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REMARK ON THE CRAMER–VON MISES–SMIRNOV CRITERION

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1. Introduction and Basic Results

Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables on the segment [0, 1],

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} 1\{X_i < x\},$$

where 1{A} is the indicator of the event A,

$$\omega_n^2 = n \int_0^1 \{F_n(x) - x\}^2\, dx,$$

$$U_n(x) = P\{\omega_n^2 < x\},$$

$$U(x) = \lim_{n\to\infty} U_n(x).$$

Let $\alpha = n/2 - 1$ if n is even, and $\alpha = (n - 1)/2$, if n is odd. We prove that the distribution function $U_n(x)$ is differentiable with respect to x α times, but not continuously differentiable $\alpha + 1$ times. In addition, the derivatives of the distribution functions $U_n(x)$ as $n \to \infty$, converge uniformly in x to the corresponding derivative of the limit distribution function U(x). In particular, one has uniform convergence of the densities $U_n'(x)$.

In this paper we also give asymptotic expansions for the derivatives of the distribution functions $U_n(x)$. The estimates of the remainders depend properly on n.

The results of the paper generalize and improve the results of Smirnov [1], Anderson and Darling [2], Kandelaki [3], Sazonov [4, 5], Rosenkrantz [6], Kiefer [7], Nikitin [8], Orlov [9], Czorgo [10], Csorgo and Stacho [11], Götze [12], Borovskikh [13], in which the convergence and rate of convergence of distribution functions $U_n(x)$ to U(x) were studied and asymptotic expansions for $U_n(x)$ were also found.

We proceed to precise formulations.

We denote by C^α the class of functions f: $R^1 \to R^1$ which have α bounded derivatives.

THEOREM 1.1. The distribution function $U_n(x)$ belongs to the class C^α but does not belong to the class $C^{\alpha+1}$, where $\alpha = n/2-1$ if n is even, and $\alpha = (n - 1)/2$ if n is odd. Moreover, for any

$$\sup_{x>0} (1 + x^m) \,|\, U_n^{(s)}(x) - U^{(s)}(x) \,|\, \leq c(s,m)/n. \tag{1.1}$$

Theorem 1.1 generalizes results of [1-13], devoted to proving (1.1) if s = 0. The first part of Theorem 1.1 on the differentiability of the distribution function $U_n(x)$ somewhat improves the result of Csorgo and Stacho [11]. One should note that in [11] the question of

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differentiability of the distribution function $U_n(x)$ was studied with the help of the representation of $U_n(x)$ as the Lebesgue measure of a certain set in the space R^n using the well-known Bruno–Minkowsky inequality. In the present paper this question is studied with the help of a detailed analysis of the characteristic function $E\exp\{it\omega_n^2\}$.

THEOREM 1.2. For any $m \geqq 0$, $s=0, 1, \ldots, p=1, 2, \ldots, n \geqq 2(s+1)$,

$$\sup_{x>0}(1+x^m)\left|\left(\frac{d}{dx}\right)^s\left\{U_n(x)-U(x)-\sum_{k=1}^{p-1} n^{-k}A_k U(x)\right\}\right| \leqq c(m, s, p)/n^p. \tag{1.2}$$

Explicit formulas for the Fourier–Stieltjes transforms of the functions $x \mapsto A_k(x)$ are given on p. 150 of [18]. It is known [4] that $U_n(x)$ can be represented in the form of the probability $P\{\|S_n\|^2<x\}$, where S_n is a sum of independent $L_2[0, 1]$-valued random elements. Asymptotic expansions for the derivatives $\left(\frac{d}{dx}\right)^k P\{\|S_n\|^2<x\}$, $k=0, 1, \ldots$, are constructed in [14-16]. There too, one can find rules for constructing the coefficients of the expansion. In particular, the functions $A_k\, U(x)$ are infinitely differentiable and their derivatives decrease at infinity faster than any power of x.

Theorem 1.2 generalizes the results of [10-13], where asymptotic expansions of the distribution functions $U_n(x)$ were studied. The proof of the theorems is based essentially on the results of [16] on asymptotic expansions in the local limit theorem in Hilbert space. It follows from the results and proofs of this paper (cf. Sec. 3 and the proof of Theorem 3.1 in [16]) that to prove Theorem 1.2 it suffices to verify the condition

$$\int_{|t| \leqq \sqrt{n}} |t|^r \left|\left(\frac{d}{dt}\right)^q E\exp\{it\omega_n^2\}\right| dt < \infty \tag{1.3}$$

for certain sufficiently large r = r(m, s, p) and q = q(m, s, p). The analysis of the characteristic function $E\exp\{it\omega_n^2\}$ and the proof of (1.3) are given in Sec. 2. The following lemma illustrates the results of this section.

LEMMA 1.3. Let $s=0, 1, \ldots, n=1, 2, \ldots, t \in R^1$. Then for any A > 0, and for sufficiently large n one has

$$\left|\left(\frac{d}{dt}\right)^s E\exp\{it\omega_n^2\}\right| \leqq c(s, A)\, n^s/(1+|t|^A). \tag{1.4}$$

A general method for estimating characteristic functions in the zone $|t| \leqq n^{1-\varepsilon}$ $(\varepsilon>0)$ was proposed by Götze [12]. In the zone $|t| \geqq n^{1/2+\varepsilon}$ $(\varepsilon>0)$, for any A > 0 and for sufficiently large n, Borovskikh [13] found the estimate

$$\left|\left(\frac{d}{dt}\right)^s E\exp\{it\omega_n^2\}\right| \leqq c(s, A)/n^A. \tag{1.5}$$

Lemma 1.3 improves this result.

2. Estimate of the Characteristic Function $E\exp\{it\omega_n^2\}$

THEOREM 2.1. There exists an absolute constant a such that for s = 0, 1, ..., n = 1, 2, ..., in the zone $|t| \geq n^2$ one has

$$\left|\left(\frac{d}{dt}\right)^s E\exp\{it\omega_n^2\}\right| \leqq |t|^{-n/2}\, n!\,(9n)^s\{a(s+1)\}^n. \tag{2.1}$$

The proof of Theorem 2.1 will be given below.

Applying the representation of Anderson and Darling for the statistic ω_n^2 (cf. [2, 17])

$$\omega_n^2=\sum_{j=1}^{n}(X_j^*-a_j)^2+1/(12n),$$

where $a_j = (j - 1/2)/n$ and $X_1^*, X_2^*, \ldots, X_n^*$ are the ordered random variables $X_1, X_2, \ldots, X_n$, we get

$$\left(\frac{d}{dt}\right)^s E\exp\{it\omega_n^2\}=i^s E\left\{\sum_{j=1}^{n}(X_j^*-a_j)^2+\frac{1}{12n}\right\}^s \exp\left\{it\sum_{j=1}^{n}(X_j^*-a_j)^2+\frac{it}{12n}\right\}=$$

$$= i^s n! \exp\left\{ \frac{it}{12n} \right\} \int \dots \int \left\{ \sum_{j=1}^{n} (x_j - a_j)^2 + \frac{1}{12n} \right\}^s \times$$

$$\times \exp\left\{ it \sum_{j=1}^{n} (x_j - a_j)^2 \right\} dx_n \dots dx_1,$$

where the integration is over all $x_1 \in [0, 1]$, $x_j \in [x_{j-1}, 1]$, $j = 2, \dots, n$. We make the change of variables $y_k = |t|^{1/2}(x_k - a_k)$, $k = 1, \dots, n$, and we note that

$$(x_1 + \dots + x_r)^s = \sum\nolimits^* C_s(k_1, \dots, k_r) x_1^{k_1} \dots x_r^{k_r},$$

where the sign Σ^* denotes summation over all $(k_1, \dots, k_r)$, $k_1 \geq 0, \dots, k_r \geq 0$, $k_1 + \dots + k_r = s$. We get

$$\left(\frac{d}{dt}\right)^s E \exp\{it\omega_n^2\} = i^s n! |t|^{-n/2-s} \exp\{it/(12n)\} \sum\nolimits^* C_s(k_0, \dots, k_n) \times$$
$$\times |t|^{k_n} (12n)^{-k_n} \int \dots \int y_0^{2k_0} \exp\{i\Theta y_0^2\} \dots y_{n-1}^{2k_{n-1}} \exp\{i\Theta y_{n-1}^2\} dy_{n-1} \dots dy_0, \qquad (2.2)$$

where we have set $\Theta = \operatorname{sgn} t$ and the integration is over all

$$y_{n-1} \in [-|t|^{1/2}(2n)^{-1}, (n - 1/2)|t|^{1/2} n^{-1}]$$

and

$$y_{p-1} \in [y_p - |t|^{1/2} n^{-1}, (p - 1/2)|t|^{1/2} n^{-1}], \quad p = 1, \dots, n-1.$$

We let $\varphi(0, x) \equiv 1$

$$\varphi(p, y) = \int x^{2k_{p-1}} \varphi(p-1, x) \exp\{i\Theta x^2\} dx, \quad p = 1, \dots, n, \qquad (2.3)$$

where the integration is over the domain $[y - |t|^{1/2}/n, (p - 1/2)|t|^{1/2}/n]$. Since $\varphi(p, y)$ also depends on $k_0, \dots, k_{p-1}$, we shall sometimes write $\varphi_{k_0, \dots, k_{p-1}}(p, y)$.

It follows quickly from (2.2) and the definition of $\varphi(p, y)$ that

$$\left(\frac{d}{dt}\right)^s E \exp\{it\omega_n^2\} = i^s n! |t|^{-n/2-s} \exp\{it/(12n)\} \times \sum\nolimits^* C_s(k_0, \dots, k_n) |t|^{k_n} (12n)^{-k_n} \varphi_{k_0, \dots, k_{n-1}}\left(n, |t|^{1/2}/(2n)\right). \qquad (2.4)$$

In estimating the sum in (2.4) we shall use the following.

<u>LEMMA 2.2.</u> Let $|t| \geq n^2$. Then there exists an absolute constant a such that for all $y \in (-\infty, (n + 1/2)|t|^{1/2} n^{-1}]$ one has

$$|\varphi_{k_0, \dots, k_{n-1}}(n, y)| \leq (s+1)^n (9|t|)^{k_{n-1} + \dots + k_0} a^n.$$

<u>Proof.</u> Let $\tau = |t|^{1/2} n^{-1}$. Then

$$\varphi(p, y) = \int_{y-\tau}^{(p-1/2)\tau} x^{2k_{p-1}} \varphi(p-1, x) \exp\{i\Theta x^2\} dx; \quad p \geq 1.$$

We set

$$T_p(A, B) = \int_A^B x^{2k_{p-1}} \varphi(p-1, x) \exp\{i\Theta x^2\} dx \qquad (2.5)$$

and let $\varphi(0, x) \equiv 1$, $\varphi(-1, x) \equiv 0$, $\varphi(-2, x) \equiv 0$.

First we prove the recurrence estimate

$$|T_p(A, B)| \leq (s+1)(9|t|)^{k_{p-1}} \left\{ \frac{|\varphi(p-1, A)|}{|A|} + \frac{|\varphi(p-1, B)|}{|B|} + \right.$$
$$\left. + \int_A^B \frac{|\varphi(p-1, x)|}{x^2} dx \right\} + (s+1)(9|t|)^{k_{p-1}+k_{p-2}} \left\{ \frac{|\varphi(p-2, A-\tau)|}{|A|\,|A-\tau/2|} + \right.$$
$$+ \frac{|\varphi(p-2, B-\tau)|}{|B|\,|B-\tau/2|} + \int_A^B |\varphi(p-2, x-\tau)| \left(\frac{1}{x^2 |x-\tau/2|} + \right. \qquad (2.6)$$
$$\left. \left. + \frac{1}{|x|(x-\tau/2)^2} + \frac{1}{|x|\,|x-\tau/2|\,|x-\tau|} \right) dx \right\} + (s+1)(9|t|)^{k_{p-1}+k_{p-2}+k_{p-3}} \int_A^B \frac{|\varphi(p-3, x-2\tau)|}{|x|\,|x-\tau/2|} dx.$$

We set

$$F(p-1,\ x)=|x|^{2k_{p-1}}|\varphi(p-1,\ x)|,$$
$$G(p-2,\ x)=|x|^{2k_{p-1}}|x-\tau|^{2k_{p-2}}|\varphi(p-2,\ x-\tau)|;\tag{2.7}$$
$$H(p-3,\ x)=|x|^{2k_{p-1}}|x-\tau|^{2k_{p-2}}|x-2\tau|^{2k_{p-3}}|\varphi(p-3,\ x-2\tau)|.$$

Integrating in (2.5) by parts and applying elementary inequalities, we get

$$|T_p(A,\ B)|\leqq\frac{F(p-1,\ A)}{|A|}+\frac{F(p-1\ B)}{|B|}+\left|k_{p-1}-\frac{1}{2}\right|\int_A^B\frac{F(p-1,\ x)}{x^2}\,dx+$$
$$+\frac{G(p-2,\ A)}{|A|\ |A-\tau/2|}+\frac{G(p-2,\ B)}{|B|\ |B-\tau/2|}+\left|k_{p-1}-\frac{1}{2}\right|\int_A^B\frac{G(p-2,\ x)}{x^2|x-\tau/2|}\,dx+\tag{2.8}$$
$$+k_{p-2}\int_A^B\frac{G(p-2,\ x)}{|x|\ |x-\tau/2|\ |x|-\tau}\,dx+\int_A^B\frac{G(p-2,\ x)}{|x|(x-\tau/2)^2}\,ax+\int_A^B\frac{H(p-3,\ x)}{|x|\ |x-\tau/2|}\,dx.$$

Since $y_{p-1}\in[y_p-\tau,\ (p-1/2)\tau]$, $p=1,\ \dots,\ n$, $y_n=\tau/2$, it is easy to verify that $y_p\in[-(n-1/2)\tau,\ (n-1/2)\tau]$ for all p = 0, 1, ..., n − 1. Hence,

$$|A|,\ |B|,\ |x|,\ |x-\tau|,\ |x-2\tau|\leqq 3|t|^{1/2}.\tag{2.9}$$

Moreover, one has

$$0\leqq k_p\leqq s\quad\forall p=0,\ 1,\ \dots,\ n.\tag{2.10}$$

Estimating the expressions (2.7) with the help of (2.9) and (2.10), we derive (2.6) from (2.8).

We shall estimate the φ(p, y) by induction on p. First we consider the case p = 1. Then

$$\varphi(1,\ y)=\int_{y-\tau}^{\tau/2}x^{2k_0}\exp\{i\Theta x^2\}\,dx.$$

We separate three subcases:

a) $y-\tau\in[1/2,\ \tau/2]$;
b) $y-\tau\in[-1/2,\ 1/2]$;
c) $y-\tau\in(-\infty,\ -1/2]$.

a) Applying (2.6) for p = 1, we get

$$|\varphi(1,\ y)|=\left|\int_{y-\tau}^{\tau/2}x^{2k_0}\exp\{i\Theta x^2\}\,dx\right|\leqq(s+1)(9|t|)^{k_0}\left\{\frac{1}{|y-\tau|}+\frac{1}{\tau/2}+\int_{y-\tau}^{\tau/2}\frac{1}{x^2}\,dx\right\}\leqq(s+1)(9|t|)^{k_0}4.$$

b) Analogously,

$$|\varphi(1,\ y)|=\left|\int_{y-\tau}^{\tau/2}x^{2k_0}\exp\{i\Theta x^2\}\,dx\right|\leqq\left|\int_{y-\tau}^{1/2}y^{2k_0}\exp\{i\Theta x^2\}\,dx\right|+$$
$$+\left|\int_{1/2}^{\tau/2}x^{2k_0}\exp\{i\Theta x^2\}\,dx\right|\leqq 4(s+1)(9|t|)^{k}+(9|t|)^{k_0}\leqq(s+1)(9|t|)^{k_0}5.$$

c) Similarly,

$$|\varphi(1,\ y)|\leqq\left|\int_{y-\tau}^{-1/2}x^{2k_0}\exp\{i\Theta x^2\}\,dx\right|+\left|\int_{-1/2}^{\tau/2}x^{2k_0}\exp\{i\Theta x^2\}\,dx\right|\leqq$$
$$\leqq\left|\int_{y-\tau}^{-1/2}x^{2k_0}\exp\{i\Theta x^2\}\,dx\right|+(s+1)(9|t|)^{k_0}5\leqq$$
$$\leqq(s+1)(9|t|)^{k_0}\left\{5+\frac{1}{|y-\tau|}+2+\int_{y-\tau}^{-1/2}\frac{1}{x^2}\,dx\right\}=(s+1)(9|t|)^{k_0}9.$$

It is clear that in case c) one gets the worst estimate. Hence, for p = 1, for all $y \in (-\infty, 3\tau/2]$ one has the estimate from the hypotheses of the lemma with a = 9.

We proceed to estimate $\varphi(p, y)$ for $p \geq 2$. Let us assume that for $\ell = 1, \ldots, p-1$ and all $y \in (-\infty, (l+1/2)\tau]$ one has

$$|\varphi(l, y)| \leqq (s+1)^l (9|t|)^{k_{l-1}+\cdots+k_0} \Phi(l, t, n) \tag{2.11}$$

with some finite $\Phi(l, t, n)$, $l \geqq 2$, and $\Phi(1, t, n) \equiv 9$. Without loss of generality, one can assume that $\Phi(0, t, n) \equiv 1$, $\Phi(-1, t, n) = \Phi(-2, t, n) \equiv 0$. First we prove that then (2.11) also holds for $\ell = p$. We prove the estimate $\Phi(p, t, n) \leq a^p$ somewhat later.

It is clear from (2.5) and (2.3) that

$$\varphi(p, y) = T_p(y-\tau, (p-1/2)\tau).$$

The points $-1/4$, $1/4$, $\tau/2-1/4$, $\tau/2+1/4$, $\tau-1/4$, $\tau+1/4$, $(p-1/2)\tau$ divide the half-line into seven intervals $I_1 = (-\infty, -1/4), \ldots, I_7 = (\tau+1/4, (p-1/2)\tau)$. The estimation of $\varphi(p, y)$ largely repeats the estimation of $\varphi(1, y)$. Hence we consider only the most laborious case $y - \tau \in I_1$. Thus, let $y - \tau \in I_1$. Then

$$|\varphi(p, y)| = |T_p(y-\tau, (p-1/2)\tau)| \leqq |T_p(y-\tau, -1/4)| + \sum_{i=2}^{7} |T_p(I_i)| \leqq$$

(we apply (2.9), (2.10), and the assumption (2.11))

$$\leqq |T_p(y-\tau, -1/4)| + (s+1)^p (9|t|)^{k_{p-1}+\cdots+k_0} \Phi(p-1, t, n)\, 3/2 + \sum_{i \in \{3, 5, 7\}} |T_p(I_i)|. \tag{2.12}$$

The points 0, $\tau/2$, τ do not belong to the intervals $(y-\tau, -1/4), I_i, i=3, 5, 7$. Hence, to the intervals $T_p(y-\tau, -1/4), T_p(I_i), i=3, 5, 7$, one can apply the reccurence estimate (2.6). Keeping (2.12) in mind, we get

$$|\varphi(p, y)| \leqq (s+1)^p (9|t|)^{k_{p-1}+\cdots+k_0} \Phi(p, t, n),$$

where

$$\begin{aligned} \Phi(p, t, n) = \Phi(p-1, t, n) &\left\{ 3/2 + U(y-\tau, -1/4) + \sum{}' U(I_i) \right\} + \\ + \Phi(p-2, t, n) &\left\{ V(y-\tau, -1/4) + \sum{}' V(I_i) \right\} + \\ + \Phi(p-3, t, n) &\left\{ W(y-\tau, -1/4) + \sum{}' W(I_i) \right\}, \end{aligned} \tag{2.13}$$

the sign Σ' denotes summation over i = 3, 5, 7,

$$U(A, B) = \frac{1}{|A|} + \frac{1}{|B|} + \int_A^B \frac{1}{x^2}\, dx,$$

$$V(A, B) = \frac{1}{|A|\,|A-\tau/2|} + \frac{1}{|B|\,|B-\tau/2|} + \int_A^B \left\{ \frac{1}{x^2|x-\tau/2|} + \frac{1}{|x|\,|x-\tau/2|\,|x-\tau|} + \frac{1}{|x|\,(x-\tau/2)^2} \right\} dx,$$

$$W(A, B) = \int_A^B \frac{1}{|x|\,|x-\tau/2|}\, dx.$$

It is clear that there exists an absolute constant $b \geq 9$ such that each of the expressions in curly brackets in (2.13) does not exceed b. Hence, from (2.13) for p = 1, 2, ..., we get

$$\Phi(p, t, n) \leqq \{\Phi(p-1, t, n) + \Phi(p-2, t, n) + \Phi(p-3, t, n)\}\, b.$$

Consequently, there exists an absolute constant a, such that $\Phi(p, t, n) \leq a^p$ (for example, one can take a = 2b).

<u>Proof of Theorem 2.1.</u> Estimating each summand in (2.4) in modulus, and applying the estimate of Lemma 2.2, we get

$$\left| \left(\frac{d}{dt}\right)^s E \exp\{it\omega_n^2\} \right| \leqq n!\, |t|^{-n/2-s} (s+1)^n a^n \times$$
$$\times \sum{}^{*} C_s(k_0, \ldots, k_n) |t|^{k_n} (12n)^{-k_n} (9|t|)^{k_{n-1}+\cdots+k_0} \leqq n!\, |t|^{-n/2-s} (s+1)^n a^n (9|t|)^s n^s.$$

The theorem is proved.

<u>Proof of Lemma 1.3.</u> Theorem 2.1 and the estimate (1.5) in the zone $|t| \geqq n^{1/2+\varepsilon}$ $(\varepsilon > 0)$. for sufficiently large n, imply

$$\left| \left(\frac{d}{dt}\right)^s E \exp\{it\omega_n^2\} \right| \leqq c(s, A)/(1+|t|^A).$$

In the zone $|t| \leqq n^{1-\varepsilon}$ $(\varepsilon > 0)$ the estimate of Lemma 1.3 is known (cf., e.g., [15, p. 37]). The lemma is proved.

3. Proofs of Theorems 1.1 and 1.2

As already noted, Theorem 1.2 follows from (1.3). The estimate of Theorem 2.1 and the above-mentioned results on estimates of characteristic functions from [12-16] guarantee that this condition holds. Theorem 1.1 is a special case (for p = 1) of Theorem 1.2. The differentiability of the functions $U_n(x)$ follows from the estimate of Theorem 2.1 and the well-known properties of the Fourier transform. It is known [19, 20] that $U_n(x) = 0$ for $x \leq 1/(12n)$ and $U_n(x) = c_n (x - 1/12)^{(n-1)/2}$, for $1/(12n) \leqq x \leqq 1/(12n) + 1/(2n^2)$, where $c_n > 0$ is a constant. Hence, $U_n \notin C^{\alpha+1}$.

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