OPERATOR SPACE STRUCTURE OF JC^*-TRIPLES AND TROS, I

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Abstract. We embark upon a systematic investigation of operator space structure of JC^*-triples via a study of the TROs (ternary rings of operators) they generate. Our approach is to introduce and develop a variety of universal objects, including universal TROs, by which means we are able to describe all possible operator space structures of a JC^*-triple. Via the concept of reversibility we obtain characterisations of universal TROs over a wide range of examples. We apply our results to obtain explicit descriptions of operator space structures of Cartan factors regardless of dimension.

1. Introduction

TROs (*ternary rings of operators*), the Banach subspaces of C^*-algebras closed under the ternary product $[a, b, c] = ab^*c$, are natural and significant objects in the category of operator spaces. We mention (see [8, 26], [5, Chapter 6]) that the algebraic isomorphisms between TROs coincide with the surjective complete isometries, the isomorphisms between operator spaces, and that each operator space E may be realised (completely isometrically) as an operator subspace of its injective envelope $I(E)$ which, in turn, may be realised as a TRO: in which case if F is an operator subspace of a C^*-algebra A then every complete isometry from F onto E is the restriction of a TRO homomorphism from the TRO generated by F onto the triple envelope $\mathcal{T}(E)$ of E, the TRO generated by E in its realisation in $I(E)$.

A complex Banach space E is said to be a JC^**-triple* if there is a surjective isometry $\phi\colon F \to E$ where F is a subspace of a TRO closed under the (Jordan) triple product $\{a, b, c\} = (1/2)([a, b, c] + [c, b, a])$. In these circumstances F is called a *concrete* JC^*-triple or, more specifically, a JC^*-subtriple of the TRO in question. By a result originating from the general theory of JB^*-triples, the triple product induced on E defined by $\{\phi(a), \phi(b), \phi(c)\} = \phi(\{a, b, c\})$ is independent

Date: Final draft July 8, 2010.

2000 *Mathematics Subject Classification.* 47L25; 17C65; 46L70.

Key words and phrases. CAR algebra; Cartan factor; JC-operator space; operator space ideal; reversibility; ternary ring of operators; universal TRO.

The work of the second and third authors was supported by the Science Foundation Ireland under grant 05/RFP/MAT0033.

of ϕ [15]. In particular, all linear isometric images of TROs are JC^*-triples. An abstract Hilbert space H is a JC^*-triple with triple product given by $\{\xi, \eta, \zeta\} = (\langle \xi, \eta \rangle \zeta + \langle \zeta, \eta \rangle \xi)/2$ as may be seen from either of the surjective isometries $H \to \mathcal{B}(H)e$ $(\xi \mapsto \xi \otimes h)$, $H \to e\mathcal{B}(H)$ $(\xi \mapsto h \otimes \bar{\xi})$ where $h \in H$ is a unit vector and e is the projection $h \otimes h$. Although linearly isometric, $\mathcal{B}(H)e$ and $e\mathcal{B}(H)$ are not completely isometric if H has dimension greater than unity [24, §10].

A contractively complemented subspace E of $\mathcal{B}(H)$ is an example of a JC^*-triple (not necessarily a JC^*-subtriple of $\mathcal{B}(H)$) [28, Corollary 2.4], [16], [6]. Operator space structure of JC^*-triples of this kind has been investigated in the important series of articles [21, 20, 22] in the case when E is reflexive, equivalently, when E is a finite ℓ_∞-sum of reflexive Cartan factors. Cartan factors also figure prominently in the study of completely contractively complemented subspaces of Schatten spaces [18].

Our aim in this paper is to explore operator space structures of an arbitrary JC^*-triple. One of our aims is to place the ground-breaking results of [21, 20, 22] into the general setting of a full theory. To this end we shall initiate the study of certain universal objects associated with a JC^*-triple and, in particular, that of the universal TRO of a JC^*-triple. We shall show that this new device (which differs from the above mentioned triple envelope — see Theorem 6.5) enables a systematic examination of the TROs generated by JC^*-triples and facilitates a general enquiry into their operator space structures. As a technical aid of possibly independent interest we shall also introduce the notion of a reversible JC^*-triple and shall apply our general theory to obtain concrete identifications of the universal TROs of all Cartan factors (those that arise as JC^*-triples — the two exceptional Cartan factors of JB^*-triple theory are not JC^*-triples) and we shall exhibit the possible operator space structures of Cartan factors of rank greater than unity.

We shall tend to use [5, 24] as our standard references for the theory of operator spaces (see also [23, 3]). Briefly, we recall that an *operator space* is a complex Banach space E together with a linear isometric embedding of E into some $\mathcal{B}(H)$ and that the corresponding operator space structure on E is determined by the matrix norms on $M_n(E)$ inherited from $M_n(\mathcal{B}(H))$. A linear map between operator spaces, $\pi \colon E \to F$, is said to be *completely bounded* with completely bounded norm given by $\|\pi\|_{cb} = \sup\{\|\pi_n\| : n \geq 1\}$ if the latter is finite, where π_n is the tensored map $\pi \otimes I_n \colon M_n(E) \to M_n(F)$, for each n. If $\|\pi\|_{cb} \leq 1$, then π is said to be a *complete contraction* and to be a *complete isometry* if every π_n is isometric.

As already noted TROs occupy a special place in operator space theory. To expand upon the triple product of an abstract JC^*-triple we remark that the the results of [15] show that $\{x, y, z\}$ on E, as was defined above, is the unique product on E that is symmetric and bilinear in x and z and conjugate linear in y for which the operator $D(x,y)$ on E given by $D(x,y)(z) = \{x,y,z\}$ satisfies

$$[D(x,y), D(a,b)] = D(\{x,y,a\},b) - D(a,\{b,x,y\}),$$

and such that $D(x,x)$ is a positive Hermitian operator in $\mathcal{B}(E)$ with norm $\|x\|^2$. We further remark that the use of abstract JC^*-triples in this paper is a matter of convenience and that nothing essential would be lost by exclusive concentration upon their concrete realisations. The fundamental difference between JC^*-triples and TROs is that the underlying structure of the former is determined by their linear isometric class while that of the latter is fixed up to complete isometry.

We have gathered background material in the next section and start on the new concepts, including universal TROs in §3. We introduce the notion of universal reversibility in §4 as an aid to the identification of the universal TROs and, in §5, we bring our methods to bear upon Cartan factors. One outcome of independent interest is a classification of universally reversible Cartan factors (Theorem 5.6). In the final section we show that the operator space structures of a JC^*-triple arising from its concrete realisations are completely determined by the *operator space ideals* (as we call them) of its universal TRO and we exhibit links with injective and triple envelopes. We apply this to elucidate a full description of the operator space strucutures of all non Hilbertian Cartan factors, answering a question raised in [19].

The Hilbertian case requires special consideration and we postpone a detailed analysis to a forthcoming paper. In addition, we plan other articles on developments of our ideas excluded from the present paper such as exactness (as a functor) of the universal TRO, its behaviour with respect to relevant tensor products and the introduction and use of the weak*-TRO of a JW^*-triple (a JC^*-triple with a predual) which shall also enable us to classify all universally reversible JC^*-triples and to analyse operator space structures of a wide range of JC^*-triples and JW^*-triples.

The methods pioneered in [21], and the techniques of [19], do not readily extend from the cases of the finite rank Cartan factors for which they were devised. In contrast, our univeral approach provides a coherent general framework for treating any JC^*-triple. In particular, for Cartan factors we do not require triple embeddings to be weak*-continuous nor the existence of norm dense grids of minimal tripotents.

2. Background

In this section, we have assembled preliminary material for later use. General references for JC^*-triples are the surveys [27, 25] and the papers [7, 11, 12, 15] whilst [30, 3, 5, 23] are helpful sources for TROs. Another class of present relevance is that of JC^*-algebras, the norm closed subspaces of C^*-algebras which are closed under the involution and Jordan product, $a \circ b = (1/2)(ab+ba)$. JC^*-algebras are the complexifications of JC-algebras, about which [10] is an extensive monograph. Since the triple product on $\mathcal{B}(H)$ can be expressed in terms of the Jordan product and $*$-operation, via $\{x, y, z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*$, it follows that JC^*-algebras are JC^*-triples.

By an *ideal* we will always mean a *norm closed ideal*. Thus an ideal of a JC^*-triple or a TRO E is a norm closed subspace I of E for which, respectively, $\{a, b, c\}$ or $[a, b, c]$ lies in I whenever a, b or c belongs to I (and $a, b, c \in E$). An ideal of a JC^*-algebra A is a norm closed subspace I such that $A \circ I \subset I$. The following removes any ambiguity.

Lemma 2.1 ([12, Proposition 5.8])**.** *The ideals of a TRO, JC^*-algebra or C^*-algebra E coincide with the ideals of E when it is regarded as a JC^*-triple.*

The quotient of a JC^*-triple by an ideal is again a JC^*-triple [7, Theorem 2] and similarly for TROs [8, Proposition 2.1]. Naturally, triple homomorphisms between JC^*-triples and TRO homomorphisms between TROs are, respectively, the linear maps preserving the underlying triple and TRO products. The image of a triple homomorphism (hence of a TRO homomorphism) is norm closed. The first part of the next statement was proved in [12, Proposition 3.4], [11, Theorem 4] and the second part in [8, Proposition 2.1].

Lemma 2.2. *(a) Between JC^*-triples, triple homomorphisms are contractive, injective triple homomorphisms are isometric and surjective linear isometries are triple isomorphisms.*

(b) Between TROs, TRO homomorphisms are completely contractive, injective TRO homomorphisms are completely isometric and surjective complete isometries are TRO isomorphisms.

By a TRO *antihomomorphism*, $\phi\colon T \to S$, between TROs we mean a linear map such that $\phi([a, b, c]) = [\phi(c), \phi(b), \phi(a)]$ whenever $a, b, c \in T$. Typically these arise as restrictions of *-antihomomorphisms between containing C^*-algebras.

Recall [15] that the JC^*-subtriple generated by an element x is a JC^*-triple E is linearly isometric to an abelian C^*-algebra, giving rise to a functional calculus and, in particular, to the existence of a 'cube root' $y \in E$ such that $x = \{y, y, y\}$. Elements $a, b \in E$ are said to be orthogonal if $\{a, a, b\} = 0$, a condition equivalent to $a^*b = ab^* = 0$ when E is realised as a JC^*-subtriple of a C^*-algebra [11, p. 18], implying $\|a + b\| = \max(\|a\|, \|b\|)$. Thus ideals I and J of E are orthogonal if and only if $I \cap J = \{0\}$, in which case $I + J$ is an ℓ_∞ sum.

If E is a subset of a C^*-algebra we write $\mathrm{TRO}(E)$ for the TRO generated by E: it the closed linear span of elements of the form $x_1x_2^*x_3 \ldots x_{2n}^*x_{2n+1}$ where the $x_i \in E$ and $n \geq 0$. We note that:

Lemma 2.3. *If E and F are orthogonal subsets of a C^*-algebra, then TRO(E) and TRO(F) are orthogonal ideals of TRO$(E{+}F)$ = TRO(E)+ TRO(F), which is an ℓ_∞ sum (where $E + F = \{a + b : a \in E, b \in F\}$).*

For a sub-TRO, T, of a C^*-algebra we denote the C^*-subalgebra of $\mathcal{B}(H)$ generated by $\{ab^* : a, b \in T\}$ and $\{a^*b : a, b \in T\}$ by $\mathscr{L}_T$ and $\mathscr{R}_T$ respectively. By definition $\mathscr{L}_T T, T\mathscr{R}_T \subseteq T$ and so, taking 'cube roots' $\mathscr{L}_T T = T = T\mathscr{R}_T$.

A tripotent in a JC^*-triple E is an element u such that $u = \{u, u, u\}$ (a partial isometry if E is realised as a JC^*-subtriple of a C^*-algebra). There is an elaborate Peirce decomposition theory associated with a tripotent u in a JC^*-triple E. For our present purposes we should recall that the Peirce 2-space $\{u, \{u, E, u\}, u\}$, denoted by $E_2(u)$, is a JC^*-algebra with identity u and with involution and Jordan product defined by

$$a^\# = \{u, a, u\}, \quad a \circ b = \{a, u, b\}.$$

Concretely, when u is a partial isometry in a JC^*-subtriple E of a TRO T in $\mathcal{B}(H)$, putting $e = uu^*$ and $f = u^*u$, we have $T_2(u) = eTf$ is a C^*-algebra [30, p. 120] with involution and product given by

$$a^\# = [u, a, u], \quad a \bullet b = [a, u, b]$$

containing $E_2(u) = eEf$ as a JC^*-subalgebra, and we note that

$$a \bullet b^\# \bullet c = ab^*c = [a, b, c] \qquad \text{for all } a, b, c \in T_2(u).$$

The following is a straightforward consequence.

Lemma 2.4. *Let E, T, u, e and f be as in the above paragraph.*

(a) If $\pi\colon A \to T_2(u)$ is a *-homomorphism of C^*-algebras, then $\pi\colon A \to T$ is a TRO homomorphism.

(b) The maps $x \mapsto u^*x$ and $x \mapsto xu^*$ are *-isomorphisms from the C^*-algebra $T_2(u)$ onto C^*-algebras of $f\mathscr{R}_T f$ and $e\mathscr{L}_T e$, respectively.

For a Banach space E, we use E^{**} for its bidual, and we consider $E \subseteq E^{**}$ canonically.

Lemma 2.5. *Let A be a JC^*-subalgebra of $\mathcal{B}(H)$. Then TRO(A) is a C^*-subalgebra.*

Proof. Put $T = \mathrm{TRO}(A)$ and let B be the C^*-subalgebra generated by A in $\mathcal{B}(H)$. Then $A^{**} \subseteq T^{**} \subseteq B^{**}$ so that the TRO T^{**} contains the common identity of A^{**} and B^{**} (A^{**} has an identity and generates B^{**} as a W^*-algebra). Hence T^{**} is a C^*-subalgebra of B^{**} implying that $T = T^{**} \cap B$ is a C^*-subalgebra of B. □

Proposition 2.6. *Let E be a JC^*-subtriple of $\mathcal{B}(H)$ where E is linearly isometric to a JC^*-algebra. Then there is a partial isometry $u \in \mathcal{B}(H)$ such that u^*TRO(E) is a C^*-subalgebra of $\mathcal{B}(H)$ and TRO(E) $\to$ u^*TRO(E) is a TRO isomorphism. If E is weakly closed, then u can be chosen in E.*

Proof. Let $\phi\colon A \to E$ be a surjective linear isometry, hence a triple isomorphism, where A is a JC^*-algebra and let $\psi\colon A^{**} \to \mathcal{B}(H)$ be the weak*-continuous extension of π. Then ψ is a triple homomorphism and letting $u = \psi(1)$, where 1 is the identity of A^{**}, we have $uu^*xu^*u = x$ implying $uu^*x = xu^*u$, for all $x \in E$, and hence for all $x \in \mathrm{TRO}(E)$. By direct calculation, u^*E is a JC^*-subalgebra of $\mathcal{B}(H)$ and $\mathrm{TRO}(u^*E) = u^*\mathrm{TRO}(E)$, which is a C^*-subalgebra of $\mathcal{B}(H)$ in view of Lemma 2.4. The final statement follows from the fact that the weak closure of E contains $\psi(E^{**})$. □

If e is a projection in a von Neumann algebra W, the weak*-closure of WeW is the weak*-closed ideal of W generated by e and equals $Wc(e)$ where $c(e)$ is the central cover of e. Given projections e and f in W we use $e \sim f$ to indicate von Neumann equivalence, that is, that $e = u^*u$, $f = uu^*$ for some (partial isometry) $u \in W$.

Corollary 2.7. *The following are equivalent for a non-zero projection e in a von Neumann algebra W with $c(e) = 1$*

(a) We is linearly isometric to a JC^*-algebra;
(b) $e \sim 1$;
(c) We is TRO isomorphic to W;
(d) We is TRO isomorphic to eW.

Proof. (a) ⇒ (b). Assume (a). By Proposition 2.6 there is a partial isometry $u \in We$ such that $u^*We = u^*uWe$ is a C^*-subalgebra of W, giving $u^*u = e$, and $\pi\colon We \to u^*We$ ($x \mapsto u^*x$) is a TRO isomorphism. Since $\pi(xe) = \pi(uu^*xe)$ for $x \in W$ we have $We = uu^*We$, thus $WeW = u^*uWeW$ and so $u^*u = 1$ since $c(e) = 1$. Therefore $e \sim 1$.

(d) ⇒ (b). Let $\pi\colon We \to eW$ be a TRO isomorphism and put $\pi(e) = u$. Given $x \in We$, $\pi(x) = \pi(x)u^*u$ so that $WeW = WeW(u^*u)$ and thus $1 = u^*u$ (since $c(e) = 1$). Since $u^*u \leq e$, we have $e \sim 1$.

The implications (b) ⇒ (c) ⇒ (a) and (b) ⇒ (d) are clear. □

Below, if S is a subset of a C^*-algebra A, $S^\#$ denotes $\{x^* : x \in S\}$ and $\langle S\rangle$ denotes the (norm closed) ideal of A generated by S. Thus $\langle S\rangle = \langle S^\#\rangle$ and coincides with the norm closure of ASA.

Proposition 2.8. *Let e and f be projections in a C^*-algebra A.*

(a) If I is an ideal in the TRO eAf, then $I = e\langle I\rangle f = \langle I\rangle \cap eAf$;
(b) if J is an ideal of A, then $\langle eJe\rangle = \langle eJ\rangle = \langle Je\rangle$;
(c) there is a bijective correspondence between the ideals of eA and of eAe given by $I \leftrightarrow Ie$.

Proof. (a) If I is an ideal of eAf, then $e(AIA)f = (eAf)I^\#(eAf) \subseteq I$ so that $e\langle I\rangle f \subseteq I$, giving (a).

(b) Since $(eJ)^\# = Je$ we have $\langle eJ\rangle = \langle Je\rangle$, which is the norm closure of AeJ and of JeA. Since $A(eJe)A = (AeJ)(JeA)$, the result follows after taking norm closures.

(c) Let J_1 and J_2 be ideals of A such that $eJ_1e = eJ_2e$, Using (a) and (b), $eJ_1 = \langle eJ_1\rangle \cap eA = \langle eJ_2\rangle \cap eA = eJ_2$, which suffices. □

Proposition 2.9. *Let T be a TRO, E a JC^*-subtriple of T and let $\mathscr{S}$ denote the set of ideals I of T such that $I \cap E = \{0\}$. Then each element I of $\mathscr{S}$ is contained in a maximal element of $\mathscr{S}$.*

Proof. Let $J \in \mathscr{S}$ and let (I_λ) be a chain in $\mathscr{S}$ with $J \subset I_\lambda$ for each λ and consider the norm closure I of $\bigcup I_\lambda$, an ideal of T. Given $x \in E$ and $z \in I$, choose (z_n) in $\bigcup I_\lambda$ with $\|z_n - z\| \to 0$. Since, for each λ, $E \to T/I_\lambda$ $(x \to x + I_\lambda)$ is isometric, for each n we have $\|x\| \le \|x - z_n\|$ implying that $\|x\| \le \|x - z\|$ and hence that $\|x\| = \|x + I\|$. Therefore $I \cap E = \{0\}$, and the result follows from Zorn's lemma. □

A non-zero tripotent u in a JC^*-triple E is said to be minimal if $\{u, E, u\} = \mathbb{C}u$. If E has a predual and contains a minimal tripotent but has no non-trivial weak*-closed ideals, then it is a Cartan factor. See §5 for a detailed discussion.

3. Universal objects

We shall examine a variety of universal objects. In order to avoid later repetition we begin with general remarks. Suppose $\mathscr{C}$ is a subcategory of a category $\mathscr{D}$ and let $E \in \mathrm{Ob}(\mathscr{D})$. Let (by a slight abuse) (F, α) be said to be a universal $\mathscr{C}$-object for E, where $F \in \mathrm{Ob}(\mathscr{C})$ and $\alpha\colon E \to F$ is a $\mathscr{D}$-morphism, if each $\mathscr{D}$-morphism $\pi\colon E \to G \in \mathrm{Ob}(\mathscr{C})$ entails the existence of a unique $\mathscr{C}$-morphism $\tilde{\pi}\colon F \to G$ such that $\tilde{\pi} \circ \alpha = \pi$. In these circumstances $\tilde{\pi} = \mathrm{id}_F$ when $\pi = \alpha$. If (H, β) is another universal $\mathscr{C}$-object for E then there is a (unique) $\mathscr{C}$-isomorphism $\tilde{\beta}\colon F \to H$ with $\tilde{\beta} \circ \alpha = \beta$, so that (F, α) and (H, β) are naturally equivalent. We may write $(F, \alpha) \equiv (H, \beta)$ to signify this. If for each $E \in \mathrm{Ob}(\mathscr{D})$ a universal $\mathscr{C}$-object $(\mathscr{C}(E), \alpha_E)$ exists, then there is a natural covariant functor $\mathscr{D} \to \mathscr{C}$ sending E to $\mathscr{C}(E)$ with corresponding (and obvious) action on morphisms.

We shall establish the existence of the universal C^*-algebra and of the universal TRO of a JC^*-triple and shall discuss connections with certain associated universal objects. We proceed by exploiting the general construction [2, II.8.3] of the universal C^*-algebra $C^*(\mathscr{G}, \mathscr{R})$ of a set of generators $\mathscr{G}$ and relations $\mathscr{R}$ when the latter is realisable among operators on a Hilbert space.

We remark that in Theorem 3.1 below, (a) is a formal consequence of (b) as is the injectivity of α_E. The same goes for the statements of Corollary 3.2.

Theorem 3.1. *Let E be JC^*-triple. Up to natural equivalence with $*$-isomorphism, there is a unique pair $(C^*(E), \alpha_E)$ where $C^*(E)$ is a*

C^*-algebra and $\alpha_E : E \to C^*(E)$ is an injective triple homomorphism, with the following properties:
(a) $\alpha_E(E)$ generates $C^*(E)$ as a C^*-algebra;
(b) for each triple morphism $\pi : E \to A$, where A is a C^*-algebra, there is a unique $*$-homomorphism $\tilde{\pi} : C^*(E) \to A$ with $\tilde{\pi} \circ \alpha_E = \pi$.

Proof. Consider a set $\mathscr{G} = \{\alpha_a : a \in E\}$ of generators and a set of relations $\mathscr{R}$ consisting of the union of the sets $\{\alpha_{\lambda a+b} - \lambda\alpha_a - \alpha_b : \lambda \in \mathbb{C}, a, b \in E\}$, $\{\alpha_{\{a,b,c\}} - (\alpha_a\alpha_b^*\alpha_c + \alpha_c\alpha_b^*\alpha_a)/2 : a, b, c \in E\}$ and $\{\|\alpha_a\| \leq \|a\| : a \in E\}$. Now put $C^*(E) = C^*(\mathscr{G}, \mathscr{R})$ and $\alpha_E(a) = \alpha_a$ for each $a \in A$. By construction, $\alpha_E(E)$ generates $C^*(E)$ and $\alpha_E : E \to C^*(E)$ is a triple homomorphism.

Let $\pi : E \to A$ be a triple homomorphism into a C^*-algebra A. Since $\{\pi(a) : a \in E\}$ satisfies the relations $\mathscr{R}$, the universal property of $C^*(\mathscr{G}, \mathscr{R})$ implies the existence of $\tilde{\pi}$ as claimed in (b). Moreover, since there is an example of an injective π, α_E is injective. Uniqueness of $(C^*(E), \alpha_E)$ up to $*$-isomorphism is clear. □

Corollary 3.2. *Let E be a JC^*-triple. Up to natural equivalence with TRO isomorphism there is a unique pair $(T^*(E), \alpha_E)$ where $T^*(E)$ is a TRO and $\alpha_E : E \to T^*(E)$ is an injective triple homomorphism with the following properties:*
(a) $\alpha_E(E)$ generates $T^*(E)$ as a TRO;
(b) for each triple morphism $\pi : E \to T$, where T is a TRO, there is a unique TRO morphism $\tilde{\pi} : T^*(E) \to T$ such that $\tilde{\pi} \circ \alpha_E = \pi$.

Proof. Let $(C^*(E), \alpha_E)$ be as in Theorem 3.1 and let $T^*(E)$ denote the TRO generated by $\alpha_E(E)$ in $C^*(E)$ (giving (a)). Given a triple homomorphism $\pi : E \to T \subset \mathcal{B}(H)$, where T is a TRO, the $*$-homomorphism $C^*(E)$ to $\mathcal{B}(H)$ satisfying Theorem 3.1 (b) restricts to the required TRO homomorphism $\tilde{\pi} : T^*(E) \to T$, the uniqueness of which being implied by (a). □

Definition 3.3. Let E be a JC^*-triple. In the notation of Theorem 3.1 and Corollary 3.2, we define $C^*(E)$ and $T^*(E)$, more formally $(C^*(E), \alpha_E)$ and $(T^*(E), \alpha_E)$, to be the *universal C^*-algebra* and *universal TRO* of E. In each case, we refer to α_E as the universal embedding.

Remarks 3.4. We may define the *universal JC^*-algebra $J^*(E)$* of a JC^*triple E to be the JC^*-algebra generated by $\alpha_E(E)$ in $C^*(E)$, with universal embedding $\alpha_E : E \to J^*(E)$ in this case, by the method of proof of Corollary 3.2. Each triple homomorphism $\pi : E \to \mathcal{B}(H)$ induces a unique Jordan $*$-homomorphism $\tilde{\pi} : J^*(E) \to \mathcal{B}(H)$ with $\tilde{\pi} \circ \alpha_E = \pi$.

If T is a TRO and A is a JC^*-algebra, we may define the universal C^*-algebras $C^*_{\mathrm{TRO}}(T)$ and $C^*_{\mathrm{J}}(A)$ by the procedure of Theorem 3.1 with transparent modifications to the set $\mathscr{R}$ of relations. Thus, via the arising universal embedding of T into $C^*_{\mathrm{TRO}}(T)$, each TRO homomorphism $T \to \mathcal{B}(H)$ 'lifts' to a *-homomorphism $C^*_{\mathrm{TRO}}(T) \to \mathcal{B}(H)$. The corresponding statement is true of $C^*_{\mathrm{J}}(A)$ in terms of Jordan $*$-homomorphisms $A \to \mathcal{B}(H)$. Of these we note that $C^*_{\mathrm{J}}(A)$ has been studied in detail in [1, 9] and [10, chapter 7] in the guise of the universal C^*-algebras of the JC-algebra A_{sa} ($= \{a \in A : a^* = a\}$).

For a JC^*-triple E, all five universal objects introduced above appear in the following single statement

$$C^*_{\mathrm{J}}(J^*(E)) = C^*(E) = C^*_{\mathrm{TRO}}(T^*(E)) \tag{3.1}$$

with universal embeddings given by the inclusions $J^*(E), T^*(E) \subset C^*(E)$.

Indeed, $J^*(E)$ generates $C^*(E)$ since $\alpha_E(E)$ does. Given a Jordan homomorphism $\pi\colon J^*(E) \to \mathcal{B}(H)$, the universal property of $\alpha_E\colon E \to C^*(E)$ guarantees a $*$-homomorphism $\psi\colon C^*(E) \to \mathcal{B}(H)$ agreeing with π on $\alpha_E(E)$. Since $J^*(E)$ is a JC^*-algebra generated by $\alpha_E(E)$, ψ must agree with π on $J^*(E)$. The remaining claim has an analogous proof.

It follows from (3.1) together with [10, Theorem 7.1.8] that there is a unique $*$-antihomomorphism Φ_0, of $C^*(E) = C^*_{\mathrm{J}}(J^*(E))$ acting identically on $J^*(E)$. Since Φ_0 has order 2 and since given $y = x_1x_2^* \cdots x_{2n}^*x_{2n+1}$ with $x_1, \ldots, x_{2n+1} \in \alpha_E(E)$, we have $\Phi_0(y) = x_{2n+1}x_{2n}^* \cdots x_2^*x_1 \in T^*(E)$, it follows that $\Phi_0(T^*(E)) = T^*(E)$. Thus, the restriction Φ of Φ_0 to $T^*(E)$ is a TRO antiautomorphism of order 2. If $\psi\colon T^*(E) \to T^*(E)$ is any TRO antiautomorphism of $T^*(E)$ with $\psi \circ \alpha_E = \alpha_E$, then $\psi \circ \Phi$ is a TRO automorphism of $T^*(E)$ acting identically on $\alpha_E(E)$, so that $\psi = \Phi^{-1} = \Phi$. We summarise these remarks in the following.

Theorem 3.5. *Let E be a JC^*-triple. Then*

(a) there is a unique $*$-antiautomorphism Φ_0 of $C^*(E)$ acting identically on $\alpha_E(E)$;
(b) the restriction Φ of Φ_0 to $T^*(E)$ is the unique TRO antiautomorphism of $T^*(E)$ acting identically on $\alpha_E(E)$;
(c) Φ_0 and Φ have order 2;
(d) Φ acts identically on $J^*(E)$;
(e) $\Phi(\mathscr{L}_T) = \mathscr{R}_T$, where $T = T^*(E)$.

We refer to Φ_0 and Φ of Theorem 3.5 as the *canonical involutions* of $C^*(E)$ and $T^*(E)$, respectively.

Proposition 3.6. *If a JC^*-triple E is the sum of orthogonal ideals I and J, then $(T^*(E), \alpha_E) \equiv (T^*(I) \oplus T^*(J), \alpha_I \oplus \alpha_J)$.*

Proof. This follows from Lemma 2.3 and the universal property of $T^*(\cdot)$. □

Proposition 3.7. *Let A be a JC^*-algebra. Then*

$$(T^*(A), \alpha_A) = (C^*_{\mathrm{J}}(A), \beta_A)$$

where $\beta_A\colon A \to C^*_{\mathrm{J}}(A)$ is the universal JC^*-algebra embedding. In this identification, the canonical involution of $T^*(A)$ is the involutory $*$-antiautomorphim of $C^*_{\mathrm{J}}(A)$ acting identically on $\beta_A(A)$.

Proof. By Lemma 2.5 we have $\mathrm{TRO}(\beta_A(A)) = C^*_{\mathrm{J}}(A)$. Given a triple homomorphism $\pi\colon A \to \mathcal{B}(H)$, consider its weak*-continuous extension $\psi\colon A^{**} \to \mathcal{B}(H)$ and put $u = \psi(1)$. Then $\pi\colon A \to \mathcal{B}(H)_2(u)$ is a Jordan $*$-homomorphism into the C^*-algebra $\mathcal{B}(H)_2(u)$, and so induces a $*$-homomorphism $\tilde{\pi}\colon C^*_{\mathrm{J}}(A) \to \mathcal{B}(H)_2(u)$ with $\tilde{\pi} \circ \beta_A = \pi$. By Lemma 2.4 (a) $\tilde{\pi}\colon C^*_{\mathrm{J}}(A) \to \mathcal{B}(H)$ is a TRO homomorphism, as required. □

4. Reversibility

By definition (see [10, 2.3.2], for example) a JC^*-subalgebra A of a C^*-algebra B is called *reversible* in B if

$$a_1 a_2 \cdots a_n + a_n \cdots a_2 a_1 \in A$$

whenever $a_1, \dots, a_n \in A$. An equivalent condition is

$$a_1 a_2^* a_3 \cdots a_{2n}^* a_{2n+1} + a_{2n+1} a_{2n}^* \cdots a_2^* a_1 \in A$$

whenever $a_1, \dots, a_{2n+1} \in A$, since if A satisfies the latter condition in B, then so does A^{**} in B^{**} implying (since A^{**} has an identity) that A^{**} is reversible in B^{**} and hence that $A = A^{**} \cap B$ is reversible in B.

A JC^*-algebra is said to be *universally reversible* if its canonical image in $C^*_{\mathrm{J}}(A)$ is reversible [9] (the reversibility of A is equivalent to that of A_{sa}).

We shall now introduce the notion of reversibility for JC^*-triples.

Definition 4.1. A JC^*-subtriple E of a TRO T is said to be *reversible* in T if

$$a_1 a_2^* a_3 \cdots a_{2n}^* a_{2n+1} + a_{2n+1} a_{2n}^* \cdots a_2^* a_1 \in E$$

whenever $a_1, \dots, a_{2n+1} \in E$

We say that a JC^*-triple E is *universally reversible* if $\alpha_E(E)$ is reversible in $T^*(E)$.

By this definition and the preamble, the reversibility of a JC^*-algebra is equivalent to its reversibility as a JC^*-triple and, via Proposition 3.7, a JC^*-algebra is universally reversible as a JC^*-triple if and only if it is universally reversible as a JC^*-algebra.

Following [9], we shall proceed to obtain a serviceable characterisation of the universal TRO of a universally reversible JC^*-triple.

Lemma 4.2. *The following are equivalent for a JC^*-triple E.*

(a) E is universally reversible;
(b) if $\pi\colon E \to T$ is a triple homomorphism into a TRO, then $\pi(E)$ is reversible in T;
(c) $\alpha_E(E) = \{a \in T^*(E) : \Phi(a) = a\}$ (where Φ is the canonical involution of $T^*(E)$).

Proof. The implication (a) $\Rightarrow$ (b) is a simple consequence of the universal property, (c) $\Rightarrow$ (a) is immediate from the fact that Φ fixes each point of $\alpha_E(E)$ and (b) $\Rightarrow$ (a) is clear. Assume (a). Then the condition $b + \Phi(b) \in \alpha_E(E)$ holds for all b of the form $a_1 a_2^* \cdots a_{2n}^* a_{2n+1}$ with the $a_i \in \alpha_E(E)$, and hence holds for all $b \in T^*(E)$. Thus if $b \in T^*(E)$ with $\Phi(b) = b$, then $b = (b + \Phi(b))/2 \in \alpha_E(E)$, proving (c). $\square$

Lemma 4.3. *Let E be a universally reversible JC^*-triple and $\mathcal{I}$ an ideal of $T^*(E)$ such that $\alpha_E(E) \cap \mathcal{I} = \{0\}$ and $\Phi(\mathcal{I}) = \mathcal{I}$. Then $\mathcal{I} = \{0\}$.*

Proof. Let $b \in \mathcal{I}$. Then $b + \Phi(b) \in \alpha_E(E) \cap \mathcal{I}$ so that $\Phi(b) = -b$. Thus for $a \in \alpha_E(E)$, since $\{a, a, b\} \in I$ we have $-\{a, a, b\} = \Phi(\{a, a, b\}) = \{a, a, b\}$ and hence that $b^* a = 0$. It follows that $b^* T^*(E) = \{0\}$, giving $b = 0$. $\square$

Theorem 4.4. *Let E be a universally reversible JC^*-triple, let $\pi \colon E \to \mathcal{B}(H)$ be an injective triple homomorphism and let $\tilde{\pi} \colon T^*(E) \to \mathcal{B}(H)$ be the TRO homomorphism so that $\tilde{\pi} \circ \alpha_E = \pi$. Suppose there is a TRO antiautomorphism ψ of TRO(E) so that $\psi \circ \pi = \pi$. Then $\tilde{\pi}$ is a isomorphism. (Hence, $(T^*(E), \alpha_E) \equiv (TRO(\pi(E)), \pi)$ with canonical involution ψ.)*

Proof. Let $\mathcal{I} = \ker \tilde{\pi}$. We have $\alpha_E(E) \cap \mathcal{I} = \{0\}$ since π is injective. By assumption, the TRO homomorphism $\psi \circ \tilde{\pi} \circ \Phi$ agrees with $\tilde{\pi}$ on $\alpha_E(E)$ and hence on $T^*(E)$, giving $\psi \circ \tilde{\pi} = \tilde{\pi} \circ \Phi$. It follows that $\Phi(\mathcal{I}) \subseteq \mathcal{I}$. Hence $\mathcal{I} = \{0\}$ by Lemma 4.3. $\square$

Corollary 4.5. *Let T be a universally reversible TRO in a C^*-algebra A. Suppose T has no ideals of codimension one and there is a TRO antiautomorphism $\theta \colon A \to A$ of order 2. Then $T^*(T) = T \oplus \theta(T)$ with universal embedding $a \mapsto a \oplus \theta(a)$.*

Proof. We have an injective triple homomorphism $\pi \colon T \to T \oplus \theta(T)$ $(a \mapsto a \oplus \theta(a))$, a TRO antiautomorphism $\psi \colon T \oplus \theta(T)$ $(a \oplus \theta(b) \mapsto b \oplus \theta(a))$, such that $\psi \circ \pi = \pi$. Put $E = \pi(T)$. By Theorem 4.4, it is enough to show TRO$(E) = T \oplus \theta(T)$.

Given $a, b, c \in T$, we have $([a, b, c] - [c, b, a], 0) = [\pi(a), \pi(b), \pi(c)] - \pi([c, b, a]) \in$ TRO(E). Thus $S \oplus \{0\} \subset$ TRO(E) where S is the set $\{[a, b, c] - [c, b, a] : a, b, c \in T\}$. The norm closure of all TRO products $[a_1 \ldots a_{2n+1}]$ $(= a_1 a_2^* \cdots a_{2n}^* a_{2n+1})$, where the $a_i \in T$ and at least one of them belongs to S, is the ideal $\mathcal{J}$ of T generated by S. Since

$$[a_1 \ldots a_{2n+1}] \oplus 0 = [\pi(a_1) \cdots (a_i \oplus 0) \cdots \pi(a_{2n+1})] \in \mathrm{TRO}(E)$$

whenever $a_i \in S$ with $1 < i < 2n + 1$, and correspondingly when $i = 1$ or $i = 2n + 1$, we have that $\mathcal{J} \oplus \{0\} \subseteq$ TRO(E). Suppose $\mathcal{J} \neq T$. We may choose a non-trivial TRO homomorphism $\phi \colon T \to \mathcal{B}(H)$ vanishing on $\mathcal{J}$, and then $\phi(x)\phi(y)^*\phi(z) = \phi(z)\phi(y)^*\phi(x)$ for $x, y, z \in T$. It follows that (see [17, Proposition 6.2]) $\phi(T)$ has an ideal of codimension one, as therefore does T, a contradiction. Hence $\mathcal{J} = T$. Therefore $T \oplus \theta(T) = \mathrm{TRO}(E)$. $\square$

5. Cartan factors

The Cartan factors in question are the JW^*-triple factors possessing minimal tripotents. (The two exceptional factors of general JB^*-triple theory are beyond our scope.) An exhaustive analysis may be found in [4]. A measure of their significance is that JC^*-triples may be realised as a weak*-dense subtriple of an ℓ_∞ sum of Cartan factors [7]. The purpose of this section is to obtain concrete descriptions of the universal TROs of Cartan factors and to determine which Cartan factors are universally reversible.

Given a Hilbert space H we may define a transposition, $x \mapsto x^t$, on $\mathcal{B}(H)$ by $x^t(\xi) = \overline{x^* \left(\bar{\xi}\right)}$ where $\xi \to \bar{\xi}$ is a conjugation on H. By the dimension, $\dim(H)$, of H we shall mean the cardinality of an orthonormal basis of H, allowing infinite cardinals.

There are four types of Cartan factors, named *rectangular*, *hermitian*, *symplectic* and *spin factors*, which up to linear isometry may be realised in the following concrete forms:

rectangular, $R_{m,n}$**:** $\mathcal{B}(H)e$ where e is a projection in $\mathcal{B}(H)$ with $m = \dim(eH) \leq \dim(H) = n$

hermitian, S_n**:** $\{x \in \mathcal{B}(H) : x^t = x\}$ for $\dim(H) = n \geq 2$

symplectic, A_n**:** $\{x \in \mathcal{B}(H) : x^t = -x\}$ for $\dim(H) = n \geq 4$

spin, $\mathrm{Sp}(n)$**:** the norm closed linear span, in a unital C^*-algebra, of the identity and a *spin system* $\{s_i : i \in I\}$ of cardinality $n \geq 2$ of anti-commuting symmetries (self adjoint unitaries with $s_i s_j + s_j s_i = 0$ if $i \neq j$).

Other than the fact that $R_{2,2}$, S_2 and A_4 are spin factors (of dimensions 4, 3 and 6 respectively) the types described above are mutually exclusive. The *rank* of a Cartan factor is the cardinality of a maximal orthogonal family of minimal tripotents it contains. By definition, Hilbert spaces are the Cartan factors of rank 1. Spin factors, $R_{m,n}$ and S_n have rank 2, m and n, respectively, and A_n has rank $\left[\frac{n}{2}\right]$ when $n < \infty$ and rank n otherwise. The reflexive Cartan factors are those of finite rank.

We shall proceed on a case by case basis roughly distinguished by rank and pathology.

Hilbert spaces. Let H be a Hilbert space. Our approach involves the CAR algebra $\mathscr{A}(H)$ acting upon antisymmetric Fock space $\mathscr{F}_-(H)$ brief details of which [14, 13, 10] are rehearsed below.

We recall that $\mathscr{F}_-(H)$ is the ℓ_2 direct sum $\bigoplus_{m=0}^{\infty} \Lambda^m H$, where $\Lambda^0 H = \mathbb{C}$ and for each $m \geq 1$, $\Lambda^m H$ is the Hilbert space of normalised anti-symmetric tensors given by

$$\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_m = (m!)^{1/2} P_m(\xi_1 \otimes \cdots \otimes \xi_m),$$

P_m being the projection on $\bigotimes^m H$ given by $P_m = \frac{1}{m!} \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) u_\sigma$, where S_m is the symmetric group on m letters and u_σ is the unitary determined by

$$u_\sigma(\xi_1 \otimes \cdots \otimes \xi_m) = \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(m)}.$$

Let $\{e_i : i \in I\}$ be an orthonormal basis. With respect to a linear ordering on I $\{e_{i_1} \wedge \cdots \wedge e_{i_m} : i_1 < \cdots < i_m\}$ is an orthonormal basis of $\Lambda^m H$. When H has finite dimension n, $\Lambda^m H = \{0\}$ for all $m > n$ and has dimension $\binom{n}{m}$ whenever $m \leq n$, and $\mathscr{F}_-(H)$ has dimension 2^n. In general, for each $\xi \in H$ the annihilation operator $a(\xi) \in \mathcal{B}(\mathscr{F}_-(H))$ (the dual of the creation operator $c(\xi)$ determined by $c(\xi)\alpha = \xi \wedge \alpha$, $\alpha \in \Lambda^m H$, $m \geq 0$) depends antilinearly on ξ and satisfies

$$\|a(\xi)\| = \|\xi\|, \quad a(\xi)(\Lambda^{m+1} H) \subset \Lambda^m H \qquad \text{(for all } \xi \in H \text{ and } m \geq 0),$$

together with the canonical anticommutation relations (CAR)

$$a(\xi)a(\eta) + a(\eta)a(\xi) = 0, \quad a(\xi)a(\eta)^* + a(\eta)^* a(\xi) = \langle \eta, \xi \rangle \operatorname{id},$$

(for all $\xi, \eta \in H$), where id is the identity element in $\mathcal{B}(\mathscr{F}_-(H))$. Writing $a_i = a(e_i)$ for each $i \in I$ the CAR specialise to

$$a_i a_j + a_j a_i = 0, \quad a_i a_j^* + a_j^* a_i = \delta_{(i,j)} \operatorname{id}, \qquad \text{for all } i, j \in I.$$

Put $\tilde{H} = \{a(\xi) : \xi \in H\}$. The C^*-algebra generated by $\tilde{H}$ is the CAR algebra $\mathscr{A}(H)$, over H. Letting $\xi \mapsto \bar{\xi}$ denote the conjugation determined by $\{e_i : i \in I\}$ we have that the map

$$\psi \colon H \to \mathscr{A}(H) \quad (\xi \mapsto a\left(\bar{\xi}\right))$$

is a triple isomorphism onto the JC^*-subtriple $\tilde{H}$ of $\mathscr{A}(H)$.

Retaining these notations, we shall show that the TRO generated by $\tilde{H}$ in $\mathscr{A}(H)$ is $T^*(H)$ and that $\psi = \alpha_H$.

Theorem 5.1. *$T^*(H)$ is TRO generated by $\tilde{H}$ in $\mathscr{A}(H)$ with α_H given by $\alpha_H(\xi) = a\left(\bar{\xi}\right)$. If H has finite dimension, n, then $T^*(H)$ coincides with the ℓ_∞ sum $\bigoplus_{m=0}^{n-1} \mathcal{B}\left(\Lambda^{m+1} H, \Lambda^m H\right)$. In addition, H is universally reversible if and only if* $\dim(H) \leq 2$.

Proof. Let $\pi \colon H \to A$ be a triple homomorphism into a C^*-algebra A and put $T = \operatorname{TRO}(\pi(H))$. Then $\pi(H)$ is a Hilbert space with orthonormal basis $\{x_i : i \in I\}$ where $x_i = \pi(e_i)$ for each $i \in I$. Given i, j and k in I we have the *rules*

$$x_i x_j^* x_k + x_k x_j^* x_i = 2\{x_i, x_j, x_k\} = \delta_{(j,k)} x_i + \delta_{(i,j)} x_k,$$

so that

$$x_i x_j^* x_i = 0 \text{ and } x_i x_j^* x_j = -x_j x_j^* x_i + x_i \text{ whenever } i \neq j,$$

and $x_i x_j^* x_k = -x_k x_j^* x_i$ whenever $i \neq j \neq k$.

To prove the statement we shall suppose, first, that H has finite dimension n with $I = \{1, \ldots, n\}$ (given the standard ordering). Then T is the norm closed linear span of TRO products of the form

$$x_{i_1} x_{j_1}^* \cdots x_{j_m}^* x_{i_{m+1}}, \tag{5.1}$$

the indices i_r and j_s ranging over $1, \ldots, n$.

Via the above *rules* appropriate shuffling of TRO products reveals that T is linearly generated by the products of the form (5.1) where

$$1 \leq i_1 < \cdots < i_{m+1} \leq n, \quad 1 \leq j_1 < \cdots < j_m \leq n. \tag{5.2}$$

Formally, there are $\sum_{m=0}^{n-1} \binom{n}{m+1} \binom{n}{m}$ such products. In the special case of $\mathrm{TRO}(\tilde{H})$ we note that the products

$$a_{i_1} a_{j_1}^* \cdots a_{j_m}^* a_{i_{m+1}}, \quad \text{where the indices satisfy (5.2)} \tag{5.3}$$

are linearly independent (as follows from the linear independence of the corresponding Wick-ordered products $a_{j_1}^* \cdots a_{j_m}^* a_{i_1} \cdots a_{i_{m+1}}$ in $\mathscr{A}(H)$ [13, 10.5.88, 12.4.40]).

Since all such products lie in $\bigoplus_{m=0}^{n-1} \mathcal{B}(\Lambda^{m+1} H, \Lambda^m H)$, a dimension count shows that this space is exactly $\mathrm{TRO}(\tilde{H})$. To establish the universal property, consider the linear map $\tilde{\pi} \colon \mathrm{TRO}(\tilde{H}) \to T$ such that

$$\tilde{\pi}(a_{i_1} a_{j_1}^* \cdots a_{j_m}^* a_{i_{m+1}}) = x_{i_1} x_{j_1}^* \cdots x_{j_m}^* x_{i_{m+1}} \tag{5.4}$$

whenever the indices satisfy (5.2). Since the a_i and x_i formally satisfy the same TRO relations, as the above *rules* show, $\tilde{\pi}$ is a TRO homomorphism satisfying the requirements. This settles the case I finite.

Suppose now that I is infinite (and linearly ordered as in the preamble). It follows from the finite case settled above that the linear map $\tilde{\pi}$ defined on the (algebraic) linear span of $\{a_{i_1} a_{j_1}^* \cdots a_{j_m}^* a_{i_{m+1}} :$ the indices satisfy $(5.2), m \geq 0\}$ by the formula (5.4) obeys the TRO rules $\tilde{\pi}([x, y, z]) = [\tilde{\pi}(x), \tilde{\pi}(y), \tilde{\pi}(z)]$ and (hence) is contractive. Thus $\tilde{\pi}$ extends to a TRO homomorphism from $\mathrm{TRO}(\tilde{H})$ onto T, as required.

We shall now turn to the question of universal reversibility. If the dimension of H does not exceed 2, then it is easy to see that H is reversible in $T^*(H)$. On the other hand if H has dimension greater than 2, consider $y = x + \Phi(x)$ where $x = a_{i_1} a_{i_1}^* a_{i_2} a_{i_2}^* a_{i_3}$ with the indices belonging to I such that $i_1 < i_2 < i_3$. By repeated application of the CAR,

$$y = 2x - a_{i_1} a_{i_1}^* a_{i_3} - a_{i_2} a_{i_2}^* a_{i_3} + a_{i_3} \tag{5.5}$$

which lies in $\tilde{H}$ if the latter is reversible in $\mathrm{TRO}(\tilde{H})$. In which case, since $a_i^* y a_i^* = 0$ for all $i \neq i_3$, the right hand side of (5.5) belongs to $\mathbb{C} a_{i_3}$ contradicting the linear independence of the terms involved. $\square$

Spin factors. In view of [10, §6.2], [24, §9.3] and Proposition 3.7 the universal TRO of a spin factor is essentially known. Let V be a spin factor generated by 1 and a spin system indexed by I in a C^*-algebra A and let A_V denote the C^*-algebra generated by V in A. We assume I has cardinality at least 2. Since V is a JC^*-subalgebra of A, by Proposition 3.7 together with the discussion in §4 we have:

Lemma 5.2. *$(T^*(V),\alpha_V) = (C^*_{\mathrm{J}}(V),\beta_V)$, and V is universally reversible (as a JC^*-triple) if and only if* $\dim(V)\le 4$.

If the dimension of V is odd (*i.e.* if I has even cardinality) or infinite then $C^*_{\mathrm{J}}(V)$ is (*-isomorphic to) the CAR algebra over $\ell_2(I)$ so that $(T^*(V),\alpha_V)\equiv(A_V, V\hookrightarrow A_V)$ and $A_V\cong\mathscr{A}(\ell_2(I))$. Thus, when V has odd or infinite dimension every concrete JC^*-triple embedding of V is universal. On the other hand when I has cardinality $2n+1<\infty$, V has universal embedding in $M_{2^{n+1}}(\mathbb{C})$ with $T^*(V)=M_{2^n}(\mathbb{C})\oplus M_{2^n}(\mathbb{C})$, so that the restrictions of the natural projections $M_{2^n}(\mathbb{C})\oplus M_{2^n}(\mathbb{C})\to M_{2^n}(\mathbb{C})$ to $\alpha_V(V)$ cannot be universal.

In order to provide details sufficient for operator space considerations (see §6) we shall recall the standard representations of finite dimensional spin factors [10, §6]. Given $x\in M_2(\mathbb{C})$ and $n\ge 1$ let the n-fold tensor product $x\otimes\cdots\otimes x\in M_{2^n}(\mathbb{C})$ be denoted x^n and let $x^0\otimes y = y\otimes x^0 = y$ for all $y\in M_{2^n}(\mathbb{C})$. Let σ_1, σ_2 and σ_3 denote the Pauli spin matrices $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\begin{pmatrix}0&i\\-i&0\end{pmatrix}$, respectively.

Fixing $n\ge 1$ and putting $1=1_2^n$, we shall write

$$V_{2n}=\mathrm{span}\{1,s_1,\dots,s_{2n}\}\text{ and }V_{2n+1}=\mathrm{span}\{V_{2n}\otimes 1_2, s_{2n+1}\}$$

where $s_1=\sigma_1\otimes 1_2^{n-1}$, $s_2=\sigma_2\otimes 1_2^{n-1}$, $s_3=\sigma_3\otimes\sigma_1\otimes 1_2^{n-2}$, $s_4=\sigma_3\otimes\sigma_2\otimes 1_2^{n-2}$, …, $s_{2n-1}=\sigma_3^{n-1}\otimes\sigma_1$, $s_{2n}=\sigma_3^{n-1}\otimes\sigma_2$, and $s_{2n+1}=\sigma_3^n\otimes\sigma_1$, noting that $\{s_1,\dots,s_{2n}\}$ and $\{s_1\otimes 1_2,\dots,s_{2n}\otimes 1_2, s_{2n+1}\}$ are spin systems in $M_{2^n}(\mathbb{C})$ and $M_{2^{n+1}}(\mathbb{C})$ respectively. With universal embeddings given by the inclusions $V_{2n}\hookrightarrow M_{2^n}(\mathbb{C})$ and $V_{2n+1}\hookrightarrow M_{2^{n+1}}(\mathbb{C})$ the upshot is that

$$T^*(V_{2n})=M_{2^n}(\mathbb{C})\text{ and }T^*(V_{2n+1})=M_{2^n}(\mathbb{C})\otimes D_2=M_{2^n}(\mathbb{C})\oplus M_{2^n}(\mathbb{C}),$$

where D_2 is the diagonal subalgebra of $M_2(\mathbb{C})$.

Putting $t_{2n+1}=\sigma_3^n$ and noting that $\{s_1,\dots,s_{2n},t_{2n+1}\}$ is a spin system in $M_{2^n}(\mathbb{C})$, we shall write

$$\tilde V_{2n+1}=\mathrm{span}\{1,s_1,\dots,s_{2n},t_{2n+1}\}.$$

We define linear maps $\beta_n\colon \tilde V_{2n+1}\to\tilde V_{2n+1}$ by

$$\beta_n(x)=x\text{ if }x\in V_{2n},\quad \beta_n(t_{2n+1})=-t_{2n+1};$$

$\mu_n\colon\tilde V_{2n+1}\to V_{2n+1}$ by

$$\mu_n(x)=x\otimes 1_2\text{ if }x\in V_{2n},\quad \mu_n(t_{2n+1})=s_{2n+1};$$

$\gamma_n\colon M_{2^{n+1}}(\mathbb{C}) \to M_{2^{n+1}}(\mathbb{C})$ by

$$\gamma_n(x) = sxs,$$

where $s = 1_2^n \otimes \sigma_2$, and we note that:

Lemma 5.3. *(a) β_n and γ_n are Jordan isomorphisms.*

*(b) γ_n is a *-isomorphism of $M_{2^{n+1}}(\mathbb{C})$ with $\gamma_n(V_{2n+1}) = V_{2n+1}$ such that γ_n acts identically on $V_{2n} \otimes 1_2$ and $\gamma_n(s_{2n+1}) = -s_{2n+1}$.*

(c) $\tilde{V}_{2n+1} \to M_{2^n}(\mathbb{C}) \oplus M_{2^n}(\mathbb{C})$ $(x \mapsto x \oplus \beta_n(x))$ is the universal embedding of $\tilde{V}_{2n+1}$ into $T^*(\tilde{V}_{2n+1})$.

Rectangular factors, rank ≥ 2.

Theorem 5.4. *Let $E = \mathcal{B}(H)e$ where e is a projection in $\mathcal{B}(H)$ of rank ≥ 2. Then E is universally reversible with $T^*(E) = E \oplus E^t$ and $\alpha_E(x) = x \oplus x^t$. It may be supposed that $e^t = e$.*

Proof. To ease notation we shall denote α_E by α throughout. We separate two cases.

(a) Suppose e has finite rank m.

Choose an orthonormal basis $\{h_j : j \in J\}$ of H so that $\{h_i : i \in I\}$ is a basis of eH, I being a subset of J of cardinality m, and denote the standard matrix units $h_k \otimes h_i$ of $\mathcal{B}(H)$ by $E_{k,i}$ as (k,j) ranges over $J \times J$.

Let $(j,i) \in J \times I$. Our first claim is that, for all $r \in I$ with $r \neq i$, the elements $e_{j,i}$ and $f_{i,j}$ of $T^*(E)$ given by

$$e_{j,i} = [\alpha(E_{j,i}), \alpha(E_{j,r}), \alpha(E_{j,i})], \quad f_{i,j} = \alpha(E_{j,i}) - e_{j,i}$$

are well-defined. To see this take any subset $K \subset J$ of cardinality m and let E_K denote the linear span of $\{E_{s,r} : (s,r) \in K \times J\}$. The latter is a linear isometric copy of $M_m(\mathbb{C})$ and hence (by Proposition 3.7 and [10, 7.4.15] or [9, Corollary 4.5]) there is a TRO homomorphism $\pi_K\colon E_K \oplus E_K^t \to T^*(E)$ such that $\pi_K(x \oplus x^t) = \alpha(x)$ for all $x \in E_K$. For any $s \in K$ and $r \in I$ with $r \neq i$, it follows that

$$[\alpha(E_{s,r}), \alpha(E_{s,r}), \alpha(E_{s,i})] = \pi_K(E_{s,i} \oplus 0).$$

Thus $e_{j,i}$ and $f_{i,j}$ are well-defined with $e_{j,i} = \pi_K(E_{j,i} \oplus 0)$ and $f_{i,j} = \pi_K(0 \oplus E_{i,j})$ whenever K is chosen so that $j \in K$. (We note that $E_{i,j} = E_{j,i}^t = E_{j,i}^*$ for $(j,i) \in J \times I$.)

We next claim that the $e_{j,i}$ satisfy the same TRO relations as the $E_{j,i}$. Indeed, let $j,k,s \in J$ and $i,\ell,r \in I$. If $\{j,k,s\}$ has cardinality $\leq m$, *a fortiori* if $3 \leq m$, then the above implies (by a suitable choice of $K \subseteq J$)

$$[e_{j,i}, e_{k,\ell}, e_{s,r}] = \delta_{(i,\ell)}\delta_{(k,s)} e_{j,r}.$$

If $m = 2$, we conclude that

$$\begin{aligned}[e_{j,i}, e_{k,\ell}, e_{s,r}] &= [e_{j,i}, [e_{k,\ell}, e_{k,\ell}, e_{k,\ell}], [e_{s,r}, e_{s,r}, e_{s,r}]] \\ &= [[e_{j,i}, e_{k,\ell}, e_{k,\ell}], [e_{s,r}, e_{s,r}, e_{k,\ell}], e_{s,r}] \\ &= \delta_{(i,\ell)}\delta_{(k,s)}e_{j,r}.\end{aligned}$$

By similar methods, we can also see that the $f_{i,j}$ satisfy the same TRO relations as the $E_{i,j}$ and are orthogonal to the $e_{k,\ell}$.

Since the linear span of $\{E_{j,i} : (j,i) \in J \times I\}$ is norm dense in E, it follows that there is a TRO isomorphism $\pi\colon E \oplus E^t \to T^*(E)$ sending each $E_{j,i} \oplus E^t_{r,s}$ to $e_{j,i} + f_{r,s}$, in particular sending $E_{j,i} \oplus E^t_{j,i}$ to $\alpha(E_{j,i})$. Thus $T^*(E)$ and α may be identified as in the statement.

The corresponding canonical involution is given by $x \oplus y^t \mapsto y \oplus x^t$, the set of fixed points being $\{x \oplus x^t : x \in E\}$, proving that E is universally reversible.

(b) Let eH be infinite dimensional.

Consider $x_1, \ldots, x_{2k+1} \in E$. Since the dimension of the Hilbert subspace generated by $x_1H + \cdots + x_{2k+1}H$ cannot exceed that of eH there is a projection f in $\mathcal{B}(H)$ such that eH and fH have the same dimension and $x_1, \ldots, x_{2k+1} \in f\mathcal{B}(H)e$, which is universally reversible since linearly isometric to $\mathcal{B}(eH)$ (via Proposition 3.7 and [10, 7.4.15] or [9, Corollary 4.5]). Hence, considering the representation of JC^*-triples $f\mathcal{B}(H)e \to \alpha(f\mathcal{B}(H)e) \subseteq \alpha(E)$ and taking $y_i = \alpha(x_i)$ for $i = 1, \ldots, 2k+1$, we have $[y_1 \ldots y_{2k+1}] \in \alpha(f\mathcal{B}(H)e) \subseteq \alpha(E)$. This shows that E is universally reversible, and the result follows from Corollary 4.5. □

Hermitian and symplectic factors.

Theorem 5.5. *Let H be a Hilbert space of dimension n, possibly infinite. Let E denote*

(a) $S_n = \{x \in \mathcal{B}(H) : x^t = x\}$ where $2 \leq n \leq \infty$, or

(b) $A_n = \{x \in \mathcal{B}(H) : x^t = -x\}$ where $5 \leq n \leq \infty$.

Then E is universally reversible and $(T^*(E), \alpha_E) = (\mathcal{B}(H), \mathrm{inclusion})$.

Proof. (a) If E is the Hermitian factor the result is immediate from Proposition 3.7 and the well known facts [9, Theorem 2.2 and p. 1070] that E is universally reversible and that $C^*_{\mathrm{J}}(E) = \mathcal{B}(H)$ with universal JC^*-algebra embedding $E \hookrightarrow \mathcal{B}(H)$.

(b) Let E be the symplectic factor A_n. (We could rely on [21, Lemmas 4.1, 4.2 & 4.3] for finite $n \geq 5$, but instead we give a different argument.)

If n is even or infinite, then (see [10, pp 167–169]) there is a conjugation $\mathsf{j}\colon H \to H$ with $\mathsf{j}^2 = -1$ inducing a quaternionic structure $H_{\mathbb{H}}$ on H and a $*$-antiautomorphism $\omega\colon \mathcal{B}(H) \to \mathcal{B}(H)$ given by $\omega(x) = -\mathsf{j}x^*\mathsf{j}$ such that the JC^*-algebra, A, of fixed points of ω is Jordan $*$ isomorphic to the complexification of the JC-algebra

$\mathcal{B}(H_{\mathbb{H}})_{sa}$. By [9, Theorem 2.2 and p. 1070] we have that A is universally reversible and that $C^*_{\mathrm{J}}(A) = \mathcal{B}(H)$ with universal embedding $A \hookrightarrow \mathcal{B}(H)$. Moreover $u^*E = A$ where u is the unitary defined by $u(\xi) = -\overline{\mathrm{j}\,(\xi)}$ for $\xi \in H$. Application of Proposition 3.7 alongside the fact that $x \mapsto u^*x$ is a TRO automorphism of $\mathcal{B}(H)$ completes the proof for $n \geq 6$ even or infinite.

For $n \geq 7$ odd, a straightforward argument applies. Let $1 \leq r \leq n$, $E = A_n$, $T = M_n(\mathbb{C})$, $T_{\hat{r}} = \{x \in T : x_{r,j} = 0 = x_{j,r}$ for $1 \leq j \leq n\}$ (a $*$-subalgebra of T isomorphic to $M_{n-1}(\mathbb{C})$) and $E_{\hat{r}} = E \cap T_{\hat{r}}$, a subtriple of E triple isomorphic to A_{n-1}. Write $E_{i,j}$ $(i, j = 1, \dots, n)$ for the standard matrix units of $M_n(\mathbb{C})$, $u_{i,j} = \alpha_E(E_{i,j} - E_{j,i})$. By the preceeding paragraph we know that $T^*(E_{\hat{r}}) = T_{\hat{r}}$ and so there is a TRO homomorphism $\pi_r \colon T_{\hat{r}} \to T^*(E)$ with $\pi_r(E_{i,j} - E_{j,i}) = u_{i,j}$ for i and j both different from r, and if $k \notin \{i, j, r\}$ we have

$$[u_{i,k}, u_{i,k}, u_{i,j}] = \pi_r(E_{i,j}),$$

hence that $f_{i,j} = \pi_r(E_{i,j})$ is independent of $r \notin \{i, j\}$ (for $i \neq j$). From $[f_{i,j}, f_{k,j}, f_{k,i}] = \pi_r(E_{i,i})$ (for distinct i, j, k and r) we see that $f_{i,i} = \pi_r(E_{i,i})$ is also independent of $r \neq i$. It follows easily that $f_{i,j}$ obey the same TRO relations as $E_{i,j}$ and hence that the map $M_n(\mathbb{C}) \to T^*(E)$ $(E_{i,j} \mapsto f_{i,j})$ is a TRO isomorphism. As $x \mapsto -x^t$ is an antiautomorphism of $M_n(\mathbb{C})$ with fixed point set A_n, we have that A_n is universally reversible.

For the remaining case, $E = A_5$, a more complicated argument is required as A_4 is triple isomorphic to the spin factor V_5. Working in $A_4 \subset M_4$, write $v_{i,j} = E_{i,j} - E_{j,i}$ $(1 \leq i \neq j \leq 4)$ and $v = v_{1,3} + v_{2,4}$. Then v is unitary, $s'_1 = v^*(v_{1,3} - v_{2,4})$, $s'_2 = v^*(v_{2,3} + v_{1,4})$, $s'_3 = iv^*(v_{2,3} - v_{1,4})$, $s'_4 = v^*(v_{1,2} + v_{3,4})$, $s'_5 = iv^*(v_{1,2} - v_{3,4})$, are 5 anticommuting symmetries in v^*A_4 which together with $s'_0 = v^*v = 1_4$, span v^*A_4. This provides a Jordan $*$ isomorphism $v^*A_4 \to V_5$ (notation as in Lemma 5.3), mapping $s'_i \mapsto s_i$ $(1 \leq i \leq 4)$ and $s'_5 \mapsto t_5$. Under the triple isomorphism $A_4 \to v^*A_4 \to V_5$, the map β_2 corresponds to the isometry of A_4 exchanging $v_{1,2}$ and $v_{3,4}$ but leaving the other $v_{i,j}$ (with $i < j$) fixed. From Lemma 5.3 we may realise $\alpha_{A_4} \colon A_4 \to M_4(\mathbb{C}) \oplus M_4(\mathbb{C})$ as the linear map given by $v_{i,j} \mapsto v_{i,j} \oplus v_{i,j}$ when $i < j$ satisfy $(i, j) \notin \{(1, 2), (3, 4)\}$, $v_{1,2} \mapsto v_{1,2} \oplus v_{3,4}$, $v_{3,4} \mapsto v_{3,4} \oplus v_{1,2}$.

Let $h_{i,j}$ denote the matrix units $E_{i,j} \oplus 0 \in T^*(A_4) = M_4(\mathbb{C}) \oplus M_4(\mathbb{C})$ and $g_{i,j} = 0 \oplus E_{i,j} \in T^*(A_4)$ $(1 \leq i, j \leq 4)$. It follows that for $w_{i,j} = \alpha_{A_4}(v_{i,j})$ we have

$$w_{i,j} = (h_{i,j} - h_{j,i}) \oplus (g_{i,j} - g_{j,i}) \quad (i < j, (i, j) \notin \{(1, 2), (3, 4)\}),$$

$w_{1,2} = (h_{1,2} - h_{1,2}) \oplus (g_{3,4} - g_{4,3})$, $w_{3,4} = (h_{3,4} - h_{3,4}) \oplus (g_{1,2} - g_{2,1})$, $w_{3,4}w^*_{3,4} + w_{2,3}w^*_{2,3} + w_{2,4}w^*_{2,4} = 2(h_{2,2} + h_{3,3} + h_{4,4}) \oplus (g_{1,1} + 3g_{2,2} + g_{3,3} + g_{4,4})$, and hence that there is a polynomial $P(z)$ without constant term so that $(1_4 - h_{1,1}) \oplus 1_4 = P(w_{3,4}w_{3,4}^* + w_{2,3}w_{2,3}^* + w_{2,4}w_{2,4}^*)$. Similarly $(1_4 - h_{1,1}) \oplus 1_4 = P(w_{3,4}^*w_{3,4} + w_{2,3}^*w_{2,3} + w_{2,4}^*w_{2,4})$.

Now we turn to the triple embedding $\pi_5\colon A_4 \to T^*(A_5)$ ($x \mapsto \alpha_{A_5}(x \oplus 0_1)$) (where 0_1 means the 1×1 zero matrix) and the induced TRO homomorphism $\tilde{\pi}_5\colon T^*(A_4) \to T^*(A_5)$. For $1 \le i \ne j \le 5$, write $u_{i,j} = E_{i,j} - E_{j,i} \in A_5$ and $W_{i,j} = \alpha_{A_5}(u_{i,j}) \in T^*(A_5)$. Note $\tilde{\pi}_5(w_{i,j}) = \tilde{\pi}_5(\alpha_{A_4}(v_{i,j})) = \pi_5(v_{i,j}) = \alpha_{A_5}(u_{i,j}) = W_{i,j}$ (for $1 \le i < j \le 4$). Since the TRO homomorphism $\tilde{\pi}_5$ induces $*$-homomorphisms on the left and right C^*-algebras,

$$p = P(W_{3,4}W_{3,4}^* + W_{2,3}W_{2,3}^* + W_{2,4}W_{2,4}^*)$$

is a projection in the left C^*-algebra of $T^*(A_5)$ and

$$q = P(W_{3,4}^*W_{3,4} + W_{2,3}^*W_{2,3} + W_{2,4}^*W_{2,4})$$

is a projection in the right C^*-algebra of $T^*(A_5)$. As $u_{1,5}$ is orthogonal in A_5 to $u_{3,4}$, $u_{2,3}$ and $u_{2,4}$, we have that $W_{1,5}$ is orthogonal in $T^*(A_5)$ to $W_{3,4}$, $W_{2,3}$ and $W_{2,4}$, hence that $pW_{1,5} = 0$. Similarly $W_{1,5}q = 0$. Let $H_{i,j} = \tilde{\pi}_5(h_{i,j})$, $G_{i,j} = \tilde{\pi}_5(g_{i,j})$ ($1 \le i, j \le 4$). As $G_{i,j}G_{i,j}^*W_{1,5} = G_{i,j}G_{i,j}^*pW_{1,5} = 0$ and $W_{1,5}G_{i,j}^*G_{i,j} = W_{1,5}qG_{i,j}^*G_{i,j} = 0$, we have that $W_{1,5}$ is orthogonal to all $G_{i,j}$ ($1 \le i, j \le 4$). As $2\{u_{1,5}, u_{1,5}, u_{1,2}\} = u_{1,2}$, we can then apply α_{A_5} to conclude that $2\{W_{1,5}, W_{1,5}, H_{1,2} - H_{2,1}\} = H_{1,2} - H_{2,1} + G_{1,2} - G_{2,1}$, then multiply on the left by $(G_{1,1} + G_{2,2})(G_{1,1} + G_{2,2})^*$ and on the right by $(G_{1,1} + G_{2,2})^*(G_{1,1} + G_{2,2})$ to get $0 = G_{1,2} - G_{2,1}$, from which it follows that $G_{i,j} = 0$ for all $1 \le i, j \le 4$.

Thus the subTRO of $T^*(A_5)$ generated by $\{W_{i,j} : 1 \le i, j \le 4\}$ is a copy of $M_4(\mathbb{C})$ and we can recover the elements $H_{i,j}$ (which satisfy the same TRO rules as $E_{i,j}$ ($1 \le i, j \le 4$)) via $H_{i,j} = [W_{i,k}, W_{i,k}, W_{i,j}]$, $H_{i,i} = [H_{i,j}, H_{i,j}, H_{i,k}]$ (for distinct i, j, k ($1 \le i, j, k \le 4$)). Similar matrix units $H_{i,j}^r$ arise from all four element subsets $K_r = \{1, 2, 3, 4, 5\} \setminus \{r\}$ (now with $(i, j) \in K_r \times K_r$) and it is quite straightforward to check that $H_{i,j}^r$ is independent of the choice of $r \in \{1, 2, 3, 4, 5\} \setminus \{i, j\}$, and then that the TRO generated by $\alpha_{A_5}(A_5)$ must be a copy of $M_5(\mathbb{C})$. Finally note that that A_5 is reversible in $M_5(\mathbb{C})$ to complete the proof. □

Reversibility. Collecting information from Theorems 5.1, 5.4 and 5.5, and Lemma 5.2, we have the following classification of the Cartan factors which are universally reverible JC^*-triples.

Theorem 5.6. *Spin factors of dimension greater than 4 and Hilbert spaces of dimension greater than 2 are not universally reversible. All other Cartan factors are universally reversible.*

6. Operator space structures

We consider operator space structures of JC^*-triples arising from concrete triple embeddings in C^*-algebras, making essential use of universal TROs and establishing links with injective envelopes and triple envelopes, concentrating upon Cartan factors in the latter part of the section. Frequent use is made of the coincidence of complete isometries between TROs and TRO isomorphisms.

Definition 6.1. A *JC-operator space structure* on a JC^*-triple E is an operator space structure determined by a linear isometry from E onto a JC^*-subtriple of $\mathcal{B}(H)$. By a *JC-operator space* is meant a JC^*-triple E together with a prescribed JC-operator space structure on E.

Let E be a JC^*-triple. Two JC-operator space structures E_1 and E_2 are deemed equal, written $E_1 = E_2$, if and only if the identity map $E_1 \to E_2$ is a complete isometry. We shall make extensive use of the (norm closed) ideals $\mathcal{I}$ of $T^*(E)$ for which $\alpha_E(E) \cap \mathcal{I} = \{0\}$ which, henceforth, we shall refer to as the *operator space ideals* of $T^*(E)$. For each operator space ideal $\mathcal{I}$ of $T^*(E)$ we have the JC-operator space structure, $E_{\mathcal{I}}$, on E determined by the isometric embedding $E \to T^*(E)/\mathcal{I}$ $(x \mapsto \alpha_E(x) + \mathcal{I})$, the JC^*-triple image of which we denote $\tilde{E}_{\mathcal{I}}$. We note that $\mathrm{TRO}(\tilde{E}_{\mathcal{I}}) = T^*(E)/\mathcal{I}$. Given operator space ideals $\mathcal{I}$ and $\mathcal{J}$ of $T^*(E)$, these notations imply that $E_{\mathcal{I}} = E_{\mathcal{J}}$ if and only if $\tilde{E}_{\mathcal{I}} \to \tilde{E}_{\mathcal{J}}$ $(\alpha_E(x) + \mathcal{I} \mapsto \alpha_E(x) + \mathcal{J})$ is a complete isometry.

By the functorial properties of the universal TRO, if $\pi\colon E \to F$ is a linear isometry onto a JC^*-triple F, then $T^*(\pi)\colon T^*(E) \to T^*(F)$ preserves operator space ideals. By Proposition 2.9 we note that every operator space ideal in $T^*(E)$ is contained in a maximal operator space ideal.

Proposition 6.2. *Let E be a JC^*-triple and let $\pi\colon E \to F$ be a linear isometry onto a JC^*-subtriple F of $\mathcal{B}(H)$ (regarded as an operator subspace of $\mathcal{B}(H)$). Then there is an operator space ideal $\mathcal{I}$ of $T^*(E)$ such that $\pi\colon E_{\mathcal{I}} \to F$ is a complete isometry. Hence every JC-operator space structure on E equals $E_{\mathcal{I}}$ for some operator space ideal $\mathcal{I}$.*

Proof. Letting $\mathcal{I} = \ker \tilde{\pi}$ where $\tilde{\pi}\colon T^*(E) \to \mathrm{TRO}(F)$ is the (surjective) TRO homomorphism with $\tilde{\pi} \circ \alpha_E = \pi$, we have that $\mathcal{I}$ is an operator space ideal of $T^*(E)$ and the induced TRO isomorphism $T^*(E)/\mathcal{I} \to \mathrm{TRO}(F)$ restricts (via Lemma 2.2) to the complete isometry $\tilde{E}_{\mathcal{I}} \to F$ $(\alpha_E(x) + \mathcal{I} \mapsto \pi(x))$. □

Proposition 6.3. *Let $\pi\colon E \to F$ be a linear isometry between JC^*-triples. Let $\mathcal{I}$ and $\mathcal{J}$ be operator space ideals of $T^*(E)$ and $T^*(F)$, respectively, and let $\mathcal{K} = T^*(\pi)(\mathcal{I})$. Then $\pi\colon E_{\mathcal{I}} \to F_{\mathcal{J}}$ is a complete isometry if and only if $F_{\mathcal{J}} = F_{\mathcal{K}}$.*

Proof. The natural TRO homomorphism $T^*(E)/\mathcal{I} \to T^*(F)/\mathcal{K}$ induced by $T^*(\pi)$ restricts to the complete isometry $\tilde{E}_{\mathcal{I}} \to \tilde{F}_{\mathcal{K}}$ $(\alpha_E(x) + \mathcal{I} \mapsto \alpha_F(\pi(x)) + \mathcal{K}$, implying that $\pi\colon E_{\mathcal{I}} \to F_{\mathcal{K}}$ is a complete isometry, whence the result. ☐

Proposition 6.4. *Let E be a JC^*-triple and let $\mathcal{I}$ and $\mathcal{J}$ be operator space ideals of $T^*(E)$ with $\mathcal{I} \subseteq \mathcal{J}$. Then the identity map $\pi\colon E_{\mathcal{I}} \to E_{\mathcal{J}}$ is a complete contraction.*

Proof. $\tilde{E}_{\mathcal{I}} \to \tilde{E}_{\mathcal{J}}$ $(\alpha_E + \mathcal{I} \mapsto \alpha_E(x) + \mathcal{J})$ is the restriction of the quotient TRO homomorphism $T^*(E)/\mathcal{I} \to T^*(E)/\mathcal{J}$ and so is a complete contraction. ☐

Given a JC^*-triple E we denote by $\mathrm{MAX}_{JC}(E)$ the JC-operator space structure determined by the zero ideal in $T^*(E)$. By Proposition 6.4 the identity map $\mathrm{MAX}_{JC}(E) \to E$ is a complete contraction for every JC-operator space structure on E. By Proposition 6.3 every surjective linear isometry $\pi\colon \mathrm{MAX}_{JC}(E) \to \mathrm{MAX}_{JC}(E)$ is a complete isometry.

Recall that the injective envelope $(I(V), j)$ of an operator space V is an injective operator space $I(V)$ together with a completely isometric injection, $j\colon V \to I(V)$, such that $I(V)$ is minimal injective containing $j(V)$. Moreover $(I(V), j)$ is unique up to complete isometry and $I(V)$ may be realised as a TRO; in which case $\mathrm{TRO}(j(V))$ is said to be the *triple envelope*, $\mathcal{T}(V)$, associated with V and it possesses the following universal property [8] (see also [3, Theorem 8.3.9])

> Given an operator space $U \subset \mathcal{B}(H)$ and a complete isometry $\phi\colon U \to V$ onto V, there is a unique TRO homomorphism $\hat{\phi}\colon \mathrm{TRO}(U) \to \mathcal{T}(V)$ such that $j \circ \phi = \hat{\phi}|_U$.

Theorem 6.5. *Let $(I(E), j)$ be an injective envelope of a JC-operator space E, where $I(E)$ is a TRO. Then*

(a) $j\colon E \to I(E)$ is a triple homomorphism;
(b) there is a unique TRO homomorphism $\tilde{j}\colon T^*(E) \to \mathcal{T}(E)$ such that $\tilde{j} \circ \alpha_E = j$, and $\tilde{j}$ is surjective;
(c) $\ker \tilde{j}$ is the largest operator space ideal $\mathcal{I}$ of $T^*(E)$ with $E = E_{\mathcal{I}}$.

Proof. (a) We may suppose E is a JC^*-subtriple (and operator subspace) of $\mathcal{B}(H)$ and may choose an injective envelope W of E in $\mathcal{B}(H)$ together with a completely contractive projection P on $\mathcal{B}(H)$ with image W. There is a complete isometry $\phi\colon W \to I(E)$ such that $\phi|_E = j$ and $\phi(P([a,b,c])) = [\phi(a), \phi(b), \phi(c)]$ for all $a, b, c \in W$ [29]. In particular, for each x in E we have $\phi(\{x,x,x\}) = \phi(P([x,x,x])) = [\phi(x), \phi(x), \phi(x)]$, proving (a).
(b) This is immediate from (a) and the universal property of $T^*(E)$.

(c) Letting $\mathcal{I} = \ker \tilde{j}$, the map $\tilde{E}_{\mathcal{I}} \to j(E)$ $(\alpha_E(x) + \mathcal{I} \mapsto j(x))$ is a complete isometry as in the proof of Proposition 6.2 so that composing with $j^{-1}\colon j(E) \to E$ we have that the identity map $E_{\mathcal{I}} \to E$ is a complete isometry.

Let $\mathcal{J}$ be any operator space ideal of $T^*(E)$ such that $E_{\mathcal{J}} = E$, and let $q\colon T^*(E) \to T^*(E)/\mathcal{J}$ be the quotient map. Since $\pi\colon \tilde{E}_{\mathcal{J}} \to E$ $(\alpha_E(x) + \mathcal{J} \mapsto x)$ is a complete isometry, the above-mentioned universal property of $\mathcal{T}(E)$ implies that there is a TRO homomorphism $\hat{\pi}\colon \mathrm{TRO}(\tilde{E}_{\mathcal{J}})(= T^*(E)/\mathcal{J}) \to \mathcal{T}(E)$ such that $\pi \circ q \circ \alpha_E = j$. Hence $\hat{\pi} \circ q = \tilde{j}$ by (b), proving that $\mathcal{J} \subseteq \ker \tilde{j}$. □

We remark that in the notation of Theorem 6.5, $\ker(\tilde{j})/\mathcal{J}$ corresponds to the Šilow boundary [8] of $E_{\mathcal{J}}$.

A natural question arises from our prior discussion. Do there exist JC^*-triples E for which there are operator space ideals $\mathcal{I} \subsetneq \mathcal{J} \subset T^*(E)$ with $E_{\mathcal{I}} = E_{\mathcal{J}}$? (Equivalently, do there exist JC-operator spaces with non-zero Šilow boundary?)

As an application we shall show that there are no such examples among non-Hilbertian Cartan factors and shall completely describe the JC-operator spaces in these cases. For the finite rank non-Hilbertian Cartan factors our results may be seen as complements of those of [21] and [19, §7].

Proposition 6.6. *Let E be a Cartan factor that is not rectangular of dimension greater than one nor an even dimensional spin factor. Then $MAX_{JC}(E)$ is the unique JC-operator space structure on E.*

Proof. By the results of §5, $T^*(E)$ has no non-zero operator space ideals. In the infinite dimensional hermitian and symplectic cases this is because for every non-zero ideal $\mathcal{J}$ of $T^*(E) = \mathcal{B}(H)$, $\mathcal{K}(H) \subseteq \mathcal{J}$ and so $\alpha_E(E) \cap \mathcal{J}$ contains non-trivial compact operators. □

By [19, Proposition 7.1] (with different notation) the isometry β_n of Lemma 5.3 is not a complete isometry, an alternative proof of which is included in the next result which is a complement of [19, Proposition 7.3 (1),(2)].

Proposition 6.7. *For $n = 1, 2, \ldots$, the spin factor V_{2n+1} has precisely three JC-operator space structures, two of which are completely isometric.*

Proof. It is convenient to recall the Jordan $*$ isomorphic copy, $\tilde{V}_{2n+1}$, of V_{2n+1} and to recall the linear isometry $\beta_n\colon \tilde{V}_{2n+1} \to \tilde{V}_{2n+1}$ of Lemma 5.3 which is not the identity map but does act identically on V_{2n}. Since $V_{2n} \subset \tilde{V}_{2n+1} \subset M_{2^n}(\mathbb{C}) = T^*(V_{2n})$, it follows from Theorem 6.5 that $M_{2^n}(\mathbb{C})$ is a triple envelope of V_{2n}. If β_n is a complete contraction, it has a completely contractive extension $\phi\colon M_{2^n}(\mathbb{C}) \to M_{2^n}(\mathbb{C})$ [5, Theorem 4.1.5] which, since it acts identically on V_{2n} must be the identity map

[5, Theorem 6.2.1], as therefore must β_n, a contradiction. Hence, β_n is not completely contractive.

Since the projection maps $M_{2^n}(\mathbb{C}) \oplus M_{2^n}(\mathbb{C}) \to M_{2^n}(\mathbb{C})$ ($x \oplus y \mapsto x$, $x \oplus y \mapsto y$) are complete contractions, it follows that $\mu_n, \pi \colon \tilde{V}_{2n+1} \to M_{2^n}(\mathbb{C}) \oplus M_{2^n}(\mathbb{C})$ are not completely contractive, where $\mu_n(x) = x \oplus \beta_n(x)$ (the canonical embedding) and $\pi(\beta_n(x)) = \mu_n(x)$, for all $x \in \tilde{V}_{2n+1}$. Therefore the inclusion $\tilde{V}_{2n+1} \subset M_{2^n}(\mathbb{C})$, β_n and μ_n determine three distinct operator space structures. In the same order, these are the structures determined by the complete set of operator space ideals $\{0\} \oplus M_{2^n}(\mathbb{C})$, $M_{2^n}(\mathbb{C}) \oplus \{0\}$ and the zero ideal. The first two mentioned JC-operator space structures are completely isometric by Proposition 6.3 since $T^*(\beta_n)$ is the exchange automorphism of $T^*(\tilde{V}_{2n+1}) = M_{2^n}(\mathbb{C}) \oplus M_{2^n}(\mathbb{C})$. $\square$

Projections e and f in $\mathcal{B}(H)$ are equivalent if and only if eH and fH have the same dimension. Since an ideal $\mathcal{I}$ of $\mathcal{B}(H)$ is the norm closed linear span of its projections and since a projection $e \in \mathcal{I}$ implies $e^t \in \mathcal{I}$, because $e^t \sim e$, we have $\mathcal{I}^t = \mathcal{I}$. Given ideals $\mathcal{I}$ and $\mathcal{J}$ of $\mathcal{B}(H)$ we have $\mathcal{I} \subseteq \mathcal{J}$ or $\mathcal{J} \subseteq \mathcal{I}$ with $\mathcal{K}(H)$ being the minimal non-zero ideal. If $\mathcal{I} \subsetneq \mathcal{J}$ we may choose equivalent projections e, f in $\mathcal{J}$ with $e, f \notin \mathcal{I}$ and hence a $*$ isomorphic copy, M, of $M_2(\mathbb{C})$ with $M \subset \mathcal{J}$ and $M \cap \mathcal{I} = \{0\}$. (If $u \in \mathcal{B}(H)$ with $e = u^*u$, $f = uu^*$, then $u \in \mathcal{J}$ and the linear span of $\{e, f, u, u^*\}$ is $*$-isometric to $M_2(\mathbb{C})$.) We further note that if U is an operator subspace of $\mathcal{B}(H)$ containing a completely isometric copy of $M_2(\mathbb{C})$ and if (temporarily) U_t and U_δ denote the operator space structures induced on U by $U \to \mathcal{B}(H)$ ($x \mapsto x^t$) and $U \to \mathcal{B}(H) \oplus \mathcal{B}(H)$ ($x \mapsto x \oplus x^t$), respectively, then the the identity maps $U \to U_t$, $U \to U_\delta$ and $U_t \to U_\delta$ are not complete isometries.

Theorem 6.8. *Let E be a Cartan factor such that E is not a Hilbert space. Then $\mathcal{I} \leftrightarrow E_{\mathcal{I}}$ is a bijective correspondence between the operator space ideals of $T^*(E)$ and the JC-operator space structures of E.*

Proof. The other cases being settled by Proposition 6.6 and 6.7, we may suppose that $E = \mathcal{B}(H)e$ where e is a projection in $\mathcal{B}(H)$ of rank not less than two. By Theorem 5.4, $T^*(E) = E \oplus E^t$ with $\alpha_E(x) = x \oplus x^t$, and we may assume that $e^t = e$. The canonical involution of $T^*(E)$ is given by $\Phi(x \oplus y^t) = y \oplus x^t)$. By Proposition 2.8 the ideals of $\mathcal{B}(H)e$ are the $\mathcal{I}e$ where $\mathcal{I}$ ranges over ideals of $\mathcal{B}(H)$, forming a chain with $\mathcal{K}(H)e$ being the minimal non-zero ideal. The operator space ideals of $T^*(E)$ are $\mathcal{A} \oplus \{0\}$ and $\{0\} \oplus \mathcal{A}$ where $\mathcal{A}$ is an ideal of E.

Let $\mathcal{I}$ and $\mathcal{J}$ be ideals of $\mathcal{B}(H)$ such that $\mathcal{I}e \subsetneq \mathcal{J}e$. The $e\mathcal{I}e \subsetneq e\mathcal{J}e$ (see Proposition 2.8). Passing to $e\mathcal{B}(H)e$ ($\equiv \mathcal{B}(eH)$) remarks preceding the statement show that we may choose a $*$-isomorphic copy, M, of $M_2(\mathbb{C})$ in $e\mathcal{J}e$ such that $M \cap e\mathcal{I}e = \{0\}$, and hence that $M \cap \mathcal{I}e = \{0\}$. The quotient map $\mathcal{B}(H)e \to \mathcal{B}(H)e/\mathcal{I}e$ restricts to a $*$-isomorphism on M.

Put $\mathcal{R} = \mathcal{I}e \oplus \{0\}$ and $\mathcal{S} = \mathcal{J}e \oplus \{0\}$. We have $\Phi(\mathcal{R}) = \{0\} \oplus e\mathcal{I}$ and $\Phi(\mathcal{S}) = \{0\} \oplus e\mathcal{J}$. Suppose $\mathcal{I}e \neq \{0\}$. The maps inducing the arising JC-operator space structures are as indicated below:

$$E_{\mathcal{R}} : x \mapsto (x + \mathcal{I}e) \oplus x^t \quad ; \quad E_{\Phi(\mathcal{R})} : x \mapsto x \oplus (x^t + e\mathcal{I})$$
$$E_{\mathcal{S}} : x \mapsto (x + \mathcal{J}e) \oplus x^t \quad ; \quad E_{\Phi(\mathcal{S})} : x \mapsto x \oplus (x^t + e\mathcal{J})$$

On $e\mathcal{K}(H)e$, $E_{\mathcal{R}}$ and $E_{\Phi(\mathcal{R})}$ induce the structures determined by $x \mapsto x^t$ and $x \mapsto x$, respectively. Thus $E_{\mathcal{R}} \neq E_{\Phi(\mathcal{R})}$ ($e\mathcal{K}(H)e$ contains a $*$-isomorphic copy of $M_2(\mathbb{C})$). Similarly $E_{\mathcal{R}} \neq E_{\Phi(\mathcal{S})}$, $E_{\mathcal{S}} \neq E_{\Phi(\mathcal{R})}$ and $E_{\mathcal{S}} \neq E_{\Phi(\mathcal{S})}$.

On M, both $E_{\mathcal{R}}$ and $E_{\Phi(\mathcal{R})}$ induce the structure determined by $x \mapsto x \oplus x^t$ whilst $E_{\mathcal{S}}$ and $E_{\Phi(\mathcal{S})}$, respectively, induce the structures determined by $x \mapsto x^t$ and $x \mapsto x$. It follows that $E_{\mathcal{R}}$, $E_{\Phi(\mathcal{R})}$, $E_{\mathcal{S}}$ and $E_{\Phi(\mathcal{S})}$ are distinct.

If $\mathcal{I}e = \{0\}$, then $E_{\mathcal{R}}$ ($= E_{\Phi(\mathcal{R})}$) is determined by $x \mapsto x \oplus x^t$ which differs from $E_{\mathcal{S}}$ and $E_{\Phi(\mathcal{S})}$ on $e\mathcal{K}(H)e$. This completes the proof. $\square$

Remark 6.9. Consider a JC^*-triple E linearly isometric to $\mathcal{B}(H)e$, where e is a non-zero projection in $\mathcal{B}(H)$ not of rank one. By Theorem 6.8 and its proof, since the ideals of E are in bijective correspondence with those of $e\mathcal{B}(H)e$ ($\equiv \mathcal{B}(eH)$), we may deduce the following, which answers a question raised in [19, Remark 7.4].

(i) *If e has finite rank then E has a total of three distinct operator space strucures.*

(ii) *If eH is separably infinite then E has a total of five distinct operator space structures.*

(iii) *If eH is non-separable, and $\dim(eH) = \aleph_\alpha$, the cardinality of the distinct operator space structures on E is the cardinality of the ordinal segment $[0, \alpha]$.*

Together with Propositions 6.6 and 6.7 this provides a complete description of the JC-operator spaces of all non-Hilbertian Cartan factors. We shall extend Theorem 6.8 to Hilbertian Cartan factors in a forthcoming paper.

Corollary 6.10. *Let E be a JC^*-subtriple of a C^*-algebra, where E is a Cartan factor of type $R_{m,n}$ with m countable and $1 < m < n$. Then every triple automorphism of E is a complete isometry.*

Proof. Let $\pi : E \to E$ and $\phi : F \to E$ be surjective linear isometries, where $F = \mathcal{B}(H)e$ for a projection e in $\mathcal{B}(H)$ such that eH is separable with $1 < \dim(eH) < \dim(H)$. Let $\theta, \psi : T^*(F) \to \mathrm{TRO}(E)$ be the TRO homomorphisms such that $\theta \circ \alpha_F = \phi$ and $\psi \circ \alpha_F = \pi \circ \phi$ and let $\mathcal{I} = \ker \theta$, $\mathcal{J} = \ker \psi$. We claim that $\mathcal{I} = \mathcal{J}$. The argument being simpler when eH has finite dimension, suppose that eH is separably infinite. The five operator space ideals of $T^*(F)$ are the zero ideal, $F \oplus \{0\}$, $\{0\} \oplus F^t$, $\mathcal{K} \oplus \{0\}$ and $\{0\} \oplus \mathcal{K}^t$, where $\mathcal{K}$ denotes $\mathcal{K}(H)e$.

The corresponding quotient TROs are $F \oplus F^t$, F^t, F, $F/\mathcal{K} \oplus F^t$ and $F \oplus F^t/\mathcal{K}^t$ which are mutually non-isomorphic as TROs since F and F^t are not TRO isomorphic (as $m \neq n$) and $F/\mathcal{K}$, $F^t/\mathcal{K}^t$ do not possess minimal tripotents. Thus, as $\mathcal{I}$ and $\mathcal{J}$ are operator space ideals with $T^*(F)/\mathcal{I}$ TRO isomorphic to $T^*(F)/\mathcal{J}$, we have $\mathcal{I} = \mathcal{J}$. Hence if $\theta_{\mathcal{I}}, \psi_{\mathcal{I}} \colon T^*(F)/\mathcal{I} \to \mathrm{TRO}(E)$ denote the induced TRO isomorphisms, $\psi_{\mathcal{I}} \circ \theta_{\mathcal{I}}^{-1}$ is a TRO automorphism of $\mathrm{TRO}(E)$ extending π. □

Theorem 6.11. *Let E be a JC^*-subtriple of a C^*-algebra where E is a Cartan factor which is not a Hilbert space. We have the following.*

(a) If $(I(E), j)$ is an injective envelope of E where $I(E)$ is a TRO, then $j \colon E \to I(E)$ extends to a TRO isomorphism $\psi \colon TRO(E) \to \mathcal{T}(E)$.
(b) $E = MAX_{JC}(E)$ if and only if $(T^*(E), \alpha_E) \equiv (TRO(E), \mathrm{inclusion})$.
(c) Suppose $E = MAX_{JC}(E)$ or E is reflexive. If E is not an infinite dimensional spin factor, then $(TRO(E), \mathrm{inclusion})$ is an injective envelope of E and $TRO(E) = \mathcal{T}(E)$.

Proof. (a) In that case, by Proposition 6.2 together with Theorems 6.5 and 6.8, we have $E = E_{\mathcal{I}}$, where $\mathcal{I} = \ker \tilde{j}$, and TRO isomorphisms $\pi \colon T^*(E)/\mathcal{I} \to \mathrm{TRO}(E)$, $\theta \colon T^*(E)/\mathcal{I} \to \mathcal{T}(E)$ such that $\pi(\alpha_E(x) + \mathcal{I}) = x$, $\theta(\alpha_E(x) + \mathcal{I}) = j(x)$ for all $x \in E$. Thus $\theta \circ \pi^{-1}$ is a TRO isomorphism extending j.

(b) If $E = \mathrm{MAX}_{JC}(E)$ then by Proposition 6.2 and Theorem 6.8 the TRO homomorphism $\pi \colon T^*(E) \to \mathrm{TRO}(E)$ such that $\pi \circ \alpha_E = \mathrm{id}_E$ is an isomorphism. The converse is clear.

(c) Suppose that E is not an infinite dimensional spin factor. Then, by the results of §5, $T^*(E)$ is an injective TRO, as is $\mathrm{TRO}(E)$ when E is reflexive. Thus (c) is a consequence of (a) and (b). □

When finalising this work we learned that, by other means, D. Bohle and W. Werner have independently observed the existence of a universal TRO of a JC^*-triple and computed the universal TRO of finite dimensional JC^*-triples.

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