University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1000 Ljubljana, Slovenia

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UNIFORM PRIMENESS OF CLASSICAL BANACH LIE ALGEBRAS OF COMPACT OPERATORS

R. Strašek B. Zalar

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R. Strašek and B. Zalar

Abstract. The concept of associative ultraprime algebras was developed by M. Mathieu who also showed that it is equivalent to a certain norm estimate which we call uniform primeness. The topic was further pursued by several authors in both associative and Jordan Banach algebras. In the present note we give a formal definition of uniformly prime Banach Lie algebra and prove that classical Banach Lie algebras of compact operators, in the sense of de la Harpe, are uniformly prime.

1. Introduction

Finite dimensional Lie algebras are much studied objects because of their connection with various parts of mathematics and even physics. The theory of Banach Lie algebras however is much less developed than that of its associative or Jordan counterparts. Perhaps the only class which has complete and satisfactory structure theory is that of Lie H^*- algebras (see [Ca6],[Cu1]). They turn out to be direct sums of simple components which can be constructed from the class of Hilbert Schmidt operators. They belong therefore to a larger class of classical Banach Lie algebras of compact operators in the sense of P. de la Harpe (see [Ha1]). We chose this framework for our present work. Some additional references for other recently treated topics in Banach Lie algebras are [Au1],[Be1],[Be2],[Be3],[Vi1].

A very interesting topic in the theory of associative Banach algebras and Jordan Banach algebras is that of ultraprimeness or uniform primeness (see [Ar1], [Ca1], [Ca2], [Ca3], [Ca4], [Ca5], [Ma1], [Ma2], [Vi1]). It was introduced by M. Mathieu. Its original definition included ultrafilters hence the name ultraprimeness. An equivalent definition can be given which involves only metric estimates and could be called uniform primeness.

Suppose that $\mathcal{A}$ is Banach algebra of a given class (associative, Jordan, alternative, Lie, ...) and $A(a,b) : \mathcal{A} \longrightarrow \mathcal{A}$ an algebraic operator, suitable for a given class, depending on two parameters. By an algebraic operator we mean an operator which can be expressed as a polynomial of left and right multiplication operators. Then $\mathcal{A}$ is called uniformly prime if the estimate

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$$\|A(a,b)\| \geq \kappa \|a\| \|b\|$$

is valid for some constant $\kappa > 0$ and all $a, b \in \mathcal{A}$. For the class of associative algebras this algebraic operator is the so called elementary operator

$$A(a,b)x = M_{a,b}\, x = axb.$$

For the class of Jordan algebras the proper $A(a,b)$ is the so called Jacobson-McCrimmon operator

$$A(a,b)x = U_{a,b}\, x = a \circ (b \circ x) + b \circ (a \circ x) - (a \circ b) \circ x$$

where $\circ$ denotes the Jordan algebra product.

A simple observation, valid for all classes of Banach algebras, is the following. The concept of primeness is always the same; namely the product of nonzero ideals must be nonzero. Now if $\mathcal{A}$ is uniformly prime and $\mathcal{I}, \mathcal{J} \subset \mathcal{A}$ nonzero ideals, we can pick nonzero $a \in \mathcal{I}$ and $b \in \mathcal{J}$. Since $\mathcal{I}, \mathcal{J}$ are ideals, algebraic operator $A(a,b)$ maps $\mathcal{A}$ into $\mathcal{I}\mathcal{J}$. Since $\|A(a,b)\| \geq \kappa \|a\| \|b\| > 0$, $A(a,b)$ is nonzero. Consequently its range and thus $\mathcal{I}\mathcal{J}$ are also nonzero. This means that uniformly prime algebra is always prime. The converse is not true. Counterexamples can be given for associative, Jordan and Lie algebras using the class of Hilbert-Schmidt operators.

The purpose of our paper is to give a formal definition of uniformly prime Banach Lie algebra and to prove that classical Banach Lie algebras of compact operators are uniformly prime.

2. Definitions and main result

We denote by $(\mathcal{A}, [,])$ a Lie algebra with Lie bracket product $[x, y]$. This product satisfies two identities

$$[x, x] = 0 \quad (\text{ anticommutativity })$$

$$[[x, y], z] + [[z, x], y] + [[y, z], x] = 0 \quad (\text{ Jacobi identity }).$$

If $\mathcal{A}$ is also a Banach space, we call it a Banach Lie algebra if $\|[a, b]\| \leq 2 \|a\| \|b\|$ holds for all $a, b \in \mathcal{A}$. Note that any associative Banach algebra gives rise to a Banach Lie algebra if we define $[x, y] = xy - yx$.

Let $(\mathcal{A}, [,])$ be a Banach Lie algebra. Given any $a, b \in \mathcal{A}$ we define the algebraic operator

$$L(a,b) : \mathcal{A} \longrightarrow \mathcal{A}$$

by

$$L(a,b)x = [\, a, [b, x]] \ .$$

We say that $\mathcal{A}$ is uniformly prime if there exists a positive constant κ such that the uniform estimate

$$\|L(a,b)\| \geq \kappa \|a\| \|b\|$$

holds for all $a, b \in \mathcal{A}$. Here $L(a,b)$ is an operator acting on a Banach space and we consider its usual operator norm.

The classical Lie algebras are built from complex $n \times n$ matrices and classical Banach Lie algebras of compact operators are their natural extension to infinite dimension (infinite matrices). We give the definitions, following the classical monograph of P. de la Harpe [Ha1], page 90.

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space. Then the classical Banach Lie algebra is the space of compact operators $C(\mathcal{H})$, equipped with the operator norm and product $[X,Y] = XY - YX$. It is denoted by $gl(\mathcal{H}, C_\infty)$.

Suppose that $\mathcal{H}$ is equipped with a conjugation $x \longmapsto \overline{x}$. This is a conjugate linear mapping which is isometric and satisfies $\overline{\overline{x}} = x$. The simplest example (on $\mathbb{C}^2$) is $(z_1, z_2) \longmapsto (\overline{z_1}, \overline{z_2})$. We define the transpose of an operator $S : \mathcal{H} \longrightarrow \mathcal{H}$ by

$$S^T x = \overline{S^*\left(\overline{x}\right)}$$

where S^* is the usual adjoint. Then the classical orthogonal Banach Lie algebra of compact operators is

$$o(\mathcal{H}, ^-, C_\infty) = \left\{ S \in C(\mathcal{H}) : S^T = -S \right\},$$

equipped with operator norm and product $[X,Y] = XY - YX$. The fact that $[X,Y]$ actually lies in $o(\mathcal{H}, ^-, C_\infty)$ is not difficult to verify (see [Ha1]).

Now suppose that Hilbert space $\mathcal{H}$ is equipped with anticonjugation J. This is a conjugate linear isometric mapping $J : \mathcal{H} \longrightarrow \mathcal{H}$ satisfying $J(Jx) = -x$. The simplest example (on $\mathbb{C}^2$) is $(z_1, z_2) \longmapsto (\overline{z_2}, -\overline{z_1})$. Then the classical symplectic Banach Lie algebra of compact operators is

$$sp(\mathcal{H}, J, C_\infty) = \{ X \in C(\mathcal{H}) : JX^*J = X \}$$

equipped with operator norm and product $[X,Y] = XY - YX$. Again it can be verified that the Lie bracket of two elements from $sp(\mathcal{H}, J, C_\infty)$ is again in $sp(\mathcal{H}, J, C_\infty)$.

It is our purpose in the sequel to prove that all classical Banach Lie algebras, defined above, are uniformly prime. More precisely we shall prove

THEOREM 1. *Let $\mathcal{H}$ be a complex Hilbert space of infinite dimension. Then we have*

$$\|L(A,B)\| \geq \frac{2}{3}\left(\sqrt{2} - 1\right) \|A\| \, \|B\| \quad \text{for} \;\; gl(\mathcal{H}, C_\infty),$$

$$\|L(A,B)\| \geq \frac{1}{6} \|A\| \, \|B\| \quad \text{for } o(\mathcal{H}, ^-, C_\infty)$$

and

$$\|L(A,B)\| \geq \frac{1}{6} \|A\| \, \|B\| \quad \text{for } sp(\mathcal{H}, J, C_\infty).$$

3. Proof for rectangular case

In this section we present the proof of Theorem 1 for the case of algebra $gl(\mathcal{H}, C_\infty)$, which is modeled on a space $C(\mathcal{H})$ of compact operators. Since $L(A,B) : C(\mathcal{H}) \longrightarrow C(\mathcal{H})$ is defined by

$$L(A,B)\, X = [A, [B, X]],$$

we have

$$L(A,B)\, X = (ABX + XBA) - (AXB + BXA). \tag{1}$$

Now we use the result on elementary operators from [St1] which is stated for all standard operator algebras and is therefore valid for $C(\mathcal{H})$.

PROPOSITION 1. *(see [16]) The norm of the operator* $X \longmapsto AXB + BXA$ *is at least*

$$2\left(\sqrt{2}-1\right)\|A\|\,\|B\|\,.$$

As the proof is not short, we do not repeat it here. From Proposition 1 and identity (**??**) we obtain immediate

COROLLARY 1. *For Lie algebra* $gl(\mathcal{H}, C_\infty)$ *the following estimate*

$$\|L(A,B)\| \geq 2\left(\sqrt{2}-1\right)\|A\|\,\|B\| - \|AB\| - \|BA\|$$

is valid for all $A, B \in gl(\mathcal{H}, C_\infty)$.

It is now necessary to provide one estimate more in order to combine it with Corollary 1. This can be done using the fact that finite rank operators are dense in $C(\mathcal{H})$ and some manipulation with rank one operators. We use rather standard notation $a \otimes b$, given $a, b \in \mathcal{H}$, for operator $(a \otimes b)(x) = \langle x, b\rangle\, a$. Here $\langle x, b\rangle$ is the inner product of $\mathcal{H}$.

PROPOSITION 2. *For Lie algebra* $gl(\mathcal{H}, C_\infty)$ *the following estimate*

$$\|L(A,B)\| \geq \max\{\|AB\|, \|BA\|\}$$

is valid for all $A, B \in gl(\mathcal{H}, C_\infty)$.

PROOF. First we assume that A, B are finite rank operators. As $\mathcal{H}$ is infinite dimensional, there exists a unit vector $e \in (\operatorname{Im} A + \operatorname{Im} B)^{\perp}$. Let $a \in \mathcal{H}$ be arbitrary nonzero vector. Then the norm of the operator $X = a \otimes e$ is $\|a\|\|e\| = \|a\|$, as is well known and easy to see. As

$$XAx = \langle Ax, e\rangle\, a \in \langle \operatorname{Im} A, e\rangle\, a = 0,$$

$$XBx = \langle Bx, e\rangle\, a \in \langle \operatorname{Im} B, e\rangle\, a = 0,$$

we have $XA = XB = 0$. Since

$$L(A,B)X = ABX + XBA - AXB - BXA = ABX,$$

we have

$$\|L(A,B)\| \geq \frac{\|ABX\|}{\|X\|} = \frac{\|ABa \otimes e\|}{\|a\|} = \frac{\|ABa\|}{\|a\|}$$

and thus, as a is arbitrary, $\|AB\| \leq \|L(A,B)\|$. This imply that

$$\|BA\| = \left\|(BA)^*\right\| = \|A^*B^*\| \leq \|L(A^*, B^*)\|\,.$$

From the definition of $L(A,B)$ we can easily calculate that

$$L(A^*, B^*)X^* = (L(A,B)X)^*\,,$$

and so the norms of $L(A^*, B^*)$ and $L(A,B)$ are the same. This concludes the proof for finite ranks. Since every compact operator is a limit of a sequence of finite rank operators, and $L(A,B)$ is continuous in A and B, we can pass to the limit and easily conclude the proof in general case. $\square$

PROOF OF THE FIRST STATEMENT OF THEOREM 1. If we add up the estimates from Corollary 1 and Proposition 2, we obtain

$$\begin{aligned} 3\|L(A,B)\| &\geq 2\left(\sqrt{2}-1\right)\|A\|\,\|B\| - \|AB\| - \|BA\| + \|AB\| + \|BA\| = \\ &= 2\left(\sqrt{2}-1\right)\|A\|\,\|B\| \end{aligned}$$

and so

$$\|L(A,B)\| \geq \kappa \|A\|\,\|B\|,$$

where $\kappa = \frac{2}{3}\left(\sqrt{2}-1\right) \doteq 0.276$. □

4. Proof for orthogonal case

Let $\mathcal{A} = o(\mathcal{H}, ^{-}, C_\infty) \subset C(\mathcal{H})$. For $A, B \in \mathcal{A}$, we know the estimate of the operator $L(A,B) : C(\mathcal{H}) \longrightarrow C(\mathcal{H})$, from the previous section. This is not enough, because the definition of uniform primeness forces us to compute the norm of its restriction $L(A,B) : \mathcal{A} \longrightarrow \mathcal{A}$. We follow the general pattern of the previous section. The results of [St1] however cannot be used so we must provide the analogous estimate as follows.

PROPOSITION 3. *Let $A, B \in \mathcal{A}$. The norm of the operator $L_1(A,B) : \mathcal{A} \longrightarrow \mathcal{A}$, defined by $L_1(A,B)X = AXB + BXA$ satisfies the inequality*

$$\|L_1(A,B)\| \geq \frac{1}{2}\|A\|\,\|B\|.$$

In order to prove this we need the following simple facts about the conjugation $a \longmapsto \overline{a}$, with respect to which $o(\mathcal{H}, ^{-}, C_\infty)$ is defined.

LEMMA 1. *Let $A \in o(\mathcal{H}, ^{-}, C_\infty)$ be arbitrary. Then*

$$\left\langle Ah, \overline{k} \right\rangle = -\left\langle Ak, \overline{h} \right\rangle \quad \textit{and}$$

$$\left\langle Ah, \overline{h} \right\rangle = 0 \quad \textit{for all } h, k \in \mathcal{H}.$$

PROOF. By the definition of $o(\mathcal{H}, ^{-}, C_\infty)$ we have $Ah = -\overline{A^*\overline{h}}$. Since $h \longmapsto \overline{h}$ is conjugate linear isometry, the identity $\langle h, h \rangle = \left\langle \overline{h}, \overline{h} \right\rangle$ can be linearized into $\langle h, k \rangle = \left\langle \overline{k}, \overline{h} \right\rangle$. Thus

$$\begin{aligned} \left\langle Ah, \overline{k} \right\rangle = \left\langle h, A^*\overline{k} \right\rangle = \left\langle \overline{\overline{h}}, \overline{\overline{A^*\overline{k}}} \right\rangle = \\ = \left\langle \overline{\overline{h}}, -\overline{Ak} \right\rangle = -\left\langle Ak, \overline{h} \right\rangle. \end{aligned}$$

The second statement is only specialization of the first one to the case $h = k$. □

LEMMA 2. *Let $h, k \in \mathcal{H}$ be orthogonal unit vectors. Then the rank 2 operator $X = h \otimes \overline{k} - k \otimes \overline{h}$ lies in the Lie algebra $o(\mathcal{H}, ^{-}, C_\infty)$ and has norm 1.*

PROOF. Given any $x \in \mathcal{H}$, we have

$$\begin{aligned} X^*\overline{x} &= \left(h \otimes \overline{k} - k \otimes \overline{h}\right)^* \overline{x} = \left(\overline{k} \otimes h - \overline{h} \otimes k\right)\overline{x} = \\ &= \langle \overline{x}, h \rangle \overline{k} - \langle \overline{x}, k \rangle \overline{h} = \left\langle \overline{h}, x \right\rangle \overline{k} - \left\langle \overline{k}, x \right\rangle \overline{h} = \\ &= \overline{\left\langle x, \overline{h} \right\rangle k - \left\langle x, \overline{k} \right\rangle h} = \overline{\left(k \otimes \overline{h} - h \otimes \overline{k}\right) x} = -\overline{Xx}. \end{aligned}$$

So $X \in o(\mathcal{H}, ^{-}, C_\infty)$ by the definition.

Since h is orthogonal to k and $\overline{h}$ is orthogonal to $\overline{k}$, the norm can be easily computed. $\square$

LEMMA 3. *Let $\mathcal{H}$ be a Hilbert space and A, B bounded operators on $\mathcal{H}$. Then*

$$\sup_{\|h\|\leq 1} \|Ah\| \, \|Bh\| \geq \frac{1}{2} \|A\| \, \|B\| \, .$$

PROOF. Without loss of generality we may assume that $\|A\| = \|B\| = 1$. Take any ε with $0 < \varepsilon < 1$. Then there exist vectors $h, k \in \mathcal{H}$ such that $\|h\| = \|k\| = 1$ and $\|Ah\|, \|Bk\| > 1 - \varepsilon$. If we multiply one of them by a suitable constant, we may assume that the quantity $r := \langle Ah, Ak \rangle$ is nonnegative. Note that this clearly implies that $h + k \neq 0$. Since

$$\|A^*Ah - h\|^2 = 1 + \|A^*Ah\|^2 - 2\|Ah\|^2 \leq 2 - 2(1-\varepsilon)^2 = 2\varepsilon(2-\varepsilon) \, ,$$

we obtain

$$\|A^*Ah - h\| < \sqrt{2\varepsilon(2-\varepsilon)}.$$

This further implies

$$|\langle Ah, Ak \rangle - \langle h, k \rangle| = |\langle A^*Ah - h, k \rangle| \leq \|A^*Ah - h\| < \sqrt{2\varepsilon(2-\varepsilon)}$$

and so

$$\langle h, k \rangle = \langle h, k \rangle - \langle Ah, Ak \rangle + \langle Ah, Ak \rangle \, ,$$

$$|\langle h, k \rangle| \leq \langle Ah, Ak \rangle + \sqrt{2\varepsilon(2-\varepsilon)} = r + \sqrt{2\varepsilon(2-\varepsilon)}$$

implies

$$\frac{\|A(h+k)\|^2}{\|(h+k)\|^2} = \frac{\|Ah\|^2 + \|Ak\|^2 + 2r}{2 + 2\operatorname{Re}(\langle h, k \rangle)} >$$

$$> \frac{(1-\varepsilon)^2 + 2r}{2 + 2|\langle h, k \rangle|} > \frac{(1-\varepsilon)^2 + 2|\langle h, k \rangle| - 2\sqrt{2\varepsilon(2-\varepsilon)}}{2 + 2|\langle h, k \rangle|} =$$

$$= \frac{1 + 2|\langle h, k \rangle| - f(\varepsilon)}{2 + 2|\langle h, k \rangle|} = \frac{1}{2} + \frac{|\langle h, k \rangle|}{2 + 2|\langle h, k \rangle|} - \frac{f(\varepsilon)}{2 + 2|\langle h, k \rangle|}.$$

We therefore obtained that given $\varepsilon > 0$, there are norm one vectors $h_\varepsilon, k_\varepsilon$ such that

$$\left\| A\left(\frac{h_\varepsilon + k_\varepsilon}{\|h_\varepsilon + k_\varepsilon\|}\right) \right\|^2 > \frac{1}{2} - \frac{f(\varepsilon)}{2 + 2|\langle h_\varepsilon, k_\varepsilon \rangle|} > \frac{1}{2} - \frac{f(\varepsilon)}{2}$$

where $f(\varepsilon)$ tends to zero as $\varepsilon \to 0$. If we interchange the roles of h_ε and k_ε we also obtain

$$\left\| B\left(\frac{h_\varepsilon + k_\varepsilon}{\|h_\varepsilon + k_\varepsilon\|}\right) \right\|^2 > \frac{1}{2} - \frac{f(\varepsilon)}{2},$$

which implies

$$\left\| A\left(\frac{h_\varepsilon + k_\varepsilon}{\|h_\varepsilon + k_\varepsilon\|}\right) \right\| \left\| B\left(\frac{h_\varepsilon + k_\varepsilon}{\|h_\varepsilon + k_\varepsilon\|}\right) \right\| > \frac{1}{2} - \frac{f(\varepsilon)}{2}$$

and by sending $\varepsilon \to 0$ we conclude the proof. $\square$

REMARK 1. *The above estimate is in general the best possible. This can be seen by taking any $\mathcal{H}$ of dimension at least 2 and orthogonal unit vectors $a, b \in \mathcal{H}$. Then, given $A = a \otimes a$ and $B = b \otimes b$ we have*

$$\|Ah\|^2 + \|Bh\|^2 = |\langle h, a\rangle|^2 + |\langle h, b\rangle|^2 \leq \|h\|^2 = 1$$

and so

$$\|Ah\| \|Bh\| \leq \frac{1}{2}\left(\|Ah\|^2 + \|Bh\|^2\right) \leq \frac{1}{2}.$$

PROOF OF PROPOSITION 3. Let $h, k \in \mathcal{H}$ be orthogonal unit vectors. By Lemma 2 the operator $X = h \otimes \overline{k} - k \otimes \overline{h}$ is in $o(\mathcal{H}, ^-, C_\infty)$ and has norm 1. Consequently $\|L_1(A, B)\| \geq \|AXB + BXA\|$. By a direct computation we get

$$U := AXB + BXA = -Ah \otimes \overline{Bk} - Bh \otimes \overline{Ak} + Ak \otimes \overline{Bh} + Bk \otimes \overline{Ah}.$$

By Lemma 1 we have $\left\langle Ah, \overline{h}\right\rangle = \left\langle Bh, \overline{h}\right\rangle = 0$ and so

$$Uh = -\left\langle h, \overline{Bk}\right\rangle Ah - \left\langle h, \overline{Ak}\right\rangle Bh.$$

This implies, using Lemma 1,

$$\begin{aligned}\left\langle Uh, \overline{k}\right\rangle &= -\left\langle Bk, \overline{h}\right\rangle \left\langle Ah, \overline{k}\right\rangle - \left\langle Ak, \overline{h}\right\rangle \left\langle Bh, \overline{k}\right\rangle = \\ &= 2\left\langle Ah, \overline{k}\right\rangle \left\langle Bh, \overline{k}\right\rangle.\end{aligned}$$

Thus

$$2\left|\left\langle Ah, \overline{k}\right\rangle \left\langle Bh, \overline{k}\right\rangle\right| = \left|\left\langle Uh, \overline{k}\right\rangle\right| \leq \|U\| \leq \|L_1(A, B)\|$$

and this is valid for all k of norm 1 which are orthogonal to h.

The case when $Ah = 0$ or $Bh = 0$ is trivial, so we assume for the moment that $Ah \neq 0$ and $Bh \neq 0$. Let us denote

$$\frac{1}{\|Ah\| \|Bh\|} \langle Ah, Bh\rangle = re^{i\varphi}.$$

Then

$$k = \frac{1}{\sqrt{2(1+r)}}\left(\frac{\overline{Ah}}{\|Ah\|} + e^{-i\varphi}\frac{\overline{Bh}}{\|Bh\|}\right)$$

is a unit vector which is, by Lemma 1, orthogonal to h. By the estimate of the previous paragraph we have

$$\begin{aligned}\|L_1(A, B)\| &\geq \frac{2}{2(1+r)}\left|\left\langle Ah, \frac{Ah}{\|Ah\|} + e^{i\varphi}\frac{Bh}{\|Bh\|}\right\rangle\right| \left|\left\langle Bh, \frac{Ah}{\|Ah\|} + e^{i\varphi}\frac{Bh}{\|Bh\|}\right\rangle\right| = \\ &= \frac{1}{1+r}\left(\|Ah\| + r\|Ah\|\right)\left(\|Bh\| + r\|Bh\|\right) = \\ &= \|Ah\| \|Bh\| (1+r) \geq \|Ah\| \|Bh\|.\end{aligned}$$

Since this estimate is trivial when $Ah = 0$ or $Bh = 0$ we finally have

$$\|L_1(A, B)\| \geq \sup_{\|h\| \leq 1} \|Ah\| \|Bh\|.$$

Using Lemma 3, we conclude the proof. ☐

In order to conclude the proof of our theorem for the orthogonal case, we must prove analogous statement to Proposition 2.

Proposition 4. *For Lie algebra* $o(\mathcal{H},^{-},C_\infty)$ *the following estimate*

$$\|L(A,B)\| \geq \max\{\|AB\|,\|BA\|\}$$

is valid for all $A,B \in o(\mathcal{H},^{-},C_\infty)$.

Remark 2. *Note that* $L(A,B)$ *in this proposition is not the same as* $L(A,B)$ *in Proposition 2 but its restriction to the subspace* $o(\mathcal{H},^{-},C_\infty) \subset C(\mathcal{H})$. *We therefore cannot use Proposition 2 directly.*

Proof. As in the proof of Proposition 2 we may assume that $A,B \in o(\mathcal{H},^{-},C_\infty)$ are finite rank operators. Since $\mathcal{H}$ has infinite dimension, there is a unit vector $h \in (\operatorname{Im} A + \operatorname{Im} B)^{\perp}$. Let $k \in \mathcal{H}$ be any unit vector orthogonal to h. Then, by Lemma 2, $X = h \otimes \overline{k} - k \otimes \overline{h}$ lies in $o(\mathcal{H},^{-},C_\infty)$. Now we compute

$$\left\langle (L(A,B)X)\left(\overline{ABk}\right), h\right\rangle,$$

using (**??**) to expand $L(A,B)$. Since h is orthogonal to images of A and B, from eight terms of the previous expression, seven are zero and the only remaining one is

$$\left\langle \left(h \otimes \overline{k}\right) BA\left(\overline{ABk}\right), h\right\rangle = \left\langle BA\left(\overline{ABk}\right), \overline{k}\right\rangle \langle h,h\rangle = \left\langle BA\left(\overline{ABk}\right), \overline{k}\right\rangle.$$

Since $A,B \in o(\mathcal{H},^{-},C_\infty)$, we have $\overline{A^*(\overline{a})} = -Aa$ and $\overline{B^*(\overline{a})} = -Ba$ for all $a \in \mathcal{H}$. This implies

$$\left\langle BA\left(\overline{ABk}\right), \overline{k}\right\rangle = \left\langle A\left(\overline{ABk}\right), B^*\overline{k}\right\rangle =$$

$$= -\left\langle A\left(\overline{ABk}\right), \overline{Bk}\right\rangle = -\left\langle \overline{ABk}, A^*\overline{Bk}\right\rangle =$$

$$= \left\langle \overline{ABk}, \overline{ABk}\right\rangle = \|ABk\|^2.$$

Hence

$$\|ABk\|^2 = \left\langle (L(A,B)X)\left(\overline{ABk}\right), h\right\rangle \leq$$

$$\leq \|L(A,B)\|\,\|X\|\,\|ABk\|\,\|h\| = \|L(A,B)\|\,\|ABk\|$$

and so

$$\|ABk\| \leq \|L(A,B)\|$$

for all those unit vectors k which can be orthogonal to some nonzero vector from the subspace $(\operatorname{Im} A + \operatorname{Im} B)$. But since this subspace has infinite dimension (hence $\dim \geq 2$), any k can occur, so we have

$$\|AB\| = \sup_{\|k\|=1} \|ABk\| \leq \|L(A,B)\|.$$

Since $o(\mathcal{H},^{-},C_\infty)$ is closed for taking adjoints, we can conclude the proof in the same way as the proof of Proposition 2. □

Proof of the second statement of Theorem 1. If we denote

$$L(A,B) = -L_1(A,B) + L_2(A,B),$$

where $L_1(A,B)X = AXB + BXA$ and $L_2(A,B)X = ABX + XBA$, we obviously have

$$\|L_2(A,B)\| \leq 2\max\{\|AB\|,\|BA\|\}$$

so, by Proposition 4,

$$\|L_2(A,B)\| \leq 2\,\|L(A,B)\|\,.$$

Thus, by Proposition 3,

$$\frac{1}{2}\,\|A\|\,\|B\| \leq \|L_1(A,B)\| \leq \|L(A,B)\| + \|L_2(A,B)\| \leq 3\,\|L(A,B)\|\,.$$

□

5. Proof for symplectic case

Let J denote an anticonjugation on an infinite dimensional complex Hilbert space $\mathcal{H}$. Recall that $J^2 = -Id$ and J is conjugate linear isometry. Recall also that $sp(\mathcal{H}, J, C_\infty) = \{X \in C(\mathcal{H}) : JX^*J = X\}$. Let us denote $X^S = JX^*J$.

Lemma 4. *Given any orthogonal unit vectors $h, k \in \mathcal{H}$, the rank two operator $X = h \otimes Jk + k \otimes Jh$ lies in $sp(\mathcal{H}, J, C_\infty)$ and has norm 1.*

Proof. The statement about the norm follows from $\|h\| = \|k\| = 1$ and $\langle h, k\rangle = \langle Jk, Jh\rangle = 0$. Given $a, b \in \mathcal{H}$, we have

$$(a \otimes b)^S x = J\,(a \otimes b)^*\,Jx = J\,(b \otimes a)\;Jx =$$

$$= J(\langle Jx, a\rangle\;b) = \overline{\langle Jx, a\rangle}\,Jb = \langle a, Jx\rangle\;Jb =$$

$$= -\langle JJa, Jx\rangle\;Jb = -\langle x, Ja\rangle\;Jb = -\,(Jb \otimes Ja)\;x$$

so

$$(a \otimes b)^S = -Jb \otimes Ja.$$

Thus

$$(h \otimes Jk + k \otimes Jh)^S = -J^2 k \otimes Jh - J^2 h \otimes Jk = k \otimes Jh + h \otimes Jk.$$

□

Proposition 5. *Let $A, B \in sp(\mathcal{H}, J, C_\infty)$. Then*

$$\|L(A,B)\| \geq \max\{\|AB)\|\,,\|BA)\|\}\,.$$

Proof. We can proof this in almost the same way as Proposition 4. First we can pass to the case when A, B have finite rank. Then we choose $h \in (\operatorname{Im} A + \operatorname{Im} B)^\perp$ and k orthogonal to h, both of norm 1. We take $X = k \otimes Jh + h \otimes Jk$ and compute expression $\left\langle (L(A,B)\,X)\left(\overline{ABk}\right), h\right\rangle$. Since almost all terms are zero, we have

$$\left\langle (L(A,B)\,X)\left(\overline{ABk}\right), h\right\rangle = \|ABk\|^2\,,$$

from which $\|ABk\| \leq \|L(A,B)\|$ follows. All other steps are the same as in the proof of Proposition 4. □

A result for $sp(\mathcal{H}, J, C_\infty)$ which is parallel to Proposition 3 for $o(\mathcal{H}, {}^-, C_\infty)$ cannot be proved in the same way. The main reason is that proof of Proposition 3 rests on the fact that for $A \in o(\mathcal{H}, {}^-, C_\infty)$ and conjugation $x \longmapsto \overline{x}$ we have $\langle Ax, \overline{x}\rangle = 0$. This is not true for anticonjugation J. Namely $\langle Ax, Jx\rangle$ need not be zero for $A \in sp(\mathcal{H}, J, C_\infty)$.

PROPOSITION 6. *Let* $A, B \in sp(\mathcal{H}, J, C_\infty)$ *and denote*

$$L_1(A,B) : sp(\mathcal{H}, J, C_\infty) \longrightarrow sp(\mathcal{H}, J, C_\infty)$$

the operator given by

$$L_1(A,B)\, X = AXB + BXA.$$

Then we have

$$\|L_1(A,B)\| \geq \frac{1}{2} \|A\| \, \|B\| \, .$$

PROOF. Take any unit vector $h \in \mathcal{H}$. Since Jh is also a unit vector, the operator $X = h \otimes Jh$ has norm 1. Since $X^S = -\left(J^2 h \otimes Jh\right) = h \otimes Jh = X$, we have $X \in sp(\mathcal{H}, J, C_\infty)$. We can therefore compute $L_1(A,B)\, X$, which is

$$Ah \otimes B^* Jh + Bh \otimes A^* Jh.$$

Since $JA^*J = A$, we have $A^*J = -JA$ and $B^*J = -JB$. This gives

$$L_1(A,B)\, X = -Ah \otimes JBh - Bh \otimes JAh.$$

Now we compute $\langle L_1(A,B)\, X\, JBh, Ah\rangle$ which turns out to be

$$-\|Ah\|^2 \|Bh\|^2 - |\langle Ah, Bh\rangle|^2 \, .$$

Thus

$$\|Ah\|^2 \|Bh\|^2 \leq |\langle L_1(A,B)\, X\, JBh, Ah\rangle| \leq$$

$$\leq \|L_1(A,B)\| \, \|Ah\| \, \|Bh\|$$

and so

$$\|L_1(A,B)\| \geq \sup_{\|h\|=1} \|Ah\| \, \|Bh\| \, .$$

An application of Lemma 3 concludes the proof. □

The third statement of Theorem 1 now follows from Proposition 5 and Proposition 6 in exactly the same way as the second statement of Theorem follows from Proposition 3 and Proposition 4.

6. A counterexample and concluding remarks

In P. de la Harpe book [Ha1] Lie algebras built from Schatten classes $C_p(\mathcal{H})$ equipped with corresponding $p-$norms are also considered. They provide examples of Banach Lie algebras which are prime but not uniformly prime. More precisely, we have

OBSERVATION 1. *Let* $\mathcal{H}$ *be an infinite dimensional complex Hilbert space and* $C_2(\mathcal{H})$ *the class of Hilbert-Schmidt operators equipped with the norm* $\|A\|_2 = \sqrt{Tr\,(AA^*)}$. *Let* $gl(\mathcal{H}, C_2)$ *denotes a Banach Lie algebra modeled on this space with the product* $[A,B] = AB - BA$. *Then* $gl(\mathcal{H}, C_2)$ *is prime but not uniformly prime.*

PROOF. Since $C_2(\mathcal{H})$ is norm dense in $C(\mathcal{H})$ Theorem 1 implies that for nonzero $A, B \in C_2(\mathcal{H})$, operator $L(A,B) : C_2(\mathcal{H}) \longrightarrow C_2(\mathcal{H})$ is also nonzero. This clearly implies that $gl(\mathcal{H}, C_2)$ is prime Lie algebra.

In order to show that $gl(\mathcal{H}, C_2)$ is not uniformly prime, it suffices to find a sequence of elements such that $\|P_n\|_2 \to \infty$ while

$$\|L(P_n, P_n)\| = \sup \frac{\|L(P_n, P_n)\, X\|_2}{\|X\|_2} \leq 4$$

for all n. This clearly makes estimate

$$4 \geq \|L(P_n, P_n)\| \geq \kappa \|P_n\|_2^2$$

impossible for any positive κ.

In fact, we can take P_n to be orthogonal projection on $n-$dimensional subspace of $\mathcal{H}$. Then $P_n = P_n^*$ and $P_n^2 = P_n$ so $\|P_n\|_2^2 = \sqrt{Tr\,(P_n)} = \sqrt{n} \longrightarrow \infty$.

Given any $X \in gl(\mathcal{H}, C_2)$ we have $L(P_n, P_n)\; X = P_n X + X P_n - 2 P_n X P_n$. If we consider the operator $L_1(X) = P_n X$, we have

$$\|L_1(X)\|_2^2 = Tr\,(P_n X X^* P_n) = Tr\,(X^* P_n X)\,.$$

Since $X^*X = X^* P_n X + X^*(1 - P_n) X$ and both operators on the right hand side are positive, we have

$$\|L_1(X)\|_2^2 = Tr\,(X^* P_n X) \leq Tr\,(X^* X) = \|X\|_2^2$$

and so $\|L_1\| \leq 1$. In a similar way one can prove that $L_2(X) = X P_n$ and $L_3(X) = P_n X P_n$ are also bounded in norm by 1, so we have $\|L(P_n, P_n)\| \leq 4$ for all n. $\square$

As our final remark we wish to note that we were not able to find operators where the proved constants $\frac{2}{3}\left(\sqrt{2} - 1\right)$ and $\frac{1}{6}$ would actually be attained, so we consider the problem of determining the best constant of uniform (ultra) primeness for Lie algebras $gl(\mathcal{H}, C)$, $o(\mathcal{H}, {}^-, C_\infty)$ and $sp(\mathcal{H}, J, C_\infty)$ still open.

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E-mail address: rok.strasek@uni-mb.si

E-mail address: borut.zalar@uni-mb.si