenvector for α then $x_\alpha \in p^+ \cong E$ and

$$[T, x_\alpha] = \alpha(T)x_\alpha = T(x_\alpha).$$

Hence $\alpha(T)$ is an eigenvalue for $T \in L(E, E)$. Since T is a projection this implies $\alpha(T) = 0$ or 1. ∎

A symmetric bilinear form on $g^{\mathbb{C}}$, *the Killing form*, is defined by $(x, y) = \mathrm{tr}(\mathrm{ad}(x)\mathrm{ad}(y))$ where tr denotes *the trace* on a space of linear operators. The Killing form is nondegenerate on $\tau^{\mathbb{C}}$. The *Weyl group* of $g^{\mathbb{C}}$ is the group of linear transformations of $\tau^{\mathbb{C}}$ generated by orthogonal reflections in the hyperplanes $\alpha = 0$ where α is a root. By duality the Weyl group acts on the dual of $\tau^{\mathbb{C}}$ and thus on the roots. We have

$$S_\alpha(\gamma) = \gamma - 2\frac{(\alpha, \gamma)}{(\alpha, \alpha)}\alpha,$$

where γ is a root and S_α is the transpose of the reflection on the hyperplane $\alpha = 0$. Define Ad: $G \to GL(g^{\mathbb{C}})$ to be the usual adjoint map ($x \in G$ goes to the derivative at the identity in G of the map $w \to xwx^{-1}$ on G) extended to $g^{\mathbb{C}}$ by $\mathbb{C}$-linearity.

LEMMA 2.2: *If* $\alpha \in \Delta_c$ *then there exists* $x \in K$ *such that*

(*i*) S_α *is the restriction of* $\mathrm{Ad}(x)$ *to* $\tau^{\mathbb{C}}$.

(*ii*) $\mathrm{Ad}(x)$ *is an automorphism of* $g^{\mathbb{C}}$ *and* $\mathrm{Ad}(x)(g^\beta) = g^{S_\alpha(\beta)}$ *for any root* β.

(*iii*) *If* $\xi \in p^+$ *then* $\mathrm{Ad}(x)(\xi) = x(\xi)$.

PROOF: First choose $x_\alpha \in g^\alpha$ and $x_{-\alpha} \in g^{-\alpha}$ such that $x_\alpha - x_{-\alpha} \in k$. Let $x = \exp(t(x_\alpha - x_{-\alpha}))$ where $t \in \mathbb{R}$ and exp: $g \to G$.

The appropriate value of t and the details of the calculation necessary to prove (i) may be found in Mostow [25, p. xxix-2] or in O. Loos [24,

(ii) is standard and (iii) follows from Lemma 2.1 using

$$\mathrm{Ad}(\exp(Y)) = \mathrm{Exp}(\mathrm{ad}(Y))$$

where Exp is the ordinary exponential map for endomorphisms. ∎

PROPOSITION 2.3: *If $E = E_1 \oplus E_2 \cdots \oplus E_s$ is a Reinhardt decomposition of E and P_j: $E \to E_j$, $1 \leqslant j \leqslant s$, are the associated projections then*

(i) the subgroup of K given by the transformations

$$(z_1, \cdots, z_s) \to \left(e^{i\theta_1} z_1, \cdots, e^{i\theta_s} z_s\right)$$

is a toral subgroup T_0 of K,

(ii) there exists $g \in K$ such that $gT_0 g^{-1} \subseteq T$,

(iii) $E = gE_1 \oplus gE_2 \cdots \oplus gE_s$ is a Reinhardt decomposition with projections $gP_j g^{-1}$ and the spaces $g(E_j) = gP_j g^{-1}(E)$ and $P_j(E)$ are isometrically isomorphic for all j, $1 \leqslant j \leqslant s$.

(iv) $iP_j \in k$ for $j = 1, 2, \cdots, s$ and $i(gP_j g^{-1}) \in \tau$ for $j = 1, 2, \cdots, s$.

PROOF: (i), (ii) and (iii) are either obvious or have already been noted. (iv) Let φ: $\mathbb{R} \to G$ be defined by

$$\varphi_t(z_1, \cdots, z_s) = \left(z_1, \cdots, z_{j-1}, e^{it} z_j, z_{j+1}, \cdots, z_s\right).$$

φ is an analytic one parameter subgroup of K. On differentiating at the origin we see that $iP_j \in k$.
The remainder of (iv) follows from general observations already made. This completes the proof. ∎

From (iii) and (iv) we may assume from now on that $iP_j \in \tau$ for $1 \leqslant j \leqslant s$.

THEOREM 2.4: *The only Reinhardt decompositions of finite dimensional Banach spaces with irreducible symmetric unit balls that can occur, up to an isometry of the space and permutation of factors, are given as follows:*

(i) Type I, $\mathrm{I}_{m,n} = m \times n$ *complex matrices*, n, $m \geqslant 1$. *The possible decompositions are*

$$\mathrm{I}_{m,n} = \sum_{j=1}^{s} \oplus \mathrm{I}_{m,n_j} \quad \text{where} \quad \sum_{j=1}^{s} n_j = n \quad \text{or}$$

$$\mathrm{I}_{m,n} = \sum_{i=1}^{r} \oplus \mathrm{I}_{m_i,n} \quad \text{where} \quad \sum_{i=1}^{r} m_i = m$$

(*ii*) *Type* II, II_n = *symmetric* $n \times n$ *matrices*, $n \geqslant 2$. *No nontrivial decompositions possible.*

(*iii*) *Type* III, III_n = *skew-symmetric* $n \times n$ *matrices*, $n \geqslant 5$.

$$\mathrm{III}_n = \mathrm{III}_{n-1} \oplus l_2^{n-1}.$$

(*iv*) *Type* IV, *the spin factors* IV_n, $n \geqslant 5$. *If n is odd then no nontrivial decompositions are possible. If n is even then*

$$\mathrm{IV}_n = l_2^{n/2} \oplus l_2^{n/2}.$$

(*v*) *The* 16 *dimensional exceptional space of* 1×2 *matrices over the complex Cayley numbers can be decomposed as a sum of two eight dimensional factors each of which is isomorphic to* IV_8.

(*vi*) *The* 27 *dimensional exceptional space of* 3×3 *"symmetric" matrices over the Cayley numbers,* $H(\mathbb{O})_3$ *or* M_3^8, *admits no decomposition.*

NOTATION: We let $\mathrm{I}_n = \mathrm{I}_{n,n}$. E_{ij} will denote the $m \times n$ matrix with 1 in the ij^{th} position and zero elsewhere. If A is an $m \times n$ matrix ${}^tA$ will denote the transpose of A, $\overline{A}$ the complex conjugate of A and we let $A^* = {}^t(\overline{A})$. Each of the above matrix spaces is considered as a subspace of $L(l_2^n, l_2^m)$; hence, if z is an $m \times n$ matrix then

$$\| z \|^2 = \sup\{ v \mid v \quad \text{is an eigenvalue of } z^*z \}.$$

PROOF: Let $E = E_1 \oplus E_2 \cdots \oplus E_3$ be a Reinhardt decomposition of E, where E is one of the spaces in cases (i)–(vi). Fix k, $1 \leqslant k \leqslant s$, and let $P = P_k$ be the projection onto E_k.

The representations for $g^{\mathbb{C}}$, $k^{\mathbb{C}}$, etc. in cases (i)–(iv) may be found in Drucker [10] or Wolf [37].

CASE (i): $g^{\mathbb{C}} = \{ z \in \mathrm{I}_{m+n} \mid \mathrm{tr}(z) = 0 \}$. Each $z \in g^{\mathbb{C}}$ may be written

$$\begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} \tag{2.1}$$

where $z_1 \in \mathrm{I}_m$, $z_2 \in \mathrm{I}_{m,n}$, $z_3 \in \mathrm{I}_{n,m}$, $z_4 \in \mathrm{I}_n$, and $\mathrm{tr}(z_1) + \mathrm{tr}(z_4) = 0$. We also have

$$k^{\mathbb{C}} = \{ z \in g^{\mathbb{C}} \mid z_2 = 0 \quad \text{and } z_3 = 0 \},$$

$$\tau^{\mathbb{C}} = \{ z \in k^{\mathbb{C}} \mid z_1 \quad \text{and } z_4 \quad \text{are diagonal matrices} \},$$

$$p^+ + p^- = \{ z \in g^{\mathbb{C}} \mid z_1 = 0 \quad \text{and } z_4 = 0 \}, \quad \text{and}$$

We identify $E_{ij} \in I_{m,n}$ with the element

$$\begin{bmatrix} 0 & E_{ij} \\ 0 & 0 \end{bmatrix}$$

in p^+ and note that each such matrix spans a root space g^α for some $\alpha \in \Delta_p^+$. Suppose P is identified with the diagonal matrix $\operatorname{diag}(\alpha_1, \cdots, \alpha_m, \beta_1, \cdots, \beta_n)$ in $\tau^{\mathbb{C}}$. Then $[P, E_{ij}] = (\alpha_i - \beta_j)E_{ij}$ for all i and j, so by Lemma 2.1 $\alpha_i - \beta_j = 0$ or 1 for all i and j. We consider the case $\alpha_1 - \beta_1 = 1$ (the case $\alpha_1 - \beta_1 = 0$ follows by applying the succeeding argument to $\mathrm{id} - P$). Since $\alpha_1 - \beta_j = \alpha_1 - \beta_1 + \beta_1 - \beta_j = 0$ or 1, it follows that $\beta_1 - \beta_j = 0$ or -1 for all j. Similarly $\alpha_i - \alpha_1 = 0$ or -1 for all i. If $\beta_1 - \beta_j = \alpha_i - \alpha_1 = -1$ for some i and j, then $\alpha_1 - \beta_j = 0$ and so $\alpha_i - \beta_j = \alpha_i - \alpha_1 + \alpha_1 - \beta_j = -1$. This is impossible, so we have either $\beta_j = \beta_1$ for all j or $\alpha_i = \alpha_1$ for all i. In the first case, $[P, E_{ij}] = (\alpha_i - \beta_1)E_{ij}$ for all i and j, i.e., P is the canonical projection onto those rows i for which $\alpha_i - \beta_1 = 1$. In the second case P is the canonical projection onto those columns j for which $\alpha_1 - \beta_j = 1$. Both occur simultaneously only when $P = 0$ or id.

Because the sum of two such projections $P_k + P_{k'}$, $1 \leqslant k,\ k' \leqslant s$, $k \neq k'$, has the same properties and because the decomposition of $I_{m,n}$ into a sum of row spaces or into a sum of column spaces is indeed a Reinhardt decomposition, we have completed the proof of (i).

CASE (ii): Elements z in $g^{\mathbb{C}}$ have the form (2.1) where $z_1,\ z_2,\ z_3 \in I_n$, $z_2 = {}^t z_2$, $z_3 = {}^t z_3$, and $z_4 = -{}^t z_1$. We also have

$$k^{\mathbb{C}} = \{ z \in g^{\mathbb{C}} \mid z_2 = 0 \quad \text{and } z_3 = 0 \},$$

$$\tau^{\mathbb{C}} = \{ z \in k^{\mathbb{C}} \mid z_1 \quad \text{is diagonal} \},$$

$$p^+ + p^- = \{ z \in g^{\mathbb{C}} \mid z_1 = 0 \}, \quad \text{and}$$

$$p^+ = \{ z \in p^+ + p^- \mid z_3 = 0 \}.$$

Suppose P is identified with $\operatorname{diag}(\alpha_1, \cdots, \alpha_n, -\alpha_1, \cdots, -\alpha_n)$ in $\tau^{\mathbb{C}}$. Since $[P, E_{ij} + E_{ji}] = (\alpha_i + \alpha_j)(E_{ij} + E_{ji})$, Lemma 2.1 implies $\alpha_i + \alpha_j = 0$ or 1 for all i and j. In particular, $\alpha_i = 0$ or $1/2$ for all i. If $\alpha_i = 1/2$ for some i, then $\alpha_j = 1/2$ for all j and P is the identity. If $\alpha_i = 0$ for some i, then $\alpha_j = 0$ for all j and P is zero. This completes the proof of (ii).

CASE (iii): $g^{\mathbb{C}}$, $k^{\mathbb{C}}$, etc. are as in (ii) except that $z_2 = -{}^t z_2$ and $z_3 = -{}^t z_3$. We have $[P, E_{ij} - E_{ji}] = (\alpha_i + \alpha_j)(E_{ij} - E_{ji})$ where $1 \leqslant i < j \leqslant n$. If $\alpha_i =$

identity or the zero projection. Observe that since $\alpha_i + \alpha_j = 0$ or 1 for all $i < j$, there can be at most two distinct values among $\alpha_1, \cdots, \alpha_n$.

If $\alpha_i + \alpha_j = 0$ for some i and j and $\alpha_i < 0$, then $\alpha_k = -\alpha_i$ for all $k \neq i$. Hence, for $k \neq i$ or j, $\alpha_j + \alpha_k = -2\alpha_i = 1$ and so

$$\alpha_i = -1/2 \quad \text{and } \alpha_k = 1/2 \quad \text{for all } k \neq i. \tag{2.2}$$

If $\alpha_i + \alpha_j = 1$ for some i and j and $\alpha_i < \alpha_j$, then $\alpha_i = 0$ and $\alpha_j = 1$ since for all $k \neq i$ or j, $\alpha_k = \alpha_i$ or α_j. Hence

$$\alpha_j = 1 \quad \text{and } \alpha_k = 0 \quad \text{for all } k \neq j. \tag{2.3}$$

Observe that if P' is another, distinct, projection associated with the Reinhardt decomposition satisfying either (2.2) for some i' or (2.3) for some j', then $P + P'$ cannot satisfy (2.2) or (2.3) for any i or j since $n \geqslant 5$. Hence $P + P' = \mathrm{id}$. Consequently, $s \leqslant 2$, i.e., E may be decomposed into at most two factors. If $E = E_1 \oplus E_2$ with associated projections P_1 and P_2 then (2.2) and (2.3) imply that, up to a permutation of coordinates,

$$P_1(z) = z_1$$

and

$$P_2(z) = \begin{bmatrix} 0_{n-1} & x \\ -{}^t x & 0_1 \end{bmatrix}$$

where

$$z = \begin{bmatrix} z_1 & x \\ -{}^t x & 0_1 \end{bmatrix},$$

z_1 is a skew-symmetric $(n-1) \times (n-1)$ matrix, and x is an $(n-1) \times 1$ matrix. Since

$$\begin{bmatrix} \mathrm{id}_{n-1} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \begin{bmatrix} z_1 & x \\ -{}^t x & 0 \end{bmatrix} \begin{bmatrix} \mathrm{id}_{n-1} & 0 \\ 0 & e^{i\theta} \end{bmatrix} = \begin{bmatrix} z_1 & e^{i\theta} x \\ -{}^t(e^{i\theta} x) & 0 \end{bmatrix}$$

it follows that $\| P_1(z) + e^{i\theta} P_2(z) \| = \| z \|$, and consequently $P_1(E) \oplus P_2(E)$ is a Reinhardt decomposition of E. Clearly $P_1(E) = \mathbb{H}_{n-1}$ and a

CASE (iv): Elements $z \in g^{\mathbb{C}}$ have the form (2.1) where $z_1 \in \mathrm{I}_n$, $z_1 = -{}^t z_1$, $z_2 \in \mathrm{I}_{n,2}$, $z_3 = -{}^t z_2$, and $z_4 \in \mathrm{I}_2$ with $z_4 = -{}^t z_4$. We also have

$$k^{\mathbb{C}} = \{ z \in g^{\mathbb{C}} \mid z_2 = 0 \quad \text{and } z_3 = 0 \},$$

$$p^+ + p^- = \{ z \in g^{\mathbb{C}} \mid z_1 = 0 \quad \text{and } z_4 = 0 \}, \quad \text{and}$$

$$p^+ = \{ z \in p^+ + p^- \mid z_2 = [w, iw], \quad \text{where } w \in I_{n,1} \}, \tag{2.4}$$

where $[w, iw]$ is the $n \times 2$ matrix whose columns are w and iw. We first consider the case when n is even. Then

$$\tau^{\mathbb{C}} = \{ \operatorname{diag}(A_1, \cdots, A_{n/2}, B) \},$$

where

$$A_j = \begin{bmatrix} 0 & \alpha_j \\ -\alpha_j & 0 \end{bmatrix}, \quad 1 \leqslant j \leqslant n/2,$$

and

$$B = \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}$$

for some constants $\alpha_1, \cdots, \alpha_{n/2}, \beta$. Let $e_j \in \mathrm{I}_{n,1}$ have a 1 in the j^{th} place and zeros elsewhere. Define f_j (resp. g_j) to be the element of p^+ obtained by setting $w = w_j = \mathrm{i}e_{2j-1} + e_{2j}$ (resp. $v_j = \mathrm{i}e_{2j-1} - e_{2j}$) for $1 \leqslant j \leqslant n/2$. Identifying P with a matrix in $\tau^{\mathbb{C}}$ we see that

$$[P, f_j] = -\mathrm{i}(\beta + \alpha_j) f_j \quad \text{and}$$

$$[P, g_j] = -\mathrm{i}(\beta - \alpha_j) g_j.$$

So $-\mathrm{i}(\beta \pm \alpha_j) = 0$ or 1 for all $1 \leqslant j \leqslant n/2$ by Lemma 2.1. If $\beta \pm \alpha_j = 0$ for all j (resp. $= i$ for all j) then P is zero (resp. the identity). Otherwise $\beta = \mathrm{i}/2$ and $\alpha_j = \mathrm{i}/2$ or $-\mathrm{i}/2$. Hence $P(E)$ is spanned by $(x_j)_{j=1}^{n/2}$, where each x_j is either f_j or g_j.

It now follows that $\operatorname{rank}(P) = n/2$ and that $s \leqslant 2$. We now check that $s = 2$ is possible.

Using (2.4) it can be shown (see for instance [13, p. 20]) that for $z = (z_1, \cdots, z_n) \in \mathrm{IV}_n$

$$\| z \|^2 = \sum_{}^{n} |z_j|^2 + \left(\left(\sum_{}^{n} |z_j|^2 \right)^2 - \left| \sum_{}^{n} z_j^2 \right|^2 \right)^{1/2}. \tag{2.5}$$

Now,

$$z = \sum_{j=1}^{n/2} \big(a_j(z)w_j + b_j(z)v_j\big),$$

where

$$a_j(z) = \frac{1}{2\mathrm{i}}\big(z_{2j-1} + iz_{2j}\big)$$

and

$$b_j(z) = \frac{1}{2\mathrm{i}}\big(z_{2j-1} - iz_{2j}\big).$$

Consequently, the element of p^+ obtained from z is $\sum_{j=1}^{n/2}(a_j(z)f_j + b_j(z)g_j)$. Define

$$h_j(z) = \begin{cases} a_j(z)w_j & \text{if } x_j = f_j \\ b_j(z)v_j & \text{if } x_j = g_j \end{cases}$$

$$k_j(z) = \begin{cases} a_j(z)w_j & \text{if } x_j = g_j \\ b_j(z)v_j & \text{if } x_j = f_j \end{cases}$$

for $1 \leqslant j \leqslant n/2$. Then $P(z) = \sum_{j=1}^{n/2} h_j(z)$ and $(\mathrm{id} - P)(z) = \sum_{j=1}^{n/2} k_j(z)$. Using (2.5),

$$\|z\|^2 = \left\| \sum_{j=1}^{n/2} \big(a_j(z)w_j + b_j(z)v_j\big) \right\|^2$$

$$= 2\sum_j |a_j(z)|^2 + 2\sum_j |b_j(z)|^2$$

$$+ 2\left(\left(\sum_j |a_j(z)|^2 + \sum_j |b_j(z)|^2 \right)^2 - 4\left| \sum_j a_j(z)b_j(z) \right|^2 \right)^{1/2}.$$

Consequently $\|\lambda_1 P(z) + \lambda_2(\mathrm{id} - P)(z)\|^2 = \|z\|^2$ for all $|\lambda_1| = |\lambda_2| = 1$, and so the decomposition $E = P(E) \oplus (\mathrm{id} - P)(E)$ is Reinhardt

each j and hence $P(E) \cong l_2^{n/2}$. Similarly $(\mathrm{id} - P(E) \cong l_2^{n/2}$. Thus

$$\mathrm{IV}_n \cong l_2^{n/2} \oplus l_2^{n/2},$$

and this completes the proof for the case when n is even.

We next consider the case when n is odd. Then

$$\tau^{\mathbb{C}} = \{\mathrm{diag}(A_1, \cdots, A_{[n/2]}, 0, B)\},$$

where A_j, $1 \leqslant j \leqslant [n/2]$ and B are as in the even case. Using the notation from the even case we find

$$[P, f_j] = -\mathrm{i}(\beta + \alpha_j) f_j,$$

$$[P, g_j] = -\mathrm{i}(\beta - \alpha_j) g_j, \quad \text{and}$$

$$[P, h] = -\mathrm{i}\beta h,$$

where h is the element in p^+ obtained from $w = ie_n$. By Lemma 2.1 we have $-\mathrm{i}\beta = 0$ or 1 and $(\beta \pm \alpha_j) = 0$ or 1 for all $1 \leqslant j \leqslant [n/2]$. This implies $\alpha_j = 0$ for all j and consequently P is either zero or the identity.

This completes the proof for the four classical domains.

CASE (v): We first recall some properties of the complex Cayley numbers $\mathbb{O}_{\mathbb{C}}$ (we refer to Drucker [10, p. 20–21] and Loos [23, p. 4.17] for details). Let $\mathrm{e}_0, \cdots, \mathrm{e}_7$ be unit vectors and $a = \sum_{j=0}^{7} a_j e_j \in \mathbb{O}_{\mathbb{C}}$. Define

$$a^* = \bar{a}_0 \mathrm{e}_0 - \sum_{j=1}^{7} \bar{a}_j e_j,$$

$$\tilde{a} = a_0 \mathrm{e}_0 - \sum_{j=1}^{7} a_j e_j$$

$$t(a) = a + \tilde{a} = 2a_0 \ (t \text{ is the trace function}),$$

$$n(a) = a\tilde{a} = \sum_{j=0}^{7} a_j^2 \quad (n \text{ is called a norm}).$$

It is easily checked that $t(aa^*) = 2 \sum_{j=0}^{7} |a_j|^2$. The 16 dimensional excep-

endowed with a norm such that the open unit ball B consists of all (a, b) such that

$$2 - t(aa^* + bb^*) > 0$$

and

$$1 - t(aa^* + bb^*) + |n(a)|^2 + |n(b)|^2 + t((\tilde{a}b)(b^*\tilde{a})) > 0$$

By noting that

$$(\lambda a)b = \lambda(ab) = a(\lambda b)$$

for any a, b in $\mathbb{O}_{\mathbb{C}}$ and $\lambda \in \mathbb{C}$ we see that $(a, b) \in B$ implies $(e^{i\theta}a, e^{i\psi}b) \in B$ for any real numbers θ and ψ. Hence V admits a nontrivial Reinhardt decomposition.

If $a = \sum_{j=0}^{7} a_j e_j$ then $(a, 0) \in B$ if and only if

$$2 - t(aa^*) > 0 \quad \text{and} \quad 1 - t(aa^*) + |n(a)|^2 > 0,$$

i.e., if and only if $\sum_{j=0}^{7} |a_j|^2 < 1$ and

$$1 - 2\sum_{j=0}^{7} |a_j|^2 + \left|\sum_{j=0}^{7} a_j^2\right| > 0.$$

Hence $\sum_{j=0}^{7} a_j e_j \in B$ if and only if $(a_0, \cdots, a_7)$ belongs to the unit ball of IV_8 (see [24, section 4.16], [37, p. 350]) and consequently

$$\mathrm{V} = \mathrm{IV}_8 \oplus \mathrm{IV}_8.$$

We now show that this is the only nontrivial decomposition of V. To obtain this result we use some further results from the general theory of Lie algebras.

It is possible to choose a linearly independent set of roots $\alpha_1, \cdots, \alpha_p$ of $g^{\mathbb{C}}$ such that every root α can be expressed as an integral linear combination

$$\alpha = \sum_{j=1}^{p} n_j \alpha_j \tag{2.6}$$

where n_j are either all nonnegative or all nonpositive. Once chosen, such

simple root, denoted by α_p, is noncompact and the noncompact positive roots can be described as those α with coefficient $n_p = 1$ in (2.6). In the present case there are six simple roots and the coefficients of the 16 noncompact positive roots, $(\beta_i)_{i=1}^{16}$, are given in the following table (taken from Drucker [10, pp. 152–154]).

Coefficients of α_i						
n_6	n_5	n_4	n_3	n_2	n_1	Δ_p^+
1	0	0	0	0	0	β_1
1	1	0	0	0	0	β_2
1	1	1	0	0	0	β_3
1	1	1	0	0	1	β_4
1	1	1	1	0	0	β_5
1	1	1	1	0	1	β_6
1	1	1	1	1	0	β_7
1	1	1	1	1	1	β_8
1	1	2	1	0	1	β_9
1	1	2	1	1	1	β_{10}
1	1	2	2	1	1	β_{11}
1	2	2	1	0	1	β_{12}
1	2	2	1	1	1	β_{13}
1	2	2	2	1	1	β_{14}
1	2	3	2	1	1	β_{15}
1	2	3	2	1	2	β_{16}

We must now solve $\beta_j(P) = 0$ or 1 for all j. We may assume without loss of generality (if necessary by replacing P by $\mathrm{id} - P$) that $\beta_1(P) = 1$. From now on we let $a_i = \alpha_i(P)$ and $b_i = \beta_i(P)$ for all i. Since a_i occurs as a difference $b_{j+1} - b_j$ for some j it follows that $a_i = 0$, 1 or -1. By considering the differences between consecutive terms in the sequences

$$b_1,\ b_2,\ b_3,\ b_5,\ b_7,\ b_8$$
$$b_1,\ b_2,\ b_3,\ b_4,\ b_6,\ b_8$$
$$b_1,\ b_2,\ b_3,\ b_5,\ b_6,\ b_8$$

we find that the nonzero terms of $\{a_5, a_4, a_3, a_2, a_1\}$, $\{a_5, a_4, a_1, a_3, a_2\}$ and $\{a_5, a_4, a_3, a_1, a_2\}$ must alternate in sign and the first nonzero term in each sequence must be -1. Hence if $a_1 \neq 0$ then a_2 and a_3 must both be zero. Also, since

$$b_{14} - b_{10} = a_5 + a_3$$
$$b_{13} - b_{11} = a_5 - a_3$$
$$b_{13} - b_9 = a_5 + a_2$$

we find that if $a_5 \neq 0$ then $a_3 = 0$ and $a_2 = 0$. By using the above relationships and by a case-by-case examination of the resulting possibilities we are led to the following nontrivial values of $(a_i)_{i=1}^6$.

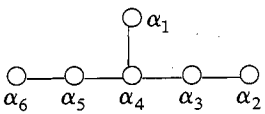
a_6	a_5	a_4	a_3	a_2	a_1
1	0	0	0	−1	0
1	0	0	−1	1	0
1	0	−1	1	0	0
1	−1	1	0	0	−1
1	−1	0	0	0	0

We call the corresponding projections φ_1, φ_2, φ_3, φ_4, φ_5. The corresponding values of $(b_i)_{i=1}^{16}$ are

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}
1	1	1	1	1	1	0	0	1	0	0	1	0	0	0	0
1	1	1	1	0	0	1	1	0	1	0	0	1	0	0	0
1	1	0	0	1	1	1	1	0	0	1	0	0	1	0	0
1	0	1	0	1	0	1	0	1	1	1	0	0	0	1	0
1	0	0	1	0	1	0	1	1	1	1	0	0	0	0	1

This implies in particular that any nontrivial Reinhardt decomposition of V has the form of $E \oplus F$ while $\dim(E) = 8$ and $\det(F) = 8$. We now show, using Lemma 2.2, that all of the above five possible decompositions are isomorphic.

To each complex semisimple Lie algebra is associated a *Dynkin diagram* which completely determines the Lie algebra. The Lie algebra associated with V is E_6 and this has the following Dynkin diagram of simple roots



where each node represents a simple root, nodes which are not joined represent orthogonal roots, $\langle \alpha_i, \alpha_i \rangle = 2$ for all i and $\langle \alpha_i, \alpha_j \rangle = -1$ if i and j are joined, where

$$\langle \alpha_i, \alpha_j \rangle = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \quad \text{for all } i \text{ and } j.$$

A simple calculation shows that S_{α_2} interchanges the pairs (β_5, β_7), (β_6, β_9), (β_9, β_{10}) (β_{12}, β_{13}) and fixes all other β in Δ^+. Hence

Consequently $\varphi_1 = S_{\alpha_2}(\varphi_2)$. Similarly,

$$\varphi_2 = S_{\alpha_3}(\varphi_3), \quad \varphi_3 = S_{\alpha_4}(\varphi_4) \quad \text{and } \varphi_4 = S_{\alpha_1}(\varphi_5).$$

By Lemma 2.2 we conclude that there exists only one nontrivial Reinhardt decomposition of V and this is $\mathrm{IV}_8 \oplus \mathrm{IV}_8$.

CASE (vi): We apply the same methods in this case as we applied in the previous case. There are seven simple roots and we are interested in roots of the form $\alpha_7 + \sum_{j=1}^{6} n_j\alpha_j$. Using the notation of the previous case for β_j, $1 \leqslant j \leqslant 16$, it can be shown (see for instance Drucker [10]) that $\alpha_7 + \beta_j$ is a root in this case, $1 \leqslant j \leqslant 16$. Hence all the considerations of the previous case apply if we consider a projection P with $\alpha_7(P) + \alpha_6(P) = 1$ and replace α_6 by $\alpha_6 + \alpha_7$. Hence we need only examine the following 12 possible values of $a_j = \alpha_j(P)$.

a_7	a_6	a_5	a_4	a_3	a_2	a_1
1	0	0	0	0	0	0
0	1	0	0	0	0	0
0	1	0	0	0	−1	0
0	1	0	0	0	−1	0
1	0	0	0	−1	1	0
1	0	0	−1	1	0	0
0	1	0	−1	1	0	0
1	0	−1	1	0	0	−1
0	1	−1	1	0	0	−1
1	0	−1	0	0	0	0
0	1	−1	0	0	0	0

The coefficients of the 11 positive noncompact roots not of the form $\alpha_7 + \beta_j$ are given in the following table (taken from Drucker [10, p. 152–154]).

n_7	n_6	n_5	n_4	n_3	n_2	n_1	
1	0	0	0	0	0	0	γ_1
1	2	2	2	1	0	1	γ_2
1	2	2	2	1	1	1	γ_3
1	2	2	2	2	1	1	γ_4
1	2	2	3	2	1	1	γ_5
1	2	2	3	2	1	2	γ_6
1	2	3	3	2	1	1	γ_7
1	2	3	3	2	1	2	γ_8
1	2	3	4	2	1	2	γ_9

By inspection it now follows that the only one of the 12 possible decompositions which is admissible is the first one and this gives the identity mapping (e.g., if the second possibility listed was admissible then it would follow that $\gamma_{11}(P) = 2$ and this is impossible). Hence $H(\mathbb{O}_3)$ admits no nontrivial Reinhardt decomposition. This completes the proof of Theorem 2.4. ∎

§3. Reinhardt decompositions of infinite dimensional symmetric spaces

If $E_1 \oplus \cdots \oplus E_s$ is a Reinhardt decomposition of a Banach space E then we shall call each E_i a factor of E. In Theorem 2.4 we have listed all factors of the irreducible finite dimensional spaces. By inspection of Theorem 2.4 we obtain the following result.

PROPOSITION 3.1: *Each factor of an irreducible finite dimensional space is irreducible.*

LEMMA 3.2: *Let $E_0 = E_1 \times E_2 \cdots \times E_s$ denote a product of symmetric finite dimensional spaces. For $0 \leqslant i \leqslant s$ let K_i, I_i, and T_i denote respectively the group of all linear isometries of E_i, the connected component of the identity in K_i, and a maximal torus in I_i (or K_i). Then $I_0 = I_1 \times I_2 \cdots \times I_s$, and $T_1 \times T_2 \cdots \times T_s$ is a maximal torus in K_0. Moreover, any maximal torus in K_0 has the form $\tilde{T}_1 \times \tilde{T}_2 \cdots \times \tilde{T}_s$ where $\tilde{T}_i$ is a maximal torus in I_i, $1 \leqslant i \leqslant s$.*

PROOF: It is easy to see that the Lie algebra of $I_1 \times I_2 \cdots \times I_s$ is a Lie direct sum $k_1 \oplus k_2 \cdots \oplus k_s$ and that any maximal abelian subalgebra of $k_1 \oplus k_2 \cdots \oplus k_s$ must have the form $\tau_1 \oplus \tau_2 \cdots \oplus \tau_s$, where τ_j is maximal abelian in k_j, $1 \leqslant j \leqslant s$. Hence $T_1 \times T_2 \cdots \times T_s$ is a maximal torus in I and every maximal torus of I_0 (or K_0) has this product form. ∎

Our next proposition enables us to treat the case of Reinhardt decompositions with reducible factors and is also used in proving Theorem 3.5.

PROPOSITION 3.3: *If $E_1 \oplus \cdots \oplus E_n$ and $\prod_{k=1}^{m} F_k$ are respectively a Reinhardt and Cartan decomposition of the finite dimensional symmetric space E then the following are true.*

(i) $\sum_{j,k} \oplus (E_j \cap F_k)$ is a Reinhardt decomposition of E and $E_j \cap F_k$ is either $\{0\}$ or irreducible for all j and k.

(ii) For each k, $1 \leqslant k \leqslant m$, $\sum_{j=1}^{n} \oplus (E_j \cap F_k)$ is a Reinhardt decomposi-

(iii) For each j, $1 \leqslant j \leqslant n$, $\prod_{k=1}^{m} (E_j \cap F_k)$ *is a Cartan decomposition of* E_j.

(iv) If each E_j *is irreducible then* $m \leqslant n$ *and for each* j *there exists* $k(j)$ *such that* $E_j \subset F_{k(j)}$.

PROOF: We first note that a result of Stachó [27, Corollary 2.4], due independently to Kaup [19], shows that each E_i is a symmetric space.

For $1 \leqslant j \leqslant n$ let P_j denote the canonical projection from E onto E_j and for each k, $1 \leqslant k \leqslant m$, let Q_k denote the canonical projection from E onto F_k. Let T be the toral subgroup of isometries of E generated by the mappings

$$\prod_{k=1}^{m} F_k \to \prod_{k=1}^{m} F_k$$

$$\sum_{k=1}^{m} x_k \to \sum_{k=1}^{m} \lambda_k x_k$$

where $|\lambda_k| = 1$ for all $1 \leqslant k \leqslant m$. Let $\tilde{T}$ be the toral subgroup generated by the isometries

$$\sum_{j=1}^{n} \oplus E_j \to \sum_{j=1}^{n} \oplus E_j$$

$$\sum_{j=1}^{n} x_j \to \sum_{j=1}^{n} \lambda_j x_j$$

where $|\lambda_j| = 1$ for all $1 \leqslant j \leqslant n$.

Now T, being central in K_0 is contained in every maximal toral subgroup, T_0 of the group of all isometries of E (see Lemma 3.2). We may choose T_0 so that $\tilde{T} \subset T_0$. Let τ be the Lie algebra of T_0. Then iP_j, $1 \leqslant j \leqslant n$, and iQ_k, $1 \leqslant k \leqslant m$ all belong to τ and hence $P_1, \cdots, P_n$, $Q_1, \cdots, Q_m$ all commute. Hence $P_j Q_k$ is a projection from E onto $E_j \cap F_k$ for all j and k. Letting $Q_k|_{F_k} = \mathrm{id}_{F_k} = \sum_{j=1}^{n} P_j Q_k|_{F_k}$ we see $\sum_{j=1}^{n} \oplus (E_j \cap F_k)$ is a Reinhardt decomposition of F_k. This proves (ii).

Since F_k is irreducible Proposition 3.1 implies that $E_j \cap F_k$ is irreducible or zero for all j and k. If $x \in E$ let $x(j, k) = P_j Q_k(x)$ for all j and k. If $|\lambda(j, k)| = 1$ for all j and k then

$$\left\| \sum_{\cdot} \sum_{\cdot}^{m} \sum_{}^{n} \lambda(j, k) x(j, k) \right\| = \max_{k} \left\| \sum_{}^{n} \lambda(j, k) x(j, k) \right\|$$

since $\prod_{k=1}^{m} F_k$ is a Cartan decomposition, and so by (ii)

$$\left\| \sum_{j,k} \lambda(j, k) x(j, k) \right\| = \max_k \left\| \sum_{j=1}^{n} x(j, k) \right\| = \|x\|.$$

Hence we have proved (i) and (iii).

If E_j is irreducible then (iii) implies that $E_j \cap F_k = \{0\}$ for all except possibly one k, say $k(j)$. Then $E_j = E_j \cap F_{k(j)}$ and $E_j \subseteq F_{k(j)}$. Since $E = \sum_{j=1}^{n} \oplus E_j$, $k(j)$ must take on all values between 1 and m and hence $m \leqslant n$. This completes the proof. ∎

COROLLARY 3.4: *Let $\{E_j\}_{j=1}^{\infty}$ be a finite-dimensional Reinhardt decomposition of the symmetric Banach space E, and let $E_j = \prod_{k=k_j+1}^{k_{j+1}} F_k$, $0 = k_1 < k_2 < \cdots$, be the Cartan decomposition of E_j, for all j. Then $\{F_k\}_{k=1}^{\infty}$ is a Reinhardt decomposition of E into irreducible factors.*

PROOF: Since $\{\sum_j x_j \in E \mid x_j \in E_j$ is zero for all but finitely many $j\}$ is dense in E, it suffices to prove the result for finite dimensional E. Let $\prod_{k=1}^{m} \tilde{F}_k$ be the Cartan decomposition of $E = E_1 \oplus \cdots \oplus E_n$. By (i) and (iii) of Proposition 3.3, $\prod_{k=1}^{m} (E_j \cap \tilde{F}_k)$ is the Cartan decomposition of E_j and $\{E_j \cap \tilde{F}_k\}_{j,k}$ is a Reinhardt decomposition of E. By the unicity of the Cartan decomposition the nonzero members of $\{E_j \cap \tilde{F}_k\}_{k=1}^{m}$ and $\{F_k\}_{k=k_j+1}^{k_{j+1}}$ agree up to a permutation. ∎

THEOREM 3.5: *Let $\{E_j\}_{j=1}^{\infty}$ be a finite dimensional Reinhardt decomposition of a symmetric Banach space E. Then E is isometrically isomorphic to $(\sum_p \oplus F_p)_{c_0}$, where each F_p is an irreducible symmetric space and is either finite dimensional or isometric to $L(H_1, H_2)$ where H_1 and H_2 are separable Hilbert spaces, one of which is finite dimensional.*

PROOF: By Vigué [34, Proposition 4.4], $L(H_1, H_2)$ is irreducible. By Corollary 3.4 we may suppose that each E_j is an irreducible finite dimensional space. We define an equivalence relationship on the positive integers by $i \sim j$ if either $i = j$ or if $E_i \oplus E_j$ is irreducible. We now show that " $\sim$ " is transitive. Suppose i_1, i_2 and i_3 are distinct positive integers and that $i_1 \sim i_2$ and $i_2 \sim i_3$. Let $\prod_m F_k$ be the Cartan decomposition of

(iv) implies that $m \leqslant 2$. Again by Proposition 3.3 (iv) $E_{i_1} \oplus E_{i_2}$ is contained in either F_1 or F_2 and $E_{i_2} \oplus E_{i_3}$ in either F_1 or F_2. Since $(E_{i_1} \oplus E_{i_2}) \cap (E_{i_2} \oplus E_{i_3}) = E_{i_2} \neq 0$ and $F_1 \cap F_2 = 0$ there exists an i such that $E_{i_1} \oplus E_{i_2} \oplus E_{i_3} \subseteq F_i$. Hence F is irreducible. By Theorem 2.4 this implies that there exist positive integers n_0, n_1, n_2 and n_3 such that either $E_{i_j} = I_{n_0,n_j}$ for $j = 1, 2, 3$ or $E_{i_j} = I_{n_j,n_0}$ for $j = 1, 2, 3$. In both cases $E_{i_1} \oplus E_{i_3}$ is irreducible and hence " $\sim$ " is an equivalence relation. Let $\mathscr{P}$ denote the set of equivalence classes. If $p \in \mathscr{P}$ and $|p| = 2$ then $\sum_{i \in p} \oplus E_i$ is an irreducible finite dimensional space. If $p \in \mathscr{P}$ and $|p| > 2$ then by the above there exists a positive integer m and a sequence (finite or infinite) of positive integers $(n_i)_i$ such that either

$$E_i = I_{m,n_i} \quad \text{for all } i \in p$$

or

$$E_i = I_{n_i,m} \quad \text{for all } i \in p.$$

If $|p|$ is finite this implies $\sum_{i \in p} \oplus E_i$ is either $I_{m,\sum_i n_i}$ or $I_{\sum_i n_i,m}$.

In both cases we obtain a finite dimensional irreducible domain. Now suppose $|p| = \infty$ and $E_i = I_{m,n_i} = L(\mathbb{C}^m; \mathbb{C}^{n_i})$ for all $i \in p$ where $\mathbb{C}^m$ and $\mathbb{C}^{n_i}$ are Hilbert spaces. Let H be a separable Hilbert space. Using a fixed basis of H we can define a sequence of linear isometries $(\varphi_i)_{i \in p}$, $\varphi_i\colon \mathbb{C}^{n_i} \to H$ such that $\varphi_i(\mathbb{C}^{n_i}) \perp \varphi_j(\mathbb{C}^{n_j})$ for all $i \neq j$ and $\bigcup_{i=1}^{\infty} \varphi_i(\mathbb{C}^{n_i})$ spans a dense subspace of H. Since $\sum_{i \in p'} \oplus E_i = L(\mathbb{C}^m; \mathbb{C}^{n(p')})$, where $n(p') = \sum_{i \in p'} n_i$, for any finite subset p' of p it follows that the mapping $\varphi\colon (T_i)_{i \in p} \in \sum_{i \in p} \oplus L(\mathbb{C}^m; \mathbb{C}^{n_i}) \to \sum_{i \in p} \varphi_i T_i$ is an isometry from $\sum_{i \in p} \oplus E_i$ onto $L(\mathbb{C}^m; H)$. The case where $E_i = I_{n_i,m}$ for all $i \in p$ is handled similarly.

For each $p \in \mathscr{P}$ let $F_p = \sum_{i \in p} \oplus E_i$. We have just shown that each F_p is irreducible and has the required form. By our construction $\sum_{p \in \mathscr{P}} \oplus F_p$ is a Reinhardt decomposition of E.

For n a positive integer and $p \in \mathscr{P}$ let $F_{p,n} = \sum_{\substack{j \in p \\ j \leqslant n}} \oplus E_j$. $F_{p,n}$ is an irreducible finite dimensional space and $\bigcup_n (\sum_p \oplus F_{p,n})$ is a dense sub-

space of E. Let $\sum_{p \in \mathscr{P}} \oplus F_p = F_{p_1} \oplus F_{p_2} \oplus \cdots$ and suppose $\prod_{i=1}^{m} G_i$

is a Cartan decomposition of $\sum_p \oplus F_{p,n}$. Since each $F_{p_i,n}$ is irreducible Proposition 3.3 (iv) implies that for each i, $1 \leqslant i \leqslant s$, there exists a $j(i)$ such that $F_{p_i,n} \subseteq G_{j(i)}$. If $j(i) = j(k)$ then Proposition 3.3 (ii) implies that $E_l \oplus E_{l'}$ is a factor of $G_{j(i)}$ for any $l \in p_i$, $l \leqslant n$, $l' \in p_k$ and $l' \leqslant n$. Since $G_{j(i)}$ is irreducible, Theorem 2.4 implies that $E_l \oplus E_{l'}$ is irreducible and hence $l \sim l'$, i.e., $p_i = p_j$. Hence $\sum_p \oplus F_{p,n} = \prod_p F_{p,n}$. Since a Banach space with a finite dimensional decomposition is separable we have that $E = (\sum_{\mathscr{P}} \oplus F_p)_{c_0}$. This completes the proof. ∎

In our first corollary we recover a result of Vigué [35] originally proved using J^* triple systems.

COROLLARY 3.6: *If E is a symmetric Banach space with a 1-unconditional basis then E is isometrically isomorphic to a c_0 sum of separable Hilbert spaces (some or all of which may be finite dimensional).*

PROOF: E has a Reinhardt decomposition $\{E_j\}$, where E_j is the span of the j^{th} basis vector. With the notation of Theorem 3.5, Theorems 2.4 and 3.5 imply that F_p is either $I_{1,n(p)}$ or $I_{n(p),1}$, where $n(p) \in \{1, 2, \cdots, \infty\}$. The result follows by noting that each of these spaces is isometrically isomorphic to a Hilbert space. ∎

COROLLARY 3.7: *If E is an infinite dimensional irreducible symmetric Banach space with a finite dimensional Reinhardt decomposition then E is isometrically isomorphic to $L(\mathbb{C}^n; H)$ for some separable infinite dimensional Hilbert space H and some positive integer n.*

PROOF: Since E is irreducible we have only one equivalence class in Theorem 3.5. The only possible infinite-dimensional irreducible factors are $L(\mathbb{C}^n, H)$ and $L(H, \mathbb{C}^n)$. These spaces are isometrically isomorphic. ∎

More specific information can also be obtained by using Theorem 2.4 and the dimensions of the subspaces which occur in the given Reinhardt decomposition. For example, if $E = \sum_{i \in I} \oplus E_i$ is a finite dimensional Reinhardt decomposition of a symmetric space and $\dim(E_i) \neq 8$, 16 or 27 for all i then E is a J^* algebra in the sense of L. Harris [14].

We now show that the decomposition given in Theorem 3.5 is unique, up to a permutation of factors, and hence may be regarded as a Cartan decomposition. To obtain this result we use J^* triple systems.

DEFINITION 3.8 [34, Definition 2.2]: *Let $(E, \mathcal{Z})$ be a J^* triple system. A subspace F of E is called a J^* ideal in E if for all $x \in E$, $y \in F$ and $a \in E$*

DEFINITION 3.9: *A symmetric Banach space is said to be strongly irreducible if it contains no nontrivial J^* ideals.*

In [34] Vigué shows that every strongly irreducible Banach space is irreducible (Proposition 2.9) and that the converse is true for finite dimensional spaces (Théorème 5.1). Also, the spaces $L(\mathbb{C}^n, H)$ and $L(H, \mathbb{C}^n)$ are strongly irreducible for Hilbert spaces H by [34, Proposition 4.4]. Hence all the irreducible factors which arise in Theorem 3.5 are strongly irreducible. If H is an infinite dimensional Hilbert space then $B(H)$, the bounded operators from H to itself, is an example of an irreducible Banach space which is not strongly irreducible ([34, Théorème 4.1]).

PROPOSITION 3.10: *Let* $E = \left(\sum_{i \in I} \oplus E_i \right)_{c_0}$ *be a symmetric Banach space and suppose* $|I| > 1$. *Then* Γ *is a minimal* J^* *ideal of* E *if and only if* Γ *is a minimal* J^* *ideal of some* E_i.

PROOF: Let Z be the Jordan triple product associated with the unit ball of E. It can be shown that

$$Z\left(\sum_I x_i, \sum_I y_i, \sum_I z_i\right) = \sum_I Z(x_i, y_i, z_i) \tag{3.1}$$

where x_i, y_i, and z_i are in E_i. Hence, if Γ is a (minimal) J^* ideal in some E_i, then it is also a (minimal) J^* ideal in E. In particular, each E_i is a J^* ideal in E.

Conversely, suppose Γ is a minimal J^* ideal in E. Let P_i denote the canonical projection from E onto E_i. Chose $i \in I$ and $x \in \Gamma$ such that $P_i(x) \neq 0$. Since the only complete constant vector field on a bounded circular domain is the zero vector field, there exist y and z in E_i such that $Z(P_i(x), y, z) \neq 0$. By (3.1)

$$P_i(Z(x, y, z)) = Z(P_i(x), y, z) = Z(x, y, z) \in \Gamma.$$

Hence $E_i \cap \Gamma$ is a nonzero J^* ideal in E. Since Γ is a minimal J^* ideal it follows that $\Gamma \cap E_i = \Gamma$ and hence $\Gamma \subseteq E_i$. This completes the proof. ■

THEOREM 3.11: *Let* E *be a symmetric Banach space with a finite dimensional Reinhardt decomposition. If* $\left(\sum_{i \in I} \oplus E_i \right)_{c_0}$ *and* $\left(\sum_{j \in J} \oplus F_j \right)_{c_0}$ *are two decompositions of* E *with* E_i *and* F_j *irreducible for all* i *and* j, *then there exists a bijection* σ *from* I *to* J *such that* E_i *is isometrically isomorphic to*

PROOF: Suppose $\left(\sum_{i\in I}\oplus E_i\right)_{c_0}$ is the decomposition of E described in Theorem 3.5. Since each E_i is strongly irreducible Proposition 3.10 implies that $\{E_i\}_{i\in I}$ is the set of all minimal J^* ideals in E. Proposition 3.10 implies that for each $i\in I$ there exists a $j\in J$, call it $\sigma(i)$, such that either

(a) $E_i = F_{\sigma(i)}$, or
(b) E_i is a non-trivial minimal J^* ideal in $F_{\sigma(i)}$,

If (a) holds for all i then the proof is complete. If (b) holds for some i_0 then letting $I' = \{i\in I \mid \sigma(i)=\sigma(i_0)\}$ we find that $\left(\sum_{i\in I'}\oplus E_i\right)_{c_0} = F_{\sigma(i_0)}$. Since $F_{\sigma(i_0)}$ is irreducible this is impossible. Hence $\sigma\colon I\to J$ is a bijection and $E_i = F_{\sigma(i)}$ for all i. This completes the proof. ∎

Now suppose $\sum_{i\in I}\oplus E_i$ and $\sum_{i\in J}\oplus F_j$ are finite dimensional Reinhardt decompositions of the symmetric Banach spaces E and F, respectively, both with irreducible factors. Let $\left(\sum_{i\in I'}\oplus \tilde{E}_i\right)_{c_0}$ and $\left(\sum_{j\in J'}\oplus \tilde{F}_j\right)_{c_0}$ be the Cartan decompositions of E and F and suppose the unit ball of E is biholomorphically equivalent to the unit ball of F. By Harris [13] (see also Kaup and Upmeier [21] for a more general result) E and F are isometrically isomorphic and hence, by Theorem 3.8, there exists a bijection $\varphi\colon I'\to J'$, such that $\tilde{E}_i$ is isometrically isomorphic to $\tilde{F}_{\sigma(i)}$ for all $i\in I'$. For fixed $i_0\in I'$ we have $\tilde{E}_{i_0} = \sum_{i\in I_0}\oplus E_i$ and $\tilde{F}_{\sigma(i_0)} = \sum_{j\in J_0}\oplus F_j$ for some subsets I_0 and J_0 of I and J respectively. The example

$$L(\mathbb{C}^2,\mathbb{C}^3) = L(\mathbb{C}^2,\mathbb{C})\oplus L(\mathbb{C}^2,\mathbb{C}^2) = L(\mathbb{C};\mathbb{C}^3)\oplus L(\mathbb{C},\mathbb{C}^3)$$

shows that the isometry from $\tilde{E}_{i_0}$ onto $\tilde{F}_{\sigma(i_0)}$ may not result in the existence of linear isometries between the factors. If, however, each factor admits no nontrivial Reinhardt decomposition (by Theorem 2.4 this is the case only if each factor is either one dimensional, II_n, IV_{2n+1} or $H(\mathbb{O}_3)$) we see easily that the following is true.

THEOREM 3.12: *If $E=\sum_{i\in I}\oplus E_i$ and $F=\sum_{j\in J}\oplus F_j$ are finite dimensional Reinhardt decompositions such that each factor in both E and F admits no nontrivial Reinhardt decomposition and if the unit balls of E and F are biholomorphically equivalent then there exists a bijection $\sigma\colon I\to J$ such that E_i and $F_{\sigma(i)}$ are linearly isometrically isomorphic for all $i\in I$.*

§4. Characterization of Reinhardt domains in spaces with a basis

Let D be a Reinhardt domain in a complex Banach space E with a basis

(Unless otherwise stated we will always assume that a Reinhardt domain in a Banach space with a basis $(e_n)_n$ is a domain which is Reinhardt with respect to the coordinate projections associated with the basis $(e_n)_n$.) It was observed in §1 that if D is bounded then $(e_n)_{n\in\mathbb{N}}$ is an unconditional basis. When this is the case the diagonal isomorphism

$$\varphi\Big(\sum_n x_n\, e_n\Big) = \sum_n \lambda_n^{-1} x_n\, e_n,$$

where

$$\lambda_n = \sup\{\lambda > 0 \mid \lambda\, e_n \in D\},$$

may be used to define an equivalent norm $\|\cdot\|$ on E for which $co(\varphi(D))$ is the unit ball. With regard to this norm $(e_n)_{\mathbb{N}}$ is a normalized 1-unconditional basis. The domain $\varphi(D)$ is also said to be normalized, i.e.,

$$e_n \in \partial\varphi(D) \quad \text{and } \lambda\, e_n \in \varphi(D) \quad \text{implies } |\lambda| < 1 \quad \text{for all } n \in \mathbb{N}.$$

We assume throughout the rest of this section that $(e_n)_{\mathbb{N}}$ is a normalized 1-unconditional basis.

Let $\phi \neq I \subseteq \mathbb{N}$, $\mathscr{P}$ be any collection of nonempty subsets of $\mathbb{N}$, A any subset of E, and $x \in E$. We'll use the following notations: $E_I = [e_i \mid i \in I]$ is the closed linear span of $(e_i)_{i\in I}$, $A_I = A \cap E_I$, $x_I = \sum_{i\in I} x_i\, e_i$ is the coordinate projection of x onto E_I, and $x_{\mathscr{P}} = (\|x_p\|)_{p\in\mathscr{P}}$.

DEFINITION 4.1: A subset D in E has normal form $(\mathscr{P}, r)$ if there exists a nonempty subset I of $\mathbb{N}$, a partition $\mathscr{P}$ of I, and real numbers $r = (r_{p,j})_{\mathscr{P},J}$, where $J = \mathbb{N}\setminus I$, such that

$$D = \left\{ x \in E \mid x_{\mathscr{P}} \in B_{c_0}, \quad \text{and } \sum_{j\in J} \prod_{p\in\mathscr{P}} \left(1 - \|x_p\|^2\right)^{-r_{p,j}} x_j\, e_j \in D_J \right\}. \tag{4.1}$$

When $I = \mathbb{N}$ we may consider $D_J = \{0\}$ and the empty sum in (4.1) to be zero.

The simplest examples (of normal form in a two-dimensional setting) were considered by Thullen [29]. They were

$$\{(x_1, x_2) \in \mathbb{C}^2 \mid |x_1|^2 + |x_2|^{2r} < 1\} \quad (0 < r < \infty)$$

and

Thullen showed that these are the only two-dimensional bounded Reinhardt domains which admit non-linear biholomorphic automorphisms.

The main result of this section is a classification of the bounded normalized Reinhardt domains in E. This will serve to describe all bounded Reinhardt domains up to a diagonal isomorphism.

THEOREM 4.2: *Let D be a bounded normalized Reinhardt domain in E. Either every biholomorphic automorphism of D is linear or there is a nonempty subset I of $\mathbb{N}$, a partition $\mathscr{P}$ of I, and real constants $r = (r_{p,j})_{p\in\mathscr{P}, j\in J}$, where $J = \mathbb{N}\setminus I$, such that*

(i) D_I is the orbit of the origin under the biholomorphic automorphisms of D,

(ii) $r_{p,j} \geqslant 0$ for all $p \in \mathscr{P}$ and $j \in J$, and $\sup_{j\in J} \sum_{p\in\mathscr{P}} r_{p,j} < \infty$,

(iii) for each $p \in \mathscr{P}$, E_p is isometrically isomorphic to a Hilbert space and E_I is isometrically isomorphic to $(\sum_{p\in\mathscr{P}} \oplus E_p)_{c_0}$,

(iv) D has normal form $(\mathscr{P}, r)$.

Furthermore, if (iii) is satisfied by the Banach space E, a set D of normal form $(\mathscr{P}, r)$ is a bounded Reinhardt domain if and only if (ii) is satisfied and D_J is a bounded Reinhardt domain in E_J, and in this case the orbit of the origin under the biholomorphic automorphisms of D contains D_I.

The idea of the proof of Theorem 4.2 will be as follows. The set I, the partition $\mathscr{P}$ and the parameters $r_{p,j}$ are given by work of Vigué [35]. However Vigué [35] only gives the conclusions (i) and (iii). The restrictions (ii) on the parameters $r_{p,j}$ will be shown in a straightforward fashion but the main difficulty is to show (iv). This we do by first establishing that the set with normal form $(\mathscr{P}, r)$ is a bounded domain and has enough of the biholomorphic properties of the original domain to allow us to apply a uniqueness lemma of Braun, Kaup and Upmeier [3]. We begin by proving the following lemma.

LEMMA 4.3: *Let D have normal form $(\mathscr{P}, r)$ and assume that D_J is a bounded (normalized) Reinhardt domain in E_J and that $E_I = \left(\sum_{\mathscr{P}} \oplus E_p\right)_{c_0}$, where $I = \cup\mathscr{P}$ and $J = \mathbb{N}\setminus I$. Then D is a bounded (normalized) Reinhardt domain if and only if $r_{p,j} \geqslant 0$ for all $p \in \mathscr{P}$ and $j \in J$ and $\sup_J \sum_{\mathscr{P}} r_{p,j} < \infty$.*

PROOF: Observe first that D is necessarily Reinhardt (and normalized) and that

$$D_I = \left\{ \sum_{\mathscr{P}} x_p \in E_I \mid x_{\mathscr{P}} \in B_{c_0} \right\}.$$

Suppose first that D is a bounded domain. Let

$$\alpha = \sup\{ \| x \| \mid x \in D \}$$

and choose $\beta > 0$ so that

$$\beta B_{E_I} \times \beta B_{E_J} \subset D.$$

This implies in particular that $\beta \leqslant \min(\alpha, 1)$ and that

$$\beta B_{E_J} \subset D_J \subset \alpha B_{E_J}.$$

For any $p \in \mathscr{P}$ choose $(\xi_n) \subset B_{E_p}$ so that $\| \xi_n \| \to 1$ as $n \to \infty$. For any $j \in J$, $\xi_n + x_j \, e_j \in D$ whenever $|x_j| = \beta(1 - \| \xi_n \|^2)^{r_{p,j}}$. Since $|x_j| \leqslant \| \xi_n + x_j \, e_j \| < \alpha$, this implies

$$\sup_n \beta\left(1 - \| \xi_n \|^2\right)^{r_{p,j}} \leqslant \alpha.$$

Hence $r_{p,j} \geqslant 0$ for all $p \in \mathscr{P}$ and $j \in J$. Now let $\mathscr{P}'$ be an arbitrary finite subset of $\mathscr{P}$. For each $p \in \mathscr{P}'$ choose $\xi_p \in E_p$ with $\| \xi_p \| = \beta/2$. By our choice of β, $\sum_{\mathscr{P}'} \xi_p + \beta/2 \, e_j \in D$ for any $j \in J$. Hence,

$$\frac{\beta/2}{\prod_{\mathscr{P}'}(1 - \beta^2/4)^{r_{p,j}}} e_j \in D_J,$$

which implies

$$\frac{\beta/2}{(1 - \beta^2/4)^{\sum_{\mathscr{P}'} r_{p,j}}} \leqslant \alpha \quad \text{for all } j \in J \quad \text{and all } \mathscr{P}'.$$

Hence

$$\sup_J \sum_{\mathscr{P}} r_{p,j} \leqslant \frac{\log \dfrac{\beta}{2\alpha}}{\log(1 - \beta^2/4)}.$$

Conversely, assume that $r_{p,j} \geqslant 0$ for all $p \in \mathscr{P}$ and $j \in J$ and that $\sup_J \sum_{\mathscr{P}} r_{p,j} = \rho < \infty$. Let

and

$$x = \sum_{\mathscr{P}} x_p + \sum_J x_j\, e_j \in D$$

be fixed. By definition of D and the hypothesis we have

$$\left\| \sum_{\mathscr{P}} x_p \right\| = \max_{\mathscr{P}} \| x_p \| < 1,$$

and

$$\prod_{\mathscr{P}} \left(1 - \| x_p \|^2\right)^{r_{p,j}} \leqslant 1.$$

Hence

$$\left\| \sum_J x_j\, e_j \right\| = \left\| \sum_J \prod_{\mathscr{P}} \left(1 - \| x_p \|^2\right)^{r_{p,j}} \cdot \frac{x_j e_j}{\prod_{\mathscr{P}} \left(1 - \| x_p \|^2\right)^{r_{p,j}}} \right\|$$

$$\leqslant \left\| \sum_J \frac{x_j e_j}{\prod_{\mathscr{P}} \left(1 - \| x_p \|^2\right)^{r_{p,j}}} \right\|$$

$$\leqslant \gamma.$$

So D is bounded. Since D_J is open and

$$\sum_J \frac{x_j\, e_j}{\prod_{\mathscr{P}} \left(1 - \| x_p \|^2\right)^{r_{p,j}}} \in D_J,$$

there is a $\beta_1 > 0$ so that

$$\sum_J \frac{x_j e_j}{\prod_{\mathscr{P}} \left(1 - \| x_p \|^2\right)^{r_{p,j}}} + 3\beta_1 B_{E_J} \subset D_J. \tag{4.2}$$

Now choose $\beta_2 > 0$ so that

$$\sum_J \alpha_j \frac{x_j e_j}{\prod \left(1 - \| x_p \|^2\right)^{r_{p,j}}} \in \beta_1 B_{E_J} \quad \text{whenever } |\alpha_j| \leqslant \beta_2. \tag{4.3}$$

Let $\alpha > 0$ be such that

$$|(1+\alpha)^{\rho} - (1-\alpha)^{\rho}| \leqslant \beta_2$$

and let

$$M = \sup_{\mathscr{P}} \| x_p \|.$$

Since $M < 1$, there exists $\delta_1 > 0$ such that

$$1 - \alpha \leqslant \frac{1 - \| x_p \|^2}{1 - \| x_p + y_p \|^2} \leqslant 1 + \alpha$$

for all $p \in \mathscr{P}$ and $y_p \in E_p$ with $\| y_p \| \leqslant \delta_1$. Finally choose $0 < \delta_2 < \delta_1$ so that

$$M + \delta_2 < 1 \quad \text{and} \quad \frac{\delta_2}{\left[1 - (M + \delta_2)^2\right]^{\rho}} \leqslant \beta_1.$$

We'll show that $x + y \in D$ for any $y \in E$ with $\| y \| \leqslant \delta_2$, demonstrating D's openness and completing the proof. Now,

$$\sup_{\mathscr{P}} \| x_p + y_p \| \leqslant \sup_{\mathscr{P}} \left(\| x_p \| + \| y_p \| \right) \leqslant M + \delta_2 < 1,$$

so it remains to show that

$$\sum_J \frac{(x_j + y_j) e_j}{\prod_{\mathscr{P}} \left(1 - \| x_p + y_p \|^2\right)^{r_{p,j}}}$$

$$= \sum_J \frac{x_j e_j}{\prod_{\mathscr{P}} \left(1 - \| x_p \|^2\right)^{r_{p,j}}} + \sum_J \frac{y_j e_j}{\prod_{\mathscr{P}} \left(1 - \| x_p + y_p \|^2\right)^{r_{p,j}}}$$

$$+ \sum_J \left[\frac{1}{\prod \left(1 - \| x_p + y_p \|^2\right)^{r_{p,j}}} - \frac{1}{\prod \left(1 - \| x_p \|^2\right)^{r_{p,j}}} \right] x_j e_j$$

is in D_J. By (4.2) and (4.3) this will be accomplished if we can establish

$$\left\| \sum_J \frac{y_j e_j}{\prod_{\mathscr{P}} \left(1 - \| x_p + y_p \|^2\right)^{r_{p,j}}} \right\| \leqslant \beta_1 \tag{4.4}$$

and

$$\left| \prod_{\mathscr{P}} \left(\frac{1 - \| x_p \|^2}{1 - \| x_p + y_p \|^2} \right)^{r_{p,j}} - 1 \right| \leqslant \beta_2 \quad \text{for all } j \in J. \tag{4.5}$$

For all $p \in \mathscr{P}$,

$$\begin{aligned} 1 - \| x_p + y_p \|^2 &\geqslant 1 - \left(\| x_p \| + \| y_p \| \right)^2 \\ &\geqslant 1 - (M + \delta_2)^2, \end{aligned}$$

so

$$\begin{aligned} \prod_{\mathscr{P}} \left(1 - \| x_p + y_p \|^2\right)^{r_{p,j}} &\geqslant \left[1 - (M + \delta_2)^2\right]^{\sum_{\mathscr{P}} r_{p,j}} \\ &\geqslant \left[1 - (M + \delta_2)^2\right]^{\rho} \\ &\geqslant \frac{\delta_2}{\beta_1} \end{aligned}$$

by our choice of δ_2. Hence,

$$\left\| \sum_J \frac{y_j}{\prod_{\mathscr{P}} \left(1 - \| x_p + y_p \|^2\right)^{r_{p,j}}} e_j \right\| \leqslant \frac{\beta_1}{\delta_2} \left\| \sum_J y_j e_j \right\| \leqslant \frac{\beta_1}{\delta_2} \| y \| \leqslant \beta_1,$$

establishing (4.4). Since $\| y_p \| \leqslant \| y \| \leqslant \delta_2 < \delta_1$ for all $p \in \mathscr{P}$,

$$1 - \alpha \leqslant \frac{1 - \| x_p \|^2}{1 - \| x_p + y_p \|^2} \leqslant 1 + \alpha$$

by our choice of δ_1. Hence,

$$(1 - \alpha)^{\sum_{\mathscr{P}} r_{p,j}} \leqslant \prod_{\mathscr{P}} \left(\frac{1 - \| x_p \|^2}{1 - \| x_p + y_p \|^2} \right)^{r_{p,j}} \leqslant (1 + \alpha)^{\sum_{\mathscr{P}} r_{p,j}} \quad \text{for all } j \in J$$

and consequently

$$(1-\alpha)^{\rho} \leqslant \prod_{\mathscr{P}} \left(\frac{1 - \| x_p \|^2}{1 - \| x_p + y_p \|^2} \right)^{r_{p,j}} \leqslant (1+\alpha)^{\rho} \quad \text{for all } j \in J.$$

This implies

$$\left| \prod_{\mathscr{P}} \left(\frac{1 - \| x_p \|^2}{1 - \| x_p + y_p \|^2} \right)^{r_{p,j}} - 1 \right| \leqslant (1+\alpha)^{\rho} - (1-\alpha)^{\rho}$$

$$\leqslant \beta_2 \quad \text{for all } j \in J,$$

establishing (4.5). ∎

PROOF OF THEOREM 4.2: Let D be any subset of E satisfying conditions (ii), (iii), and (iv) for which D_J is a bounded normalized Reinhardt domain in E_J. By Lemma 4.3 D is a bounded normalized Reinhardt domain in E. We claim that the orbit of the origin under $G(D)$ contains D_I. To show this, we define a continuous triple product Z: $E_I \times E \times E \to E$ and show that the vector fields

$$x \to X_n(x) = e_n + Z(e_n, x, x), \; n \in I$$

are complete (n.b. the map Z we define is not known, a priori, to be the Jordan triple product associated with D). Having done this, by (1.5) and (1.6) we conclude that $E_I \subseteq F$ and so by (1.4) the orbit of the origin under $G(D)$ contains D_I.

We define Z on $E_I \times E_I \times E_I$ to be the Jordan triple product associated with the bounded symmetric domain B_{E_I} (see for instance [35]), i.e.,

$$Z\Big(\sum_{\mathscr{P}} \xi_p, \sum_{\mathscr{P}} x_p, \sum_{\mathscr{P}} y_p \Big) = -\tfrac{1}{2} \sum_{\mathscr{P}} \big((x_p \mid \xi_p) y_p + (y_p \mid \xi_p) x_p \big),$$

where, without fear of confusion, we have let $(\cdot \mid \cdot)$ denote the inner product on E_p for any $p \in \mathscr{P}$. Z is clearly conjugate linear in the first variable and symmetric bilinear in the second and third variables and continuous.

For any finitely supported vectors $\sum_{\mathscr{P}} \xi_p \in E_I$, $\sum_{\mathscr{P}} x_p + \sum_J x_j \, e_j \in E$, and $\sum_J y_j \, e_j \in E_J$, we let

$$Z\Big(\sum \xi_p, \sum x_p + \sum x_j \, e_j, \sum y_i \, e_i \Big) = \sum \Big[- \sum (x_p \mid \xi_p) r_{p,i} \Big] y_i \, e_i$$

and extend Z to all finitely supported vectors in $E_I \times E \times E$ using conjugate linearity, linearity, and symmetry. Since

$$\left\| Z\Big(\sum_{\mathscr{P}} \xi_p, \sum_{\mathscr{P}} x_p, \sum_J y_j\, e_j\Big)\right\|$$

$$= \left\| \sum_J \Big[-\sum_{\mathscr{P}} r_{p,j}(x_p \mid \xi_p)\Big] y_j\, e_j \right\| \leqslant \Big(\sup_J \Big| \sum_{\mathscr{P}} r_{p,j}(x_p \mid \xi_p)\Big|\Big) \Big\| \sum_J y_j\, e_j \Big\|$$

$$\leqslant \Big(\sup_{\mathscr{P}} |(x_p \mid \xi_p)|\Big)\Big(\sup_J \sum_{\mathscr{P}} r_{p,j}\Big) \Big\| \sum_J y_j\, e_j \Big\|$$

$$\leqslant \Big(\sup_{\mathscr{P}} \| x_p \|\Big)\Big(\sup_{\mathscr{P}} \| \xi_p \|\Big)\Big(\sup_J \sum_J r_{p,j}\Big) \Big\| \sum_J y_j e_j \Big\|$$

$$= \Big(\sup_J \sum_{\mathscr{P}} r_{p,j}\Big) \Big\| \sum_{\mathscr{P}} x_p \Big\| \cdot \Big\| \sum_{\mathscr{P}} \xi_p \Big\| \cdot \Big\| \sum_J y_j\, e_j \Big\|,$$

Z is continuous on the finitely supported vectors of $E_I \times E \times E$, and hence has a unique continuous extension to all of $E_I \times E \times E$.

We now show that the vector fields $(X_n)_{n \in I}$ are complete. Fix $i \in I$. For each $x = \sum_{n \in \mathbb{N}} x_n\, e_n$ in D we must find a differentiable map $\varphi(\cdot) = \sum_{n \in \mathbb{N}} \varphi_n(\cdot)\, e_n \colon \mathbb{R} \to D$ such that

$$\varphi'(t) = e_i + Z(e_i, \varphi(t), \varphi(t)) \quad \forall t \in \mathbb{R} \tag{4.6}$$

and $\varphi(0) = x$.

Let $\tilde{p}$ denote the element of $\mathscr{P}$ to which i belongs. Rewriting (4.6) as a system of scalar equations we obtain

$$\varphi_i'(t) = 1 - \varphi_i^2(t)$$

$$\varphi_n'(t) = -\varphi_i(t)\varphi_n(t), \; n \in \tilde{p}, \; n \neq i$$

$$\varphi_n'(t) = 0, \; n \in I \setminus \tilde{p}$$

We find the solutions

$$\varphi_i(t) = \tanh(t+c), \quad \text{where } \tanh(c) = x_i$$

$$\varphi_n(t) = x_n \frac{\operatorname{sech}(t+c)}{\operatorname{sech}(c)}, \quad n \in \tilde{p},\ n \neq i$$

$$\varphi_n(t) = x_n, \quad n \in I \setminus \tilde{p}$$

$$\varphi_j(t) = x_j \left[\frac{\operatorname{sech}(t+c)}{\operatorname{sech}(c)} \right]^{2r_{\tilde{p},j}}, \quad j \in J,$$

where we take the principal banch of $z \to z^{r_{\tilde{p},j}}$. We have only to verify that $\varphi(t) \in D$ for all $t \in \mathbb{R}$. Let $\varphi_{\tilde{p}}(t) = \sum_{n \in \tilde{p}} \varphi_n(t)\, e_n$. Then

$$1 - \| \varphi_{\tilde{p}}(t) \|^2$$

$$= 1 - |\tanh(t+c)|^2 - \sum_{\substack{n \in \tilde{p} \\ n \neq i}} |x_n|^2 \left| \frac{\operatorname{sech}(t+c)}{\operatorname{sech}(c)} \right|^2$$

$$= 1 - |\tanh(t+c)|^2 - \left(\| x_{\tilde{p}} \|^2 - |x_i|^2 \right) \left| \frac{\operatorname{sech}(t+c)}{\operatorname{sech}(c)} \right|^2$$

$$= \cos(2\,\mathrm{Im}(c)) \cdot |\operatorname{sech}(t+c)|^2 \left(1 - \frac{\| x_{\tilde{p}} \|^2 - |\tanh(c)|^2}{1 - |\tanh(c)|^2} \right)$$

$$= \cos(2\,\mathrm{Im}(c))\, |\operatorname{sech}(t+c)|^2 \left(\frac{1 - \| x_{\tilde{p}} \|^2}{(1 - |\tanh(c)|^2)} \right)$$

$$= \left| \frac{\operatorname{sech}(t+c)}{\operatorname{sech}(c)} \right|^2 \left(1 - \| x_{\tilde{p}} \|^2 \right)$$

$$\geqslant 1 - \| x_{\tilde{p}} \|^2, \tag{4.7}$$

where we have used the identity

Hence $\varphi_{\mathscr{P}}(t) \in B_{c_0}$ for all $t \in \mathbb{R}$. Using (4.7), we see that

$$\sum_J \frac{\varphi_j(t) e_j}{\prod_{\mathscr{P}} \left(1 - \| \varphi_p(t) \|^2\right)^{r_{p,j}}}$$

$$= \sum_J \left[\frac{x_j \left(\frac{\operatorname{sech}(t+c)}{\operatorname{sech}(c)} \right)^{2r_{\tilde{p},j}} e_j}{\left(1 - \| x_{\tilde{p}} \|^2\right)^{r_{\tilde{p},j}} \left| \frac{\operatorname{sech}(t+c)}{\operatorname{sech}(c)} \right|^{2r_{\tilde{p},j}} \prod_{p \neq \tilde{p}} \left(1 - \| x_p \|^2\right)^{r_{p,j}}} \right]$$

$$= \sum_J \frac{\lambda_j(t) x_j e_j}{\prod_{\mathscr{P}} \left(1 - \| x_p \|^2\right)^{r_{p,j}}},$$

where $|\lambda_j(t)| = 1$ for all $j \in J$, $t \in \mathbb{R}$.

Since D_J is Reinhardt, this last expression is in D_J for all $t \in \mathbb{R}$. This completes the first half of the proof.

Now suppose $\tilde{D}$ is a bounded normalized Reinhardt domain in E. By Vigué [35] there exists a subset I of $\mathbb{N}$ such that $G(\tilde{D})$. $0 = \tilde{D} \cap E_I$. The required partition $\mathscr{P}$ of I is also given in [35], and this partition satisfies the isometric properties in (iii). Let $\tilde{Z}$: $E_I \times E \times E \to E$ denote the Jordan triple product associated with $\tilde{D}$ and let

$$M = \sup\left\{ \| \tilde{Z}(\xi, x, y) \| \mid \xi \in B_{E_I},\ x,\ y \in B_E \right\}.$$

By Lemma 5.3 of [35], for each $p \in \mathscr{P}$ and $j \in J$ there exists $r_{p,j} \geqslant 0$ such that

$$\tilde{Z}(e_n, e_n, e_j) = -r_{p,j} e_j \quad \text{for all } n \in p.$$

We also note that if $p_1, \cdots, p_n$ are distinct elements of $\mathscr{P}$ and $m_1, \cdots, m_n$ are elements of $p_1, \cdots, p_n$, respectively, then $\| \sum_{i=1}^{n} e_{m_i} \| \leqslant 1$ by (iii). Consequently,

$$\tilde{Z}\left(\sum_{i=1}^{n} e_{m_i}, \sum_{i=1}^{n} e_{m_i}, e_j \right) = -\left(\sum_{i=1}^{n} r_{p_i,j} \right) e_j$$

and so

$$\sum_{i=1}^{n} r_{p_i,j} \leqslant M \quad \text{for all } j \in J \text{ and all } n.$$

where we've also used Lemma 3.4 (ii) of [35]. Hence $\sup_J \sum_{\mathscr{P}} r_{p,j} < \infty$. Define

$$D = \left\{ \sum x_p + \sum_J x_j\, e_j \mid x_{\mathscr{P}} \in B_{c_0} \quad \text{and} \quad \sum_J \frac{x_j e_j}{\prod_{\mathscr{P}} \left(1 - \| x_p \|^2\right)^{r_{p,j}}} \in \tilde{D}_J \right\}.$$

By the first half of the proof,

$$G(D).0 \supseteq D_I = \tilde{D}_I = G(\tilde{D}).0.$$

We also have $D_J = \tilde{D}_J$ and, by [35], $\tilde{Z}$ has the same values on the basis vectors as does Z, where Z is the Jordan triple product defined in the first half of the proof. Hence $Z = \tilde{Z}$, and $D = \tilde{D}$ follows upon applying Lemma 2.5 of [3]. ∎

REMARKS: 1. Vigué [35] gives an example of a Reinhardt domain in a Banach space with unconditional basis for which the span of the orbit of the origin under the biholomorphic automorphisms has codimension one. This example motivated our approach in this section.

2. Various results on Reinhardt domains in a more general setting are to be found in Braun et al. [3]. For instance, the authors characterize, in example 2.10, the bounded Reinhardt domains relative to a decomposition of E into two subspaces, one of which is one-dimensional and contained in the span of the orbit of the origin, in a manner generalizing the analogous result in $\mathbb{C}^2$ given by Thullen [29].

§5. Normal form of the unit ball of some Tsirelsohn spaces

Let B be the open unit ball of a Banach space E with a 1-symmetric basis $(e_n)_{n \in N}$. It is easily seen that either every biholomorphic automorphism of B is linear or else B is symmetric, in which case E must be isometric to either c_0 or l_2 by Vigué [35]. For, using the notation of Theorem 4.2, if I is not empty, then the subspace E_I is invariant under $G(B)$. In particular, any permutation map

$$\sum_n x_n\, e_n \to \sum_n x_n\, e_{\sigma(n)}$$

must leave E_I fixed and so $E_I = E$, i.e., $I = \mathbb{N}$.

In the remainder of this section we'll exhibit Banach spaces whose

and let $(e_n)_{n\in\mathbb{N}}$ be the usual unit vectors of the space T_0 of all finitely nonzero sequences. For $x = \sum_n x_n e_n \in T_0$ define

$$\|x\|_0 = \max_n |x_n|$$

$$\|x\|_{m+1} = \max\left\{ \|x\|_m, \sup \theta \sum_{l=1}^{k} \left\| \sum_{n=p_l+1}^{p_{l+1}} x_n e_n \right\|_m \right\}, \quad m \geqslant 0,$$

where the sup is taken over all choices $1 \leqslant k \leqslant p_1 < \ldots < p_{k+1}$. The sequence $(\|x\|_m)_{m=0}^{\infty}$ is clearly nondecreasing and bounded above by $\sum_n |x_n|$. Thus, we may define

$$\|x\| = \lim_m \|x\|_m.$$

The completion of T_0 with respect to $\|\cdot\|$ is called the Tsirelsohn space with parameter θ, which we'll denote simply by T without fear of confusion. It is well known (see [22]) that $(e_n)_{n\in\mathbb{N}}$ is a 1-unconditional basis for T, and that T is reflexive and contains no subspace isomorphic to l_p, $1 \leqslant p < \infty$. An elementary argument establishes that $\|\cdot\|$ (uniquely) satisfies

$$\|x\| = \max\left\{ \|x\|_0, \sup \theta \sum_{l=1}^{k} \left\| \sum_{n=p_l+1}^{p_{l+1}} x_n e_n \right\| \right\} \tag{5.1}$$

where the sup, here and hereafter in this section, is taken over all choices $1 \leqslant k \leqslant p_1 < \ldots < p_{k+1}$.

Denote the integer part of a real number α by $[\alpha]$. The main result of this section is

THEOREM 5.1: *Let T be the Tsirelsohn space with parameter θ and let B be its open unit ball. Then*

$$G(B).0 = \left[e_i \,|\, 1 \leqslant i \leqslant [1/\theta] + 1 \right] \cap B$$

and

$$B = \left\{ x \in T \,|\, |x_i| < 1 \quad \textit{for all } 1 \leqslant i \leqslant [1/\theta]+1, \right.$$

$$\left. \textit{and} \left\| \sum_{n=[1/\theta]+2}^{\infty} x_n e_n \right\| < 1 \right\}.$$

PROPOSITION 5.2: *There exists a nonempty finite subset I of $\mathbb{N}$ so that*

$$G(B).0 = [e_i \mid i \in I] \cap B \tag{5.2}$$

and

$$B = \left\{ x \in T \mid |x_i| < 1 \quad \textit{for all } i \in I, \quad \textit{and} \left\| \sum_{n \notin I} x_n e_n \right\| < 1 \right\} \tag{5.3}$$

PROOF: By (5.1)

$$B = \left\{ x \in T \mid |x_1| < 1 \quad \text{and} \left\| \sum_{n=2}^{\infty} x_n e_n \right\| < 1 \right\}.$$

Hence, by Theorem 4.2, $G(B).0 \supseteq [e_1] \cap B$, and consequently there is a nonempty subset I of $\mathbb{N}$ satisfying (5.2) and a partition $\mathscr{P}$ of I and constants $r = (r_{p,j})_{\mathscr{P},J}$ so that B has normal form $(\mathscr{P}, r)$. Consider the two-dimensional subspace $[e_i, e_j]$, $i \neq j$. If $0 < \theta \leqslant \frac{1}{2}$, then by (5.1) $[e_i, e_j] \cong l_\infty^2$. If $\frac{1}{2} < \theta < 1$, then $[e_1, e_j]$ and $[e_2, e_j]$ are both isometric with l_∞^2 by (5.1). For i and j both greater than 2, let

$$x = \frac{1-\theta}{\theta} e_i + e_j. \tag{5.4}$$

Then $\|x\| = 1$ by (5.1), and so $[e_i, e_j]$ is not isometric with l_2^2.

Since no two-dimensional coordinate subspace is isometric with l_2^2, it follows from Theorem 4.2 that $\mathscr{P}$ consists of the singletons $\{i\}$, $i \in I$. Hence $B_I \cong B_{c_0}$. Since T is reflexive it cannot contain c_0, so I must be finite. Let $i \in I$ and $j \in J$. Then

$$B \cap [e_i, e_j] = \left\{ (x_i, x_j) \mid |x_i| < 1 \quad \text{and} \frac{|x_j|}{\left(1 - |x_i|^2\right)^{r_{p,j}}} < 1 \right\}$$

where $p = \{i\}$. If $0 < \theta \leqslant \frac{1}{2}$, then $[e_i, e_j] \cong l_\infty^2$, so $r_{p,j} = 0$. In case $\frac{1}{2} < \theta < 1$, the vector x of (5.4) is in ∂B. Hence

$$\frac{1}{\left(1 - \left(\frac{1-\theta}{\theta}\right)^2\right)^{r_{p,j}}} = 1,$$

which again implies $r_{p,j} = 0$. Thus B has normal form $(\{\{i\}: i \in I\}, 0)$, i.e., (5.3) is satisfied by I. ∎

PROOF OF THEOREM 5.1: Let $I' = \{1, 2, \ldots, [1/\theta]+1\}$. To complete the proof we must show that I' is the set I of Proposition 5.2. By Theorem 4.2 it will suffice to prove that I' is the largest set satisfying (5.3).

Let $x \in T$ have its support in I'. Then

$$\sup \theta \sum_{l=1}^{k} \| \sum_{n=p_l+1}^{p_{l+1}} x_n e_n \| / \leqslant \theta \sum_{n=2}^{[1/\theta]+1} |x_n| \leqslant \theta[1/\theta] \|x\|_0 \leqslant \|x\|_0,$$

so $\|x\| = \|x\|_0$ by (5.1). Hence $B_{I'}$ is isometric to the unit ball of $l_\infty^{[1/\theta]+1}$.

We now prove that

$$\|x\|_m \leqslant \max\left\{ \left\| \sum_{n \in I'} x_n e_n \right\|_0, \left\| \sum_{n \notin I'} x_n e_n \right\|_m \right\} \tag{5.5}$$

for all m. This is obvious for $m = 0$. Assume (5.5) holds for some $m \geqslant 0$. Let

$$\alpha = \theta \sum_{l=1}^{k} \left\| \sum_{n=p_l+1}^{p_{l+1}} x_n e_n \right\|_m$$

for any $1 \leqslant k \leqslant p_1 < \ldots < p_{k+1}$. If $[1/\theta]+1 \leqslant p_1$, then $\alpha \leqslant \left\| \sum_{n \notin I'} x_n e_n \right\|_{m+1}$. If $p_{k+1} \leqslant [1/\theta]+1$, then $\alpha \leqslant \left\| \sum_{n \in I'} x_n e_n \right\|_0$. Otherwise find j so that

$$p_j < [1/\theta]+1 \leqslant p_{j+1}.$$

Then

$$\begin{aligned}
\alpha &\leqslant \theta(j-1) \left\| \sum_{n \in I'} x_n e_n \right\|_0 + \theta(k-j) \left\| \sum_{n \notin I'} x_n e_n \right\|_m \\
&\quad + \theta \left\| \sum_{n=p_j+1}^{p_{j+1}} x_n e_n \right\|_m \\
&\leqslant \theta(k-1) \max\left\{ \left\| \sum_{n \in I'} x_n e_n \right\|_0, \left\| \sum_{n \notin I'} x_n e_n \right\|_m \right\} \\
&\quad + \theta \max\left\{ \left\| \sum_{n=p_j+1}^{[1/\theta]+1} x_n e_n \right\|_0, \left\| \sum_{n=[1/\theta]+2}^{p_{j+1}} x_n e_n \right\|_m \right\} \\
&\leqslant k\theta \max\left\{ \left\| \sum x_n e_n \right\|_0, \left\| \sum x_n e_n \right\|_m \right\}
\end{aligned}$$

where we've used the induction hypothesis and 1-unconditionality. Since $k \leqslant p_1 < [1/\theta] + 1$, $k\theta \leqslant \theta[1/\theta] \leqslant 1$. Hence,

$$\alpha \leqslant \max\left\{ \left\| \sum_{n \in I'} x_n e_n \right\|_0, \left\| \sum_{n \notin I'} x_n e_n \right\|_{m+1} \right\},$$

and (5.5) holds. Letting $m \to \infty$ in (5.5) we have

$$\| x \| \leqslant \max\left\{ \left\| \sum_{n \in I'} x_n e_n \right\|_0, \left\| \sum_{n \notin I'} x_n e_n \right\| \right\}.$$

Since the reverse inequality also holds by 1-unconditionality, I' satisfies (5.3). Hence $I' \subseteq I$.

Now suppose there exists $i \in I \setminus I'$. Choose distinct $j_1, \cdots, j_{[1/\theta]}$ in J. Then $j_n > [1/\theta] + 1$ for each n, so by (5.1)

$$\left\| \sum_{n=1}^{[1/\theta]} e_{j_n} \right\| = \max\{1, \theta[1/\theta]\} = 1,$$

but

$$\left\| e_i + \sum_{n=1}^{[1/\theta]} e_{j_n} \right\| \geqslant \theta([1/\theta] + 1) > 1$$

since $i > [1/\theta] + 1$. This example shows that $I' \cup \{i\}$ fails (5.3), so $I' = I$. ∎

REMARK: It follows from Theorem 5.1 that every biholomorphic automorphism of B has the form $f \oplus S$, where f is a biholomorphic automorphism of the unit ball of $l_\infty^{[1/\theta]+1}$ and S is a linear isometric isomorphism of $[e_j \mid j > [1/\theta] + 1]$.

The space originally described by Tsirelsohn [30] is the dual of T. T^* shares the property of containing no l_p subspace and has a 1-unconditional basis $(e_n^*)_{n \in N}$, the coefficient functionals of $(e_n)_{n \in N}$. The next result shows that T^* does not enjoy the same holomorphic properties that T does.

PROPOSITION 5.3: *Let T have parameter θ. Then*

$$G(B_{T^*}).0 = \{0\},$$

PROOF: If, to the contrary, there is a nonlinear biholomorphic automorphism of B_{T^*}, Theorem 4.2 assures us that there must be some two-dimensional coordinate subspace $[e_i^*, e_j^*]$ so that $B_{T^*} \cap [e_i^*, e_j^*]$ also has this property. We'll show in fact that every such domain has only linear automorphisms.

From (5.1) and the proof of Proposition 5.2 the subspace $[e_i, e_j]$ of T is either isometric with l_∞^2 or its norm is given by

$$\| z\, e_i + w\, e_j \| = \max\{ |z|, |w|, \theta(|z| + |w|) \} \tag{5.6}$$

for arbitrary complex numbers z and w. In both cases $\| z\, e_i + w\, e_j \| = \| w\, e_i + z\, e_j \|$, and by duality this symmetry of the norm occurs also in $[e_i^*, e_j^*]$. Therefore, as discussed in the beginning of this section, $B_{T^*} \cap [e_i^*, e_j^*]$ is either a symmetric domain or has only linear automorphisms. In the former case $[e_i^*, e_j^*]$ must be isometric with l_∞^2 or l_2^2, i.e., $[e_i, e_j]$ must be isometric with l_1^2 or l_2^2. It was observed in the proof of Proposition 5.2 that it isn't l_2^2, and (5.6) shows that it isn't l_1^2. This completes the proof. ∎

"Modified" and "convexified" Tsirelsohn spaces have been described by Johnson [17] and Figiel and Johnson [11]. The two-dimensional isometric structure of these spaces is also reasonably straightforward, and the above methods may be applied to obtain results analogous to Theorem 5.1 and Proposition 5.3 for these spaces. We refer to Barton [2] for further details.

§6. Convexity of bounded Reinhardt domains

Having obtained the results of the previous two sections our attention was drawn to the recent interesting article of Stachó [27]. A portion of that article (pp. 110–124) may be regarded as being complementary, in the following sense, to our results of §4. In §4 we described the bounded Reinhardt domains containing the origin which support a non-linear biholomorphic automorphism using Jordan theoretic techniques. In [27] the biholomorphic automorphisms of bounded convex Reinhardt domains containing the origin are described using [26] and convexity (see particularly [26, Lemma] and [27, pp. 111–113]) [1]. Moreover, the problem of when a bounded Reinhardt domain is convex has not, to our knowledge, received any attention in the literature.

This problem turns out to be both complex and interesting. For the general case we obtain necessary (Proposition 6.1) and sufficient (Proposition 6.4) conditions but no characterizations. With the notation of

Definition 4.1, we then consider the case of $D_J = B_{c_0}$ and $D_J = B_{l_1}$ and find that for these particular cases both Propositions 6.1 and 6.4 lead to necessary and sufficient conditions. Next we obtain necessary and sufficient conditions for $D = B_{l_q}$, $1 < q < \infty$, when D contains one equivalence class (i.e., $|\mathscr{P}| = 1$; see Theorem 4.2). This also leads to a further sufficient condition for the convexity of a bounded Reinhardt domain.

We continue with the notation of Theorem 4.2 and in addition we set

$$X = [e_i \mid i \in I], \quad Y = [e_j \mid j \in J],$$

$$A = D \cap X = G(D).0, \quad B = D \cap Y.$$

We have precise information about A from Theorem 4.2 including the fact that A is convex. We also recall at this point that in view of Theorem 4.2 we shall always assume $r_{p,j} \geqslant 0$ and $\sup_J \sum_{\mathscr{P}} r_{p,j} < \infty$.

PROPOSITION 6.1: *If D is convex then*

(6.1) *B is convex*
(6.2) $\sup_J \sum_{\mathscr{P}} r_{p,j} \leqslant 1$.

PROOF: Clearly B must be convex. To prove (6.2) fix $j \in J$ and choose $x_p \in E_p$ with $\| x_p \| = 1$ for each p in some finite subset S of $\mathscr{P}$. Then $\sum_{p \in S} x_p \in \partial D$ and thus, for $0 \leqslant \alpha \leqslant 1$, we have

$$\alpha \sum_{p \in S} x_p + (1 - \alpha)\, e_j \in \overline{D}.$$

By Theorem 4.2

$$(1 - \alpha) \prod_{p \in S} \left(1 - \alpha^2 \| x_p \|^2\right)^{-r_{p,j}} \leq 1.$$

Taking logarithms we get

$$\sum_{p \in S} r_{p,j} \leqslant \frac{\log(1 - \alpha)}{\log(1 - \alpha^2)} \quad \text{for } 0 < \alpha < 1.$$

Letting α tend to 1 yields $\sum_{p \in S} r_{p,j} \leqslant 1$. Since S was arbitrary this completes the proof. ∎

In proving sufficiency of conditions for convexity of D we can assume without loss of generality that both $\mathscr{P}$ and J are finite. We make this

LEMMA 6.2: *Assume $\mathscr{P}$ and J are finite. Let*

$$D^{\mathbb{R}} \equiv \left\{ \left(\| x_p \| \right)_{p \in \mathscr{P}} \oplus \left(|x_j| \right)_{j \in J} \mid \sum_{\mathscr{P}} x_p + \sum_{J} x_j \, e_j \in D \right\}.$$

Then D is convex if and only if $D^{\mathbb{R}}$ is a convex domain in $\mathbb{R}^N (N = |\mathscr{P}| + |J|)$.

PROOF: It is easily checked that D convex implies $D^{\mathbb{R}}$ convex. For the converse we first observe that if $D^{\mathbb{R}}$ is convex then $B^{\mathbb{R}}$ is convex and consequently B is convex. Hence, if $|x_j| \leqslant |\hat{x}_j|$ and $\sum_J \hat{x}_j \, e_j \in B$ then $\sum_J x_j \, e_j \in B$. Secondly we observe that the functions

$$\phi_j(x) = \phi_j\left(\left(\| x_p \|_{p \in \mathscr{P}} \right) \right) \equiv \prod_{p \in \mathscr{P}} \left(1 - \| x_p \|^2 \right)^{r_{p,j}}$$

are modularly decreasing, i.e., if $\| x_p \| \leqslant \| \hat{x}_p \|$ for all $p \in \mathscr{P}$ then $\phi_j(\hat{x}) \leqslant \phi_j(x)$. Now choose x^0 and x^1 in D and $0 \leqslant \alpha \leqslant 1$. For $j \in J$ let

$$y_j = \frac{\alpha |x_j^0| + (1-\alpha) |x_j^1|}{\phi_j\left(\left(\alpha \| x_p^0 \| + (1-\alpha) \| x_p^1 \| \right)_{p \in \mathscr{P}} \right)}.$$

Our second observation and the triangle inequality show

$$\frac{|\alpha x_j^0 + (1-\alpha) x_j^1|}{\phi_j\left(\alpha x^0 + (1-\alpha) x^1 \right)} \leqslant y_j.$$

Hence our first observation and the convexity of $D^{\mathbb{R}}$ show that $\alpha x^0 + (1 - \alpha) x^1 \in D$ and this completes the proof. ∎

In the remainder of this section we shall use elementary properties of convex and concave functions. If $\psi: \mathbb{R}^n \to \mathbb{R}$ is a C^2 function then the Hessian of ψ, H_ψ, is the symmetric $n \times n$ matrix

$$\left(\frac{\partial^2 \psi}{\partial x_i \, \partial x_j} \right)_{i=1, j=1}^{n,n}.$$

ψ is concave if $-\psi$ is convex, i.e., if and only if H_ψ is negative semi-definite. If ψ is a smooth positive function defined on an interval I in $\mathbb{R}_+^n$ then the set $\{(\lambda, x) \in \mathbb{R}_+ \times I \mid \lambda < \psi(x)\}$ is convex if and only if ψ is concave, and the set $\{x \mid \psi(x) \leqslant c\}$ is convex if ψ is a convex

LEMMA 6.3: If $\sum_{i=1}^{n} r_i \leqslant 1$ and $r_i \geqslant 0$ for all i then the function

$$\phi(t_1, \cdots, t_n) = \prod_{i=1}^{n} \left(1 - t_i^2\right)^{r_i}$$

is concave on the set $\{(t_i)_{i=1}^n \mid 0 < t_i < 1 \text{ for all } i\}$.

PROOF: Using subscripts to denote partial derivatives we find

$$\phi_i(t) = -2r_i t_i \left(1 - t_i^2\right)^{-1} \phi(t)$$

$$\phi_{ii}(t) = \phi_i(t)^2 \phi(t)^{-1} \left(1 - \frac{1 + t_i^2}{2r_i t_i^2}\right)$$

$$\phi_{ij}(t) = \phi_i(t)\phi_j(t)\phi(t)^{-1}, \quad i \neq j.$$

Hence $H_\phi = \frac{1}{\phi} \operatorname{diag}(\phi_i)\ M \operatorname{diag}(\phi_i)$ where $\operatorname{diag}(\phi_i)$ is the diagonal matrix with entries $\phi_1, \cdots, \phi_n$ and M is the matrix with diagonal entries

$$1 - a_i = 1 - \frac{1 + t_i^2}{2r_i t_i^2}$$

and off diagonal entries equal to 1. Hence H_ϕ is negative semi-definite if and only if M is negative semi-definite. Since the principal minors of M are of the same form as M we need only show $(-1)^n \det(M) \geqslant 0$.

By induction on n it is not difficult to show

$$\det(M) = (-1)^n \prod_{i=1}^{n} a_i \left(1 - \sum_{i=1}^{n} \frac{1}{a_i}\right).$$

Since $a_i > 0$ and $1 \leqslant a_i r_i$ for all i we have

$$1 - \sum_{i=1}^{n} \frac{1}{a_i} \geqslant 1 - \sum_{i=1}^{n} r_i \geqslant 0$$

and $\det(M)$ has the required sign. This completes the proof. ■

We now give a sufficient condition for convexity.

PROPOSITION 6.4: *Suppose B is convex. The following are equivalent.*

(6.3) $r_{p,j} = r_p$ *is independent of* j.
(6.4) *For every closed subspace* F *of* E *which contains* X, $G(D \cap F).0 \supseteq A = G(D).0$.

Moreover, if (6.3) is satisfied and if $\sum r_p < 1$ *then* D *is convex.*

PROOF: Assume (6.3). let Z be the triple product constructed in the proof of Theorem 4.2. Then

$$Z\Big(\sum_{\mathscr{P}}\xi_p, \sum_{\mathscr{P}}x_p + \sum_J x_j\, e_j, \sum_{\mathscr{P}}y_p + \sum_J y_j\, e_j\Big)$$

$$= Z\Big(\sum_{\mathscr{P}}\xi_p, \sum_{\mathscr{P}}x_p, \sum_{\mathscr{P}}y_p\Big) + \sum_{\mathscr{P}} r_p(x_p \mid \xi_p)\sum_J y_j\, e_j$$

$$+ \sum_{\mathscr{P}} r_p(y_p \mid \xi_p)\sum_J x_j\, e_j.$$

Clearly Z maps $X \times F \times F$ into F and the argument given in the proof of Theorem 4.2 results in an automorphism of D which leaves $D \cap F$ invariant and takes 0 to any desired point of $D \cap X = A$. Hence (6.4) is satisfied.

Assume (6.4). Fix j and k distinct numbers in J and a scalar $\lambda \neq 0$. Let $F = X \oplus [e_j + \lambda\, e_k]$. Then $D \cap F$ is a Reinhardt domain with respect to the 1-unconditional basis $\{e_j + \lambda e_k\} \cup \{e_i \mid i \in I\}$ and we have $G(D \cap F).0 \supseteq A$. Since A has codimension 1 either $G(D \cap F).0 = A$ or $G(D \cap F).0 = D \cap F$ by [35, Théorème 3.2].

By Theorem 4.2 there exist non-negative parameters $(s_p)_{p \in \mathscr{P}}$ such that

$$D \cap F = \left\{ \sum_{\mathscr{P}} x_p + y(e_j + \lambda\, e_k) \mid \sup_p \| x_p \| < 1 \right.$$

$$\left. \text{and } |y| \frac{\| e_j + \lambda\, e_k \|_B}{\prod_{\mathscr{P}} \left(1 - \| x_p \|^2\right)^{s_p}} < 1 \right\}. \tag{6.5}$$

Here $\| \circ \|_B$ denotes the norm on Y whose unit ball is B. Now fix $p \in \mathscr{P}$ and $i \in p$. Using (6.5) and Theorem 4.2 we see that, for $|x| < 1$, $x\, e_i + y(e_j + \lambda\, e_k) \in D \cap F$ if and only if the following two equivalent conditions are satisfied:

$$|y|\, \| e_j + \lambda\, e_k \|_B \left(1 - |x|^2\right)^{-s_p} < 1,$$

$$|y|\, \| e_j \left(1 - |x|^2\right)^{-r_{p,j}} + \lambda\, e_k \left(1 - |x|^2\right)^{-r_{p,k}} \|_B < 1.$$

Hence

Letting x increase to 1 we conclude that

$$s_p = \max(r_{p,j},\ r_{p,k})$$

is independent of the value of λ. If $s_p = r_{p,j} > r_{p,k}$ then letting x increase to 1 we have $\| e_j + \lambda\, e_k \|_B = \| e_j \|_B$ for all $\lambda \neq 0$. This is impossible and hence $r_{p,k} \leqslant r_{p,j}$. Similarly $r_{p,j} \leqslant r_{p,k}$ and hence $r_{p,j} = r_{p,k}$. Since j and k were arbitrary this shows that (6.3) is satisfied.

Now suppose (6.3) holds and $\sum_{\mathscr{P}} r_p \leqslant 1$. Then $\sum_{\mathscr{P}} x_p + \sum_J x_j\, e_j \in D$ if and only if $\max_p \| x_p \| < 1$ and $\left\| \sum_J x_j\, e_j \right\|_B < \phi\left(\sum_{\mathscr{P}} x_p \right)$. If $\sum_{\mathscr{P}} x_p^0 + \sum_J x_j^0\, e_j$ and $\sum_{\mathscr{P}} x_p^1 + \sum_J x_j^1\, e_j$ belong to D and if $0 < \alpha < 1$ then

$$\begin{aligned}
&\left\| \sum_J \left(\alpha x_j^0 + (1-\alpha) x_j^1 \right) e_j \right\|_B \\
&\leqslant \alpha \left\| \sum_J x_j^0\, e_j \right\|_B + (1-\alpha) \left\| \sum_J x_j^1\, e_j \right\|_B \\
&< \alpha \phi\left(\sum_{\mathscr{P}} x_p^0 \right) + (1-\alpha) \phi\left(\sum_{\mathscr{P}} x_p^1 \right) \\
&\leqslant \prod_{\mathscr{P}} \left(1 - \left(\alpha \| x_p^0 \| + (1-\alpha) \| x_p^1 \| \right)^2 \right)^{r_p}
\end{aligned}$$

by Lemma 6.3. By choosing $x_j^0 \geqslant 0$ and $x_j^1 \geqslant 0$ for all $j \in J$ we conclude that $D^{\mathbb{R}}$ is convex and hence, by Lemma 6.2, D is convex. This completes the proof. ■

PROPOSITION 6.5: (*a*) *If B is the unit ball of c_0 then D is convex if and only if* $\sup_j \sum_{p \in \mathscr{P}} r_{p,j} \leqslant 1$.

(*b*) *If B is the unit ball of l_1 then D is convex if and only if $r_{p,j} = r_p$ for all p, j, and* $\sum_{\mathscr{P}} r_{p,j} \leqslant 1$.

PROOF. (a) By Proposition 6.1 the condition $\sum_{\mathscr{P}} r_{p,j} \leqslant 1$ is necessary for convexity. The structure of D and of the unit ball of c_0 shows that it suffices to consider the case where $|J| = 1$. An application of Proposition 6.4 completes the proof.

(b) Proposition 6.4 shows that the conditions $r_{p,j} = r_p$ for all p, j, and $\sum_{\mathscr{P}} r_p \leqslant 1$ are sufficient for convexity. Suppose conversely that D is

convex. Proposition 6.1 shows that $\sum_{\mathscr{P}} r_{p,j} \leqslant 1$. Fix $p \in \mathscr{P}$ and $j,\ k \in J$ with $j \neq k$. Passing to $D^{\mathbb{R}}$ we need only consider the three dimensional case

$$D^{\mathbb{R}} = \{(x,\ y,\ z) \mid x,\ y,\ z \geqslant 0,\ x < 1, \quad \text{and } y(1-x^2)^{-r} + z(1-x^2)^{-s} < 1\}$$

where $r = r_{p,j}$ and $s = r_{p,k}$. If $D^{\mathbb{R}}$ is convex then

$$z(x,\ y) \equiv (1-x^2)^s - y(1-x^2)^{s-r}$$

must be a concave function since its graph bounds $D^{\mathbb{R}}$. Since $\dfrac{\partial^2 z}{\partial y^2} = 0$ the determinant of H_z is

$$-\left(\frac{\partial^2 z}{\partial x\, \partial y}\right)^2 = -\left(2(s-r)x(1-x^2)^{s-r-1}\right)^2.$$

Hence the Hessian matrix cannot be negative semi-definite unless $r = s$, i.e., unless $r_{p,j} = r_{p,k}$. Since j and k were arbitrary this completes the proof. ∎

We now consider the case where B is the unit ball of l_q, $1 < q < \infty$. Our method is to show that the Hessian of a certain function is negative semi-definite (as in Proposition 6.5). This leads to rather complicated expressions and consequently we omit certain details which the reader may verify directly. We first consider the case where $|\mathscr{P}| = 1$.

THEOREM 6.6. *Let* $1 < q < \infty$ *and let*

$$D = \left\{ x \oplus (y_j)_{j=1}^{\infty} \mid x \in l_2,\ y_j \in \mathbb{C} \quad \text{all } j,\ \|x\| < 1, \right.$$
$$\left. \text{and } \sum_j \frac{|y_j|^q}{(1-\|x\|^2)^{qr_j}} < 1 \right\},$$

where $0 \leqslant r_j \leqslant 1$ *for all* j. *Then* D *is convex if and only if no pair* $(r_j,\ r_k)$ *satisfies the conditions*

$$\left(q(r_j + r_k) + 1 - q\right)^2 > 4qr_jr_k \tag{6.6}$$

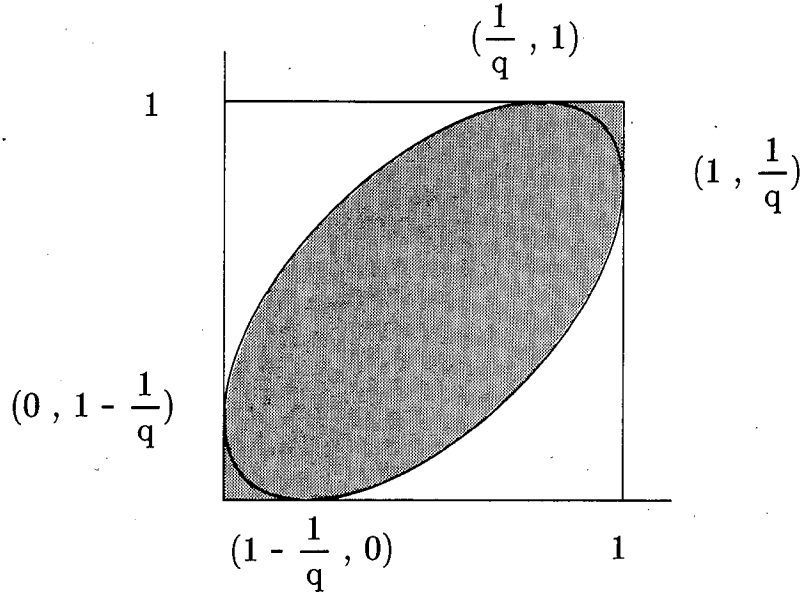


Figure 1.

REMARK: Condition (6.6) specifies the exterior of an ellipse tangent to the unit square at $\left(1-\frac{1}{q}, 0\right), \left(0, 1-\frac{1}{q}\right), \left(\frac{1}{q}, 1\right), \left(1, \frac{1}{q}\right)$, and condition (6.7) specifies the connected components of the exterior (relative to the unit square) which contain (1, 0) and (0, 1). The set of (r_j, r_k) which do not satisfy (6.6) and (6.7) lies in the shaded portion of figure 1. For $q = 1$ the ellipse degenerates to the diagonal $r_j = r_k$ and in the limiting case $q = \infty$ the ellipse fills the whole square (compare with Proposition 6.4).

PROOF: As noted previously it suffices to show $D^{\mathbb{R}}$ is convex and to consider finite dimensional Y. This reduces the problem to the finite dimensional case. Hence D is convex if and only if the set

$$D^{\mathbb{R}} = \left\{ (x, y_1, \cdots, y_{n+1}) \mid 0 \leqslant x < 1,\ y_j > 0, \right.$$
$$\left. \text{and} \sum_{j=1}^{n-1} \frac{y_j^q}{(1-x^2)^{qr_j}} < 1 \right\}$$

is convex.

Let $z = y_{n+1}$ and $s = r_{n+1}$. On the curved boundary of D (i.e., for $x > 0$) we have

$$z^q = (1-x^2)^{qs} - \sum_{j=1}^{n} (1-x^2)^{qs-qr_j} y_j^q.$$

Let $\lambda_j = (1-x^2)^{-qr_j} y_j^q$ for $j = 1, \cdots, n$. Then D is convex if and only if

semi-definite. We shall show that $(-1)^{n+1}\det(H_z) \geqslant 0$ if and only if (6.6) and (6.7) are satisfied. Since n is arbitrary (6.6) and (6.7) then imply that H_z is negative semi-definite (by considering minors of H_z).

Some lengthy calculations show that

$$\det(H_z) = (-1)^{n+1} A(x, \lambda) B(x, \lambda),$$

where

$$A(x, \lambda) = 4(q-1)^{n-1}(1-x^2)^{(n+1)s-2-2\sum_{j=1}^{n} r_j} \times \left(1 - \sum_{j=1}^{n} \lambda_j\right)^{(n+1)(-1+\frac{1}{q})-1} \prod_{j=1}^{n} \lambda_j^{1-2/q}$$

and

$$B(x, \lambda) = x^2 \left(\sum_{j=1}^{n} (r_j - s)\lambda_j\right)^2 + \sum_{j=1}^{n} \lambda_j (r_j - s)\left(\tfrac{1}{2}(q-1)(1+x^2) - q(r_j+s)x^2 + 2sx^2\right) + (q-1)s\left(\tfrac{1}{2}(1+x^2) - sx^2\right).$$

Now $A(x, \lambda) \geqslant 0$ for $0 \leqslant x \leqslant 1$ and hence $(-1)^{n+1}_{n} \det(H_z) \geqslant 0$ if and only if $B(x, \lambda) \geqslant 0$ for $0 < x < 1$, $\lambda_j > 0$, and $\sum_{j=1}^{n} \lambda_j < 1$. Since $B(x, \lambda)$ is linear in x^2 the minimum value of $B(x, \lambda)$ for fixed λ occurs at $x = 0$ or $x = 1$. Now

$$B(0, \lambda) = (q-1)\sum_{j=1}^{n} r_j \lambda_j + (q-1)s\left(1 - \sum_{j=1}^{n} \lambda_j\right) \geqslant 0$$

and

$$B(1, \lambda) = \left(\sum_{j=1}^{n} (r_j - s)\lambda_j\right)^2 + \sum_{j=1}^{n} (r_j - s)\left(q - 1 - q(r_j + s) + 2s\right)\lambda_j$$

If $n=1$, $B(1, \lambda)$ is a quadratic form in λ_1 and the minimum will occur at one of $\lambda_1=0$, $\lambda_1=1$, or

$$\lambda_1=\alpha\equiv\frac{q-1-q(r_1+s)+2s}{r_1-s}$$

if $0<\alpha<1$. This is quivalent to $(r_1, s)=(r_1, r_2)$ satisfying (6.6) and (6.7).

If $n>1$ then the minimum of $B(1, \lambda)$ occurs on the boundary where $\lambda_j=0$ for some j (a lower dimensional case) or where $\sum_{j=1}^{n}\lambda_j=1$. In the latter case we have

$$B(1, \lambda)=\left(\sum_{j=1}^{n}(r_j-r_n)\lambda_j\right)^2+\sum_{j=1}^{n}(r_j-r_n)$$
$$\times\left(q-1-q(r_j+r_n)+2r_n\right)+(q-1)r_n(1-r_n)$$

which is again a lower dimensional case. Hence $(-1)^{n+1}\det(H_z)\geqslant 0$ if and only if the conditions of the theorem are satisfied. This completes the proof. ∎

Using Theorem 6.6 we get a sufficient condition for convexity in the case $|\mathscr{P}|>1$ which is also obviously not necessary in general.

PROPOSITION 6.7: *If* $1<q<\infty$ *and* $\sup_j\sum_{\mathscr{P}}r_{p,j}\leqslant 1-\frac{1}{q}$ *then*

$$D=\left\{\sum_{\mathscr{P}}x_p+(y_j)_{j\in J}\mid\sup_{\mathscr{P}}\|x_p\|<1\right.$$
$$\left.\text{and}\ \sum_j\frac{|y_j|^q}{\prod_{\mathscr{P}}\left(1-\|x_p\|^2\right)^{qr_{p,j}}}<1\right\}$$

is a convex domain.

PROOF: Let

$$\psi(x_1, \cdots, x_m, y_1, \cdots, y_n)=\sum_{j=1}^{n}\frac{y_j^q}{\prod^{m}(1-x_i^2)^{qr_{i,j}}}.$$

By Lemma 6.2 it suffices to show that the domain

$$\{(x_1,\cdots,x_m,\ y_1,\cdots,y_n)\,|0\leqslant x_i<1,\ y_j\geqslant 0,$$

$$\psi(x_1,\cdots,x_m,\ y_1,\cdots,y_n)<1\}$$

is convex for all n and m. Since the sum of convex functions is convex it suffices to take $n=1$ and to show that

$$\psi(x_1,\cdots,x_m,\ y)=\left(\prod_{i=1}^{m}\left(1-x_i^2\right)^{-qr_i}\right)y^q$$

is convex when $\sum_{i=1}^{m} r_i\leqslant 1-\frac{1}{q}$.

Using subscripts to denote partial derivatives we have

$$\psi_{x_i,x_i}=\frac{2qr_i\psi}{\left(1-x_i^2\right)^2}\left(1+x_i^2+2qr_ix_i^2\right)$$

$$\psi_{x_i,x_j}=\frac{2qr_i\psi}{\left(1-x_i^2\right)\left(1-x_j^2\right)}\left(2qr_jx_ix_j\right),\ i\neq j$$

$$\psi_{x_i,y}=\frac{2qr_i\psi}{\left(1-x_i^2\right)y}qx_i$$

$$\psi_{yy}=\frac{q\psi}{y^2}(q-1).$$

Hence $H_\psi=\psi A(K+B)A$ where

$$A=\operatorname{diag}\left(2qr_1/\left(1-x_1^2\right),\ldots,2qr_n/\left(1-x_n^2\right),\ q/y\right),$$

K is the matrix with all its entries equal to one and

$$B=\operatorname{diag}\left(\left(1+x_1^2\right)/\left(2qr_1x_1^2\right),\ldots,\left(1+x_n^2\right)/\left(2qr_nx_n^2\right),\ -1/q\right).$$

Since $B\geqslant\tilde{B}=\operatorname{diag}(1/qr_1,\ldots,1/qr_n,\ -1/q)$ it is sufficient to show that $K+\tilde{B}$ is positive definite. By the formula for $\det(M)$ in the proof of Lemma 6.3

$$\det(K+\tilde{B})=\left(1-\frac{1}{q}-\sum_{j=1}^{n}r_j\right)/(q^nr_1r_2\ldots r_n)\geqslant 0$$

and the other principal minors are also positive.

REMARK: Combining Theorem 6.6 and Proposition 6.7 one can easily see that if B is the unit ball of l_q and $r_{p,j}=0$ for all p and some fixed j then D will be convex if and only if $\sum_{\mathscr{P}} r_{p,k} \leqslant 1-\frac{1}{q}$ for all $k \neq j$.

Our final theorem resulted from our experience with the l_q case. We state this result for finite dimensional spaces. The modifications necessary for the infinite dimensional case are obvious.

THEOREM 6.8: *Let* $\|\cdot\|_B$ *be a fixed norm on* $\mathbb{C}^{n+1}$, $n \geqslant 0$, *with unit ball B and suppose*

$$\|(y_1, \cdots, y_{n+1})\|_B = \|(|y_1|, \cdots, |y_{n+1}|)\|_B$$

(hence B is a convex Reinhardt domain). Let S_+ *be the portion of the unit sphere contained in the positive orthant* $\mathbb{R}_+^{n+1}$ *and suppose* S_+ *is the graph of a* C^2 *function* $y_{n+1}=\psi(y_1, \cdots, y_n)$. *Let*

$$D(r_{i,j}) = \left\{ (x_1, \cdots, x_m, y_1, \cdots, y_{n+1}) \in \mathbb{R}_+^{n+m+1} \mid x_i < 1 \right.$$

$$\left. \text{and} \left\| \left(\prod_{i=1}^{m} (1-x_i^2)^{-r_{i,j}} y_j \right)_{j=1}^{n+1} \right\|_B < 1 \right\}$$

where $(r_{i,j})$ *are non-negative parameters. Then*

(*i*) *the* $(r_{i,j})$ *for which D is convex form a convex set of* $m \times (n+1)$ *matrices* Γ,

(*ii*) Γ *contains the "diagonal"* $r_{i,j}=r_{i,k}$ *for all j and k,*

(*iii*) Γ *is contained in the simplex*

$$\sup_j \sum_i r_{i,j} \leqslant 1,$$

(*iv*) *if* $\|\circ\|_B$ *is the* l_q *norm,* $1<q<\infty$, *then* Γ *contains the simplex*

$$\sup_j \sum_i r_{i,j} \leqslant 1-\frac{1}{q}.$$

PROOF: We have already proved (ii) (Proposition 6.4), (iii) (Proposition 6.1), and (iv) (Theorem 6.7) and hence it remains only to show that (i) is true.

Let $z=y_{n+1}$ and $s_i=r_{i,n+1}$. The boundary of $D^{\mathbb{R}}(D=D(r_{i,j}))$ is given by

$$z = \prod_{i=1}^{m} (1-x_i^2)^{s_i} \psi\left(\left(\prod_{i=1}^{m} (1-x_i^2)^{-r_{i,j}} y_j \right)_{j=1}^{n} \right)$$

The Hessian of z is equal to

$$\prod_{i=1}^{m}\left(1-x_i^2\right)^{s_i} AM({}^tA)$$

where

$$A=\operatorname{diag}\Big(\left(1-x_1^2\right)^{-1},\cdots,\left(1-x_m^2\right)^{-1},\ \prod_{i=1}^{m}\left(1-x_i^2\right)^{r_{i,1}},\cdots,$$

$$\prod_{i=1}^{m}\left(1-x_i^2\right)^{r_{i,n}}\Big)$$

and the entries of M are

$$M(x_a,\ x_b)=-2\delta_{a,b}\left(1+x_a^2\right)\Big(s_a\psi(\alpha)-\sum_{j=1}^{n} r_{a,j}\alpha_j\psi_j\Big)$$

$$+4x_a x_b\Big(s_a s_b\psi(\alpha)$$

$$+\sum_{j=1}^{n}\left(r_{a,j}r_{b,j}-s_a r_{b,j}-s_b r_{a,j}\right)\alpha_j\psi_j$$

$$+\sum_{j,k} r_{a,j}r_{b,j}\alpha_j\alpha_k\psi_{j,k}(\alpha)\Big)$$

$$M(x_a,\ y_j)=M(y_j,\ x_a)=-2x_a\Big(\left(s_a-r_{a,j}\right)\psi_j-\sum_{k} r_{a,k}\alpha_k\psi_{j,k}\Big)$$

$$M(y_j,\ y_k)=\psi_{j,k}$$

where we have used the abbreviations

$$\alpha_j=\prod_{i=1}^{m}\left(1-x_i^2\right)^{-r_{i,j}} y_j,$$

$$\alpha=(\alpha_1,\cdots,\alpha_n),$$

and

$$\psi_j=\psi_j(\alpha),\quad \psi_{j,k}=\psi_{j,k}(\alpha)$$

Now, on M, we perform successively the following row and column operations:

$$\mathrm{Col}(x_a) - \sum_{j=1}^{n} 2x_a r_{a,j}\alpha_j \,\mathrm{Col}(y_j)$$

$$\mathrm{Row}(x_a) - \sum_{j=1}^{n} 2x_a r_{a,j}\alpha_j \,\mathrm{Row}(y_j).$$

This results in a new matrix $\tilde{M}$ which is related to M by $M = B\tilde{M}({}^tB)$ for some invertible matrix B. The entries of $\tilde{M}$ are

$$\tilde{M}(x_a, x_b) = -2\delta_{a,b}(1+x_a^2)\left(s_a\psi(\alpha) - \sum_{j=1}^{n} r_{a,j}\alpha_j\psi_j\right)$$

$$+4x_a x_b\left(s_a s_b\psi(\alpha) - \sum_{j=1}^{n} r_{a,j}r_{b,j}\alpha_j\psi_j\right)$$

$$\tilde{M}(x_a, y_j) = \tilde{M}(y_j, x_a) = -2x_a(s_a - r_{a,j})\psi_j$$

$$\tilde{M}(y_j, y_k) = \psi_{j,k}.$$

Hence z is a concave function if and only if $\tilde{M} = \tilde{M}((s_i), (r_{i,j}), x, \alpha)$ is negative semidefinite for all choices x, α with $0 \leqslant x_a < 1$ and $\alpha = (\alpha_1, \cdots, \alpha_n, 0) \in B$. Consider two collections $(s_i, r_{i,j})$ and $(\hat{s}_i, \hat{r}_{i,j})$ of parameters for which z is concave. Fix x, α and let

$$N = \tfrac{1}{2}\left(\tilde{M}(s, r) + \tilde{M}(\hat{s}, \hat{r})\right) - \tilde{M}\left(\frac{s+\hat{s}}{2}, \frac{r+\hat{r}}{2}\right).$$

Now

$$N(x_a, x_b) = x_a x_b\left((s_a - \hat{s}_a)(s_b - \hat{s}_b)\psi(\alpha)\right.$$

$$\left. + \sum_{j=1}^{n} (r_{a,j} - \hat{r}_{a,j})(r_{b,j} - \hat{r}_{b,j})(-\alpha_j\psi_j)\right)$$

and all other entries of N are zero. For $1 \leqslant j \leqslant n$ let

$$f_j(t) = \psi(\alpha_1, \cdots, \alpha_{j-1}, t, \alpha_{j+1}, \cdots, \alpha_n)$$

hence is concave, i.e., $f_j''(t) \leqslant 0$ and f_j' is decreasing. Also the even function obtained by putting $f_j(t) = f_j(-t)$ is concave. This implies that $f_j'(0) \leqslant 0$ and consequently $f_j'(\alpha_j) = \psi_j(\alpha) \leqslant 0$. Using this fact it is easily seen that N is a positive definite matrix. Hence

$$\tilde{M}\left(\frac{s+\hat{s}}{2}, \frac{r+\hat{r}}{2}\right) = \tfrac{1}{2}\tilde{M}(s, r) + \tfrac{1}{2}\tilde{M}(\hat{s}, \hat{r}) - N$$

is a sum of three negative semi-definite matrices and so it also must be negative semidefinite. This proves (i) and completes the proof of the theorem. ∎

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