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Alternative and Minimax Theorems beyond Vector Spaces

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The main results of this paper concern the minimax equality without algebraic structure of the underlying spaces. They include some classical minimax theorems as special cases and are independent of many other recent results of the same type. The proofs of our minimax theorems are based on some special alternative theorems established under some general connectedness conditions. © 2001 Elsevier Science

1. INTRODUCTION

Since the first proof of the minimax theorem due to J. von Neumann, numerous authors have extended the original result in several ways. The different versions of the theorem are based on various topological and algebraic conditions.

For most of them the convexity is a basic assumption, both for the strategy spaces and for the payoff function. The well-known minimax theorems of H. Nikaido [7] and M. Sion [9] exemplify this approach.

However, there are two points of discussion concerning the role of convexity in the minimax theory.

In many choice problems, particularly in games, the algebraic operations could not be naturally defined on the alternatives set, so that the vector space structure is not always desirable.

On the other hand, several authors have observed that the minimax equality may be obtained by some connectedness properties, so that, technically, the convexity seems to be unnecessary.

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One of the main themes in the literature around the minimax theorem originated in the seminal paper of Ky Fan [1]. There Fan proved the first minimax result without linear structure of the underlying spaces, but the payoff function was assumed to be concave-convex-like. Like classical convexity or the quasi-convexity, this new concept arose from the primary idea of the convexity of the preferences. A slight extension of Fan's convexity appears later in Konig [5] and Terkelsen [12]. More recently, Kindler [3] introduced the concept of *convexity with respect to a mean function*, which includes many convexity-type properties. Several "pure topological" minimax theorems, without convexity of the payoff function, are found in Wu [14], Tuy [13], Stacho [10], Stefanescu [11], and Yu and Yuan [15].

The main result of this paper deals with the minimax equality in the form

$$\sup_{x\in X}\inf_{y\in Y} f(x, y) = \inf_{y\in Y}\sup_{x\in X} f(x, y). \tag{1}$$

One does not require either the linear structure of the strategy spaces or the convexity (concavity) of the payoff function. The method of proof is new and makes use of topological properties of the multivalued functions (here called *correspondences*).

2. CONVEXITY WITHOUT VECTOR SPACE STRUCTURE AND RELATED PROPERTIES

As we have already mentioned in the Introduction, the reference point of our approach is Fan's concept of convexity (concavity) and its implications for the minimax theory.

In the present section we introduce a new property which is directly comparable to the Fan convexity and its earlier extensions. Other new properties refer rather to the connectedness, but in a further discussion we will prove some surprising connections with convexity.

Let Y be an arbitrary nonvoid set and let $\mathscr{G} = (g_\theta)_{\theta\in\Theta}$ be a family of real-valued functions defined on Y. By $\mathscr{D}_{[0,1]}$ one denotes the set of all dyadic numbers in the interval $[0, 1]$.

Definition 1. Let $t \in [0, 1]$. The family $\mathscr{G}$ is said to be t-convexlike on Y, if for any $y^1, y^2 \in Y$, there exists $y^0 \in Y$ such that

$$g_\theta(y^0) \le t g_\theta(y^1) + (1-t) g_\theta(y^2) \tag{2}$$

for all $\theta \in \Theta$.

$\mathscr{G}$ is t-concavelike on Y if $-\mathscr{G} = \{-g | g \in \mathscr{G}\}$ is t-convexlike on Y.

Definition 2. $\mathscr{G}$ is said to be convexlike on Y if it is t-convexlike on Y for every $t \in [0, 1]$. $\mathscr{G}$ is concavelike on Y if it is t-concavelike on Y for every $t \in [0, 1]$.

The first author to use t-convexlike functions (for $t = \frac{1}{2}$) seems to be König. The same property appears in the minimax theorem due to Terkelsen.

One can easily verify that

Proposition 2.1. *If $\mathscr{G}$ is $\frac{1}{2}$-convexlike ($\frac{1}{2}$-concavelike) it is t-convexlike (t-concavelike) for every $t \in \mathscr{D}_{[0,1]}$.*

Moreover, in a special topological framework one obtains the following equivalence:

Proposition 2.2. *Let Y be a compact in a topological (Hausdorff) space.*

(a) *A family $\mathscr{G}$ of lower semi-continuous functions is convexlike if and only if it is t-convexlike for every $t \in \mathscr{D}_{[0,1]}$.*

(b) *A family $\mathscr{G}$ of upper semi-continuous functions is concavelike if and only if it is t-concavelike for every $t \in \mathscr{D}_{[0,1]}$.*

Now, let us define a more general property.

Definition 3. $\mathscr{G}$ is said to be weakly convexlike (w.c.l.) on Y if

$$\inf_{y\in Y} \sup_{\theta\in\Theta} g_\theta(y) \le \sup_{\theta\in\Theta}[t g_\theta(y^1) + (1-t) g_\theta(y^2)], \tag{3}$$

for any $y^1, y^2 \in Y$ and any $t \in \mathscr{D}_{[0,1]}$.

$\mathscr{G}$ is weakly concavelike on Y if $-\mathscr{G}$ is weakly convexlike on Y, i.e.,

$$\sup_{y\in Y} \inf_{\theta\in\Theta} g_\theta(y) \ge \inf_{\theta\in\Theta}[t g_\theta(y^1) + (1-t) g_\theta(y^2)], \tag{4}$$

for any y^1, $y^2 \in Y$, and any $t \in \mathscr{D}_{[0,1]}$.

Remark 1. Since the right-hand members in (3) and (4) are continuous functions of t, one can replace $\mathscr{D}_{[0,1]}$ in the above definition with the whole interval $[0, 1]$ (or, with any dense subset of $[0, 1]$.)

Obviously, if $\mathscr{G}$ is $\frac{1}{2}$-convexlike (or, convexlike), it must be w.c.l. Moreover, one can argue that, in the framework of the two-person zero-sum game theory, the latter property seems to be more adequate than the former.

In its typical representation, a two-person zero-sum game consists of the triple (X, Y, f), where X and Y are the strategy sets of the two players and $f\colon X \times Y \mapsto \mathbf{R}$ is the utility function of the first player. As the expression of the convexity of preferences, Fan's minimax theorem assumed that the family of functions $(f(.,y))_{y\in Y}$ is concavelike on X, i.e., "For any $t \in [0,1]$ and any two (pure) strategies $x^1, x^2 \in X$, there exists $x^0 \in X$ such that $f(x^0,y) \geq tf(x^1,y) + (1-t)f(x^2,y)$, for all $y \in Y$."

Or, in terms of strategical dominance, this means that any mixed strategy whose support consists of two points is dominated by some pure strategy. Obviously, this is a strong and rather unrealistic assumption, because, generally, the mixed strategies make the players better than the pure strategies.

In contrast with this situation, if $(f(.,y))_{y\in Y}$ is weakly concavelike one assumes only that the maximin value of the first player $(\sup_{x\in X}\inf_{y\in Y} f(x,y))$ is at least equal to his minimum payoff guaranteed by any two-point mixed strategy. Moreover, if the minimax equality holds (equivalently, the game admits pure equilibrium), this condition is necessarily satisfied. Indeed, in this case one has

$$\sup_{x\in X}\inf_{y\in Y} f(x,y) = \inf_{y\in Y}\sup_{x\in X} f(x,y),$$

and, since $\sup_{x\in X} f(x,y) \geq [tf(x^1,y) + (1-t)f(x^2,y)]$, for all $t \in [0,1]$, $x^1, x^2 \in X$, and $y \in Y$, it results that

$$\sup_{x\in X}\inf_{y\in Y} f(x,y) \geq \inf_{y\in Y}[tf(x^1,y) + (1-t)f(x^2,y)].$$

The following two examples show that a family of functions can be weakly convexlike (weakly concavelike) but not t-convexlike (t-concavelike).

EXAMPLE 1. $\Theta = \mathbf{Z}^*$ (the set of all non-zero integers), $Y = \{\frac{1}{z} | z \in \mathbf{Z}\}$, $g_\theta(y) = \frac{y}{\theta}$.

One can easily verify that $\mathscr{G} = (g_\theta)_{\theta\in\Theta}$ is weakly convexlike. Obviously, $\inf_{y\in Y}\sup_{\theta\in\Theta} g_\theta(y) = 0$. On the other hand, if $t \in [0,1]$ and $y^1, y^2 \in Y$, then $\sup_{\theta\in\Theta}[tg_\theta(y^1) + (1-t)g_\theta(y^2)] = |ty^1 + (1-t)y^2| \geq 0$.

$\mathscr{G}$ is not $\frac{1}{2}$-convexlike. Indeed, $\frac{1}{2}g_\theta(-1) + \frac{1}{2}g_\theta(1) = 0$, but there are no $y \in Y$ such that $g_\theta(y) \leq 0$ for all $\theta \in \Theta$.

Note also that $\mathscr{G}$ is weakly concavelike but not $\frac{1}{2}$-concavelike.

EXAMPLE 2. $\Theta = Y = \mathbf{N}^*$ (the set of all positive integers), and

$$g_\theta(y) = \begin{cases} y & \text{if } \ y < \theta \\ \frac{1}{\theta} & \text{if } \ y \geq \theta. \end{cases}$$

$\mathscr{G} = (g_\theta)_{\theta\in\Theta}$ is weakly concavelike. Obviously, $\sup_{y\in Y}\inf_{\theta\in\Theta} g_\theta(y) = 1$ and $\inf_{\theta\in\Theta}[tg_\theta(y^1) + (1-t)g_\theta(y^2)] = \min\{1/y^1,\ ty^1 + (1-t)/y^2\} \leq 1/y^1 \leq 1$, whenever $t \in [0,1]$ and $y^1, y^2 \in Y$, $y^1 \leq y^2$.

One can see that there are no $y \in Y$ such that $g_\theta(y) \geq \frac{1}{2}g_\theta(2) + \frac{1}{2}g_\theta(3)$, for all $\theta \in \Theta$, so that $\mathscr{G}$ is not $\frac{1}{2}$-concavelike.

In the following $\mathcal{F}(A)$ will denote the class of all finite subsets of a set A.

If $\alpha \in \mathbf{R}$, denote by $Y_\alpha(\theta) = \{y \in Y | g_\theta(y) \leq \alpha\}$, for any $\theta \in \Theta$ and by $Y_\alpha(\Theta') = \bigcap_{\theta\in\Theta'} Y_\alpha(\theta)$, for any $\Theta' \in \mathcal{F}(\Theta)$ $(Y_\alpha(\emptyset) = Y)$.

Furthermore, if $t \in [0, 1]$, $\theta^1, \theta^2 \in \Theta$, and $\alpha \in \mathbf{R}$, define the following three subsets of Y:

$$C_{t,\alpha}(\theta^1, \theta^2) = \{y \in Y \mid tg_{\theta^1}(y) + (1-t)g_{\theta^2}(y) \leq \alpha\}$$

$$W_{t,\alpha}(\theta^1, \theta^2) = \{y \in Y \mid tg_{\theta^1}(y) + (1-t)g_{\theta^2}(y) < \alpha\}$$

$$A_{t,\alpha}(\theta^1, \theta^2) = \{y \in Y \mid tg_{\theta^1}(y) + (1-t)g_{\theta^2}(y) = \alpha\}.$$

Obviously, these sets are contained in the union $Y_\alpha(\theta^1) \cup Y_\alpha(\theta^2)$. The properties defined below ask for them to be enclosed in one of $Y_\alpha(\theta^1)$, $Y_\alpha(\theta^2)$, whenever these two sets are disjoint.

DEFINITION 4. $\mathcal{G} = (g_\theta)_{\theta\in\Theta}$ is said to be α-affine-connected (α-a.c.) on Y if, for any $t \in [0, 1]$, at most one of the two sets $C_{t,\alpha}(\theta^1, \theta^2) \cap Y_\alpha(\theta^i)$, $i = 1, 2$ is non-empty, whenever $\inf_{y\in Y} \max_{i=1,2} g_{\theta^i}(y) > \alpha$, for $\theta^1, \theta^2 \in \Theta$.

Two weaker versions of this property are also defined here

DEFINITION 5. $\mathcal{G}$ is said to be weakly α-affine-connected (α-w.a.c.) (respectively, almost α-affine-connected (α-a.a.c.)) on Y, if for any $t \in [0, 1]$, θ^1, $\theta^2 \in \Theta$, at most one of the two sets $W_{t,\alpha}(\theta^1, \theta^2) \cap Y_\alpha(\theta^i)$, $i = 1, 2$ $(A_{t,\alpha}(\theta^1, \theta^2) \cap Y_a(\theta^i)$, $i = 1, 2)$ is non-empty, whenever $\inf_{y\in Y} \max_{i=1,2} g_{\theta^i}(y) > \alpha$, for $\theta^1, \theta^2 \in \Theta$.

PROPOSITION 2.3. *If $\mathcal{G}$ is convexlike, then it must be α-affine-connected on $Y_\alpha(\Theta')$, for every $\alpha \in \mathbf{R}$ and $\Theta' \in \mathcal{F}(\Theta)$.*

Proof. Let there be $\Theta' \in \mathcal{F}(\Theta)$, $\theta^1, \theta^2 \in \Theta$, and $\alpha \in \mathbf{R}$ such that

$$\inf_{y\in Y_\alpha(\Theta')} max_{i=1,2} g_{\theta^i}(y) > \alpha. \tag{5}$$

Put Y_i for $Y_\alpha(\Theta' \cup \{\theta^i\})$, $i = 1, 2$. Clearly, $Y_1 \cap Y_2 = \emptyset$.

By way of contradiction, let us assume that there exist $t \in [0, 1]$, $y^1 \in Y_1$, $y^2 \in Y_2$ such that

$$tg_{\theta^1}(y^1) + (1-t)g_{\theta^2}(y^1) \leq \alpha \tag{6}$$

and

$$tg_{\theta^1}(y^2) + (1-t)g_{\theta^2}(y^2) \leq \alpha. \tag{7}$$

Obviously, $t \in (0, 1)$. (Otherwise, from (6) and (7) one contradicts (5)).

Since $g_{\theta^1}(y^1) \leq \alpha$ and $g_{\theta^1}(y^2) > \alpha$, one can find $t_0 \in [0, 1]$ such that

$$t_0 g_{\theta^1}(y^1) + (1-t_0)g_{\theta^1}(y^2) = \alpha. \tag{8}$$

Multiplying (6) and (7) by t_0 (respectively, $1 - t_0$) and summing the resulting inequalities, one obtains

$$t\alpha + (1-t)[t_0 g_{\theta^1}(y^1) + (1-t_0) g_{\theta^1}(y^2)] \le \alpha. \tag{9}$$

Since $t < 1$ it results that

$$t_0 g_{\theta^2}(y^1) + (1-t_0) g_{\theta^2}(y^2) \le \alpha. \tag{10}$$

On the other hand, there exists $y^0 \in Y$ such that

$$g_\theta(y^0) \le t_0 g_\theta(y^1) + (1-t_0) g_\theta(y^2), \qquad \text{for all } \theta \in \Theta. \tag{11}$$

Since y^1 and y^2 belong to $Y_\alpha(\Theta')$, it follows that $y^0 \in Y_\alpha(\Theta')$.

From (8) and (11) one has $y^0 \in Y_\alpha(\theta^1)$.

From (10) and (11) one has $y^0 \in Y_\alpha(\theta^2)$.

Thus, $y^0 \in Y_1 \cap Y_2$, a contradiction. ∎

COROLLARY 1. *Let Y be a compact in a topological Hausdorff space, and let $\mathscr{G}$ a family of lower semi-continuous functions on Y. If $\mathscr{G}$ is $\frac{1}{2}$-convexlike on Y, then it must be α-affine-connected on $Y_\alpha(\Theta')$, for every $\alpha \in \mathbf{R}$ and $\Theta' \in \mathscr{F}(\Theta)$.*

Proof. The proof follows immediately from Proposition 2.2. ∎

The next example shows that the converse of the above proposition does not hold.

EXAMPLE 3. $\Theta = \mathbf{N}^*$, $Y = \{\frac{1}{n} | n \in \mathbf{N}^*\}$,

$$g_\theta(y) = \begin{cases} \theta & \text{if} y < \frac{1}{\theta} \\ y & \text{if} y \ge \frac{1}{\theta}. \end{cases}$$

One has

$$Y_\alpha(\theta) = \begin{cases} \varnothing & \text{if } \alpha < \frac{1}{\theta} \\ \{\frac{1}{\theta}, \frac{1}{\theta-1}, \ldots, \frac{1}{n(\alpha)}\} & \text{if } \frac{1}{\theta} \le \alpha < 1 \\ \{\frac{1}{\theta}, \frac{1}{\theta-1}, \ldots, 1\} & \text{if } 1 \le \alpha < \theta \\ Y & \text{if } \alpha \ge \theta. \end{cases}$$

(Here $n(\alpha)$ stands for $\frac{1}{\alpha}$ if $\frac{1}{\alpha} \in \mathbf{N}^*$, or for $[\frac{1}{\alpha}]$, otherwise.)

Since $Y_\alpha(\theta^1) \cap Y_\alpha(\theta^2) = \varnothing$ if and only if at least one of the two sets is empty, then it trivially follows that $\mathscr{G} = (g_\theta)_{\theta \in \Theta}$ is α-affine-connected on $Y_\alpha(\Theta')$, for every $\alpha \in \mathbf{R}$ and $\Theta' \in \mathscr{F}(\Theta)$.

$\mathscr{G}$ is not $\frac{1}{2}$-convexlike. For instance, there is no $y^0 \in Y$ such that $g_\theta(y^0) \le \frac{1}{2} g_\theta(1) + \frac{1}{2} g_\theta(\frac{1}{2})$, for all $\theta \in \Theta$.

DEFINITION 6. $\mathcal{G} = (g_\theta)_{\theta\in\Theta}$, is said to be α-flatless on Y if for any $\theta^1, \theta^2 \in \Theta$, and any $t \in [0,1]$, $\mathrm{cl}\{y \in Y | tg_{\theta^1}(y) + (1-t)g_{\theta^2}(y) < \alpha\} = \{y \in Y | tg_{\theta^1}(y) + (1-t)g_{\theta^2}(y) \le \alpha\}$, whenever the first set in the equality is not empty.

(Here by $\mathrm{cl}\, A$ we denote the closure in Y of the set A.)

The above-defined property is not directly comparable to the convexity or connectedness properties considered in this section. The next two examples prove the relative independence of these concepts.

EXAMPLE 4. Let $\mathcal{G}$ be as in Example 3.

$\mathcal{G}$ is α-affine connected on Y for every $\alpha \in \mathbf{R}$, but it is not α-flatless if α is a positive integer. It suffices to observe that, for any $\theta \in \Theta$, $\{g_\theta(y) \le \theta\} = Y$, while $\mathrm{cl}\{g_\theta(y) < \theta\} = \{\frac{1}{\theta}, \frac{1}{\theta-1}, \ldots, 1\}$.

EXAMPLE 5. $\Theta = \mathbf{Z}^*$, $Y = [-2,-1] \cup [1,2]$, $g_\theta(y) = \frac{y}{\theta}$.

One can easily verify that $\mathcal{G} = (g_\theta)_{\theta\in\Theta}$ is α-flatless for every $\alpha \le 0$, but it is not 0-affine connected on Y. (Take $\theta^1 = -1$, $\theta^2 = 1$, $t = 1/2$.) Consequently, $\mathcal{G}$ is not $\frac{1}{2}$-convexlike.

3. THE MINIMAX THEOREM

The main results of the paper concern the minimax equality in a usual framework of the two-person zero-sum games theory. One assumes X be an arbitrary nonempty set and Y to be a subset of a topological space. All three theorems of this section are stated only in terms of X, Y, and f, where $f\colon X \times Y \mapsto \mathbf{R}$ is the payoff function.

THEOREM 3.1. *Let Y be any topological space and let $f(x,.)$ be lower semi-continuous on Y, for every $x \in X$. Assume that for every $\alpha < \inf_{y\in Y}\sup_{x\in X} f(x,y)$ the sets $Y_\alpha(x) = \{y \in Y | f(x,y) \le \alpha\}$ $x \in X$ are compact in Y, the family $(f(.,y))_{y\in Y_\alpha(X')}$ is weakly concavelike, and the family $(f(x,.))_{x\in X}$ is weakly α-affine-connected on $Y_\alpha(X')$, for every $X' \in \mathcal{F}(X)$. Then* (1) *holds.*

This result yields a generalization of some classical minimax theorems, including the well-known theorems of Fan and König. As an immediate consequence of the theorem and the results of the previous section, one obtains the next two corollaries.

COROLLARY 2 (Fan [1]). *Assume Y to be a compact set and $f(x,.)$ to be lower semi-continuous on Y for every $x \in X$. If the family $(f(.,y))_{y\in Y}$ is concavelike and the family $(f(x,.))_{x\in X}$ is convexlike* (*i.e.*, *the payoff function f is concave–convexlike*), *then* (1) *holds.*

COROLLARY 3 (König [5]). *Assume Y to be a compact set and $f(x,.)$ to be lower semi-continuous on Y for every $x \in X$. If the family $(f(.,y))_{y\in Y}$ is $\frac{1}{2}$-concavelike and the family $(f(x,.))_{x\in X}$ is $\frac{1}{2}$-convexlike, then* (1) *holds.*

The following example shows that Theorem 3.1 actually generalizes the above-cited results and is independent of several other known minimax theorems.

EXAMPLE 6. $X = \mathbf{N}^*$, $Y = \{\frac{1}{n} | n \in \mathbf{N}^*\}$,

$$f(x,y) = \begin{cases} x & \text{if } y < \frac{1}{x} \\ y & \text{if } y \geq \frac{1}{x}. \end{cases}$$

The minimax equality (1) holds ($\sup_{x\in X} \inf_{y\in Y} f(x,y) = \inf_{y\in Y} \sup_{x\in X} f(x,y) = 1$.)

One can show that all assumptions of the theorem are verified. If Y is endowed with the topology induced by the usual topology of the real line, then it is easy to see that $f(x,.)$ is lower semi-continuous on Y, for every $x \in X$. ($Y_\alpha(x)$ is closed in Y, for every $\alpha \in \mathbf{R}$ and $x \in X$.)

For $\alpha < 1$ and $x \in X$,

$$Y_\alpha(x) = \begin{cases} \varnothing & \text{if } \alpha < \frac{1}{x} \\ \{\frac{1}{x}, \frac{1}{x-1}, \ldots, \frac{1}{n(\alpha)}\} & \text{if } \alpha \geq \frac{1}{x}. \end{cases}$$

($n(\alpha)$ is defined as in Example 3.)

Obviously, the compactness requirement of the theorem is satisfied. For any $X' \in \mathcal{F}(X)$, $Y_\alpha(X') = Y_\alpha(x')$, where $x' = \min X'$. Let us verify that $\sup_{x\in X} \inf_{y\in Y_\alpha(X')} f(x,y) \geq \inf_{y\in Y_\alpha(X')}[tf(x^1,y) + (1-t)f(x^2,y)]$, for any $t \in [0,1]$ and $x^1, x^2 \in X$. ($(f(.,y))_{y\in Y_\alpha(X')}$ is weakly concavelike.) The non-trivial case is when $Y_\alpha(X') \neq \varnothing$ ($x' \geq \frac{1}{\alpha}$). In this case $\sup_{x\in X} \inf_{y\in Y_\alpha(X')} f(x,y) = n(\alpha) - 1$. Assume that $x^1 \leq x^2$. Then,

$$\inf_{y\in Y_\alpha(X')}[tf(x^1,y) + (1-t)f(x^2,y)]$$

$$= \begin{cases} \frac{1}{x^1} & \text{if } x^1 \geq n(\alpha) \\ tx^1 + (1-t)\max\{\frac{1}{x'}, \frac{1}{x^2}\} & \text{if } x^1 < n(\alpha) \leq x^2 \\ tx^1 + (1-t)x^2 & \text{if } x^2 < n(\alpha) \end{cases}$$

Thus, the above inequality holds.

Finally, one easily verifies that $(f(x,.))_{x\in X}$ is α-affine connected on $Y_\alpha(X')$ (see Example 3).

On the other hand, one can remark that several minimax theorems known in the literature fail in this example.

The family $(f(x,.))_{x\in X}$ is not $\frac{1}{2}$-convexlike, as was shown in Example 3. One can also see that $(f(.,y)_{y\in Y}$ is not $\frac{1}{2}$-concavelike. (There are no $x \in X$

such that $f(x, y) \geq \frac{1}{2}f(2, y) + \frac{1}{2}f(3, y)$, for all $y \in Y$.) Therefore, minimax theorems of the Fan or König type fail.

Obviously, the sets $Y_\alpha(x)$ are not connected, so that minimax theorems like those in [12–15] also fail.

Since there are no non-constant continuous mappings from the unit interval to Y, some connectedness properties used by several authors (α-connectedness of [13], submaximum set property of [2], etc.) are not satisfied.

Note also that the minimax theorem of [10] fails ($Y_\alpha(1)$ is not compact if $\alpha \geq 1$).

The second result of this section uses another variant of our connectedness property, by strengthening the topological framework.

THEOREM 3.2. *Let Y be a connected compact set and let $f(x, .)$ be continuous on Y for every $x \in X$. If the family $(f(., y))_{y \in Y_\alpha(X')}$ is weakly concavelike and the family $(f(x, .))_{x \in X}$ is almost α-affine-connected on $Y_\alpha(X')$, for every $X' \in \mathcal{F}(X)$ and $\alpha < \inf_{y \in Y} \sup_{x \in X} f(x, y)$,* (1) *holds.*

In the third minimax theorem the connectedness is replaced by the flatless property.

THEOREM 3.3. *Let Y be any topological space and let $f(x, .)$ be lower semi-continuous on Y for every $x \in X$. Assume that for every $\alpha < \inf_{y \in Y} \sup_{x \in X} f(x, y)$ the sets $Y_\alpha(x)$, $x \in X$ are compact in Y, the family $(f(., y))_{y \in Y_\alpha(X')}$ is weakly concavelike, and the family $(f(x, .))_{x \in X}$ is α-flatless on $Y_\alpha(X')$, for every $X' \in \mathcal{F}(X)$. Then* (1) *holds.*

EXAMPLE 7. $X = \mathbf{Z}^*$, $Y = [-2, -1] \cup (1, 2]$ $f(x, y) = \frac{y-1}{x}$.

The equality (1) holds: $\sup_{x \in X} \inf_{y \in Y} f(x, y) = \inf_{y \in Y} \sup_{x \in X} f(x, y) = 0$. One can see that all assumptions of the above theorem are satisfied. As in Example 6 some conditions required by other minimax theorems established in [1–3, 5, 6, 8, 10, 12–15] are not satisfied.

The theorems are proved in Section 5.

4. THREE SPECIAL THEOREMS OF THE ALTERNATIVE

Everywhere in this section g_1 and g_2 are two real-valued functions defined on a topological space Y.

The main results of this section deal with the following problem: If $\max\{g_1(y), g_2(y)\} > 0$ for all $y \in Y$, when can be found a convex combination $g = tg_1 + (1 - t)g_2$ of g_1 and g_2 with the property that $g(y) \geq 0$ for all $y \in Y$?

All three theorems of this section respond to this question.

For any real number α, define the correspondences (multi-valued functions, point-to-set mappings) φ, ψ, $\bar{\psi}$ from the interval $[0, 1]$ to the family 2^Y of all subsets of Y by $\varphi(t) = \{y \in Y \mid tg_1(y) + (1-t)g_2(y) \leq 0\}$, $\psi(t) = \{y \in Y \mid tg_1(y) + (1-t)g_2(y) < 0\}$, $\bar{\psi}(t) = \operatorname{cl} \psi(t)$ (here $\operatorname{cl} A$ stands for the closure in Y of the subset A).

For any correspondence φ: $[0, 1] \mapsto 2^Y$, denote by φ^{-l} and φ^{-u} the lower inverse (respectively, the upper inverse): $\varphi^{-l}(B) = \{t \in [0, 1] \mid \varphi(t) \cap B \neq \emptyset\}$, $\varphi^{-u}(B) = \{t \in [0, 1] \mid \varphi(t) \subseteq B\}$.

THEOREM 4.1. *Let Y be any topological space and let g_1, g_2 be lower semi-continuous. Assume that the sets $\{y \in Y | g_i(y) \leq 0\}$, $i = 1, 2$, are compact in Y and*

$$\inf_{y \in Y} \max_{i=1,2} g_i(y) > 0. \tag{12}$$

If, in addition,

$$\emptyset \in \{\psi(t) \cap \varphi(0), \psi(t) \cap \varphi(1)\} \qquad \text{for every } t \in [0, 1], \tag{13}$$

then there exists $t \in [0, 1]$ such that

$$tg_1(y) + (1-t)g_2(y) \geq 0, \qquad \text{for all } y \in Y. \tag{14}$$

Remark 2. By (13), the family $\mathcal{G} = \{g_1, g_2\}$ is weakly 0-affine connected on Y.

THEOREM 4.2. *Let Y be any topological space and let g_1, g_2 be lower semi-continuous. Assume that the sets $\{y \in Y | g_i(y) \leq 0\}$, $i = 1, 2$, are compact in Y and* (12) *holds. If, in addition,*

$$\bar{\psi}(t) = \varphi(t), \qquad \text{whenever } \psi(t) \neq \emptyset, \tag{15}$$

then there exists a dyadic number $t \in [0, 1]$ such that (14) *holds.*

Remark 3. By (15), the family $\mathcal{G} = \{g_1, g_2\}$ is 0-flatless on Y.

The proof of both theorems follows after the next four lemmas.

LEMMA 1. *For any topological space Y and any two functions g_1, g_2 the correspondences ψ and $\bar{\psi}$ are lower semi-continuous on* $[0, 1]$.

Proof. Let $t_0 \in [0, 1]$. Assume that $\psi(t_0) \cap G \neq \emptyset$, for some open set $G \subseteq Y$. Pick an $y_0 \in \psi(t_0) \cap G$. Then, $t_0 g_1(y_0) + (1 - t_0)g_2(y_0) < 0$. Obviously, there exists a neighborhood $V(t_0)$ of t_0 in $[0, 1]$ such that $tg_1(y_0) + (1-t)g_2(y_0) < 0$, for all $t \in V(t_0)$; i.e., $\psi(t) \cap G \neq \emptyset$ for all $t \in V(t_0)$. Thus, ψ is lower semi-continuous at t_0. Since ψ is lower semi-continuous, so is $\bar{\psi}$ [4, Proposition 7.3.3]. ∎

LEMMA 2. *Let g_1, g_2 be lower semi-continuous on Y. Assume that $\varphi(0)$ and $\varphi(1)$ are compact in Y. Then, the correspondence φ is upper semi-continuous on* $[0, 1]$.

Proof. Since φ is valued in the compact $Y' = \varphi(0) \cup \varphi(1)$, it suffices to show that φ is closed [4, Theorem 7.1.16].

Let $(t_0, y_0) \in [0, 1] \times Y'$ such that $y_0 \notin \varphi(t_0)$. Then, $t_0 g_1(y_0) + (1 - t_0) g_2(y_0) = \delta$ for some $\delta > 0$. Since g_1, g_2 are lower semi-continuous, one can find a neighborhood $V(y_0)$ of y_0 such that $g_i(y) \geq g_i(y_0) - \delta/4$ for all $y \in V(y_0)$, $i = 1, 2$.

Denote by $V(t_0) = (t_0 - \epsilon,\ t_0 + \epsilon) \cap [0, 1]$, where $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ and

$$\epsilon_i = \begin{cases} \dfrac{\delta}{4|g_i(y_0)|} & \text{if } g_i(y_0) \neq 0 \\ \delta & \text{otherwise} \end{cases}, \qquad \text{for} \quad i = 1, 2.$$

Obviously, $V(t_0)$ is a neighborhood of t_0 in $[0, 1]$, and if $(t, y) \in V(t_0) \times V(y_0)$ one has

$$\begin{aligned} t g_1(y) + (1 - t) g_2(y) &\geq t g_1(y_0) + (1 - t) g_2(y_0) - \delta/4 \\ &= (t - t_0)(g_1(y_0)) + (t_0 - t)(g_2(y_0)) + 3\delta/4 \\ &\geq \delta/4 > 0. \end{aligned}$$

Thus, $y \notin \varphi(t)$. ∎

LEMMA 3. *Let all assumptions of Lemma* 2 *hold. If* (12) *also holds and the sets $\psi(0)$ and $\psi(1)$ are non-empty, then there exist $\underline{t}, \bar{t} \in (0, 1)$ such that $\varphi^{-l}(\varphi(0)) = [0, \bar{t}]$ and $\varphi^{-l}(\varphi(1)) = [\underline{t}, 1]$.*

Proof. Since $\psi(0) \neq \emptyset$, then there exists a neighborhood $V(0)$ of 0 in $[0, 1]$ such that $V(0) \subseteq \varphi^{-l}(\varphi(0))$. ($g_2(y) < 0$ for some y; hence $y \in \varphi(0)$ for all $t \leq -g_2(y)/(g_1(y) - g_2(y))$.) Denote by $\bar{t} = \sup \varphi^{-l}(\varphi(0))$. Obviously, $\bar{t} > 0$. Let us show that $t \in \varphi^{-l}(\varphi(0))$ whenever $0 < t < \bar{t}$.

Pick a $t' \in \varphi^{-l}(\varphi(0))$, $t < t'$. Then, there exists $y \in Y$ such that $t' g_1(y) + (1 - t') g_2(y) \leq 0$ and $g_2(y) \leq 0$. It follows by (12) that $g_1(y) > 0$. Then, $t g_1(y) + (1 - t) g_2(y) < t' g_1(y) + (1 - t') g_2(y) \leq 0$. Thus, $y \in \varphi(t) \cap \varphi(0)$; i.e., $t \in \varphi^{-1}(\varphi(0))$.

In summary, we have proved that $\varphi^{-l}(\varphi(0))$ is an interval. Since φ is upper semi-continuous (Lemma 2), and $\varphi(0)$ is closed in Y, this interval must be closed. Thus, $\varphi^{-l}(\varphi(0)) = [0, \bar{t}]$. ∎

Remark 4. One can see from the proof of the above lemma that $[0, \bar{t}) \subseteq \psi^{-l}(\varphi(0))$ (and $(\underline{t}, 1] \subseteq \psi^{-l}(1)$).

LEMMA 4. *Let g_1 and g_2 satisfy* (12). *If $\psi(0)$ and $\psi(1)$ are non-empty, then there exist $\underline{\tau}, \bar{\tau} \in (0, 1)$ such that $\psi^{-l}(\varphi(0)) = [0, \bar{\tau})$ and $\psi^{-l}(\varphi(1)) = (\underline{\tau}, 1]$.*

Proof. As in the proof of the previous lemma, one can show that $\psi^{-l}(\varphi(0))$ is an interval. Since $\psi(t) \subseteq \varphi(0) \cup \varphi(1)$, it follows by (12) that $\psi^{-l}(\varphi(0)) = [0,1] \setminus \psi^{-u}(\varphi(1))$. Since ψ is lower semi-continuous and $\varphi(1)$ is closed in Y, it follows that $\psi^{-l}(\varphi(0))$ must be open in $[0,1]$. Hence, $\psi^{-l}(\varphi(0)) = [0, \bar{\tau})$, for some $\bar{\tau} > 0$. ∎

COROLLARY 4. *If the assumptions of Lemma* 3 *are fulfilled, then* $\underline{\tau} = \underline{t}$ *and* $\bar{\tau} = \bar{t}$. *Moreover,* $\bar{\psi}^{-l}(\varphi(0)) = \psi^{-l}(\varphi(0))$ *and* $\bar{\psi}^{-l}(\varphi(1)) = \psi^{-l}(\varphi(1))$.

Proof. From Remark 1 it follows that $\bar{t} \leq \bar{\tau}$. On the other hand, $\psi(t) \subseteq \bar{\psi}(t) \subseteq \varphi(t)$ for all $t \in [0,1]$. Thus, $\bar{t} = \bar{\tau}$. The identity $\bar{\psi}^{-l}(\varphi(0)) = \psi^{-l}(\varphi(0))$ follows now from the lower semi-continuity of $\bar{\psi}$. ∎

Proof of Theorem 4.1. If $\psi(0) = \emptyset (\psi(1) = \emptyset)$, then (14) holds for $t = 0$ $(t = 1)$. The non-trivial case is where $\psi(0)$ and $\psi(1)$ are both non-empty. By (13) it follows that if $\psi(t) \neq \emptyset$ then either $\psi(t) \subseteq \varphi(0)$ or $\psi(t) \subseteq \varphi(1)$, but these two situations are incompatible. Thus, in Lemma 4 one has $\bar{\tau} \leq \underline{\tau}$. ∎

Proof of Theorem 4.2. As in the above, let us consider the case where $\psi(0) \neq \emptyset$, $\psi(1) \neq \emptyset$. It suffices to show that $\bar{t} < \underline{t}$ in Lemma 3.

To the contrary, assume that $\underline{t} \leq \bar{t}$. Then, $\psi(t) \neq \emptyset$ for every $t \in [0,1]$, and by Lemma 3 it follows that $\varphi^{-u}(\varphi(0)) = [0, \underline{t})$ and $\bar{\psi}^{-u}(\varphi(0)) = [0, \underline{t}]$. Since $\varphi(t) = \bar{\psi}(t)$ for all $t \in [0,1]$ and $0 < \underline{t} < 1$ we have arrived at a contradiction. ∎

For the next result, define the correspondence F from Y to the family of subsets of $[0,1]$ by

$$F(y) = \{t \in [0,1] \mid tg_1(y) + (1-t)g_2(y) > 0\}.$$

Obviously, if $F(y) \neq \emptyset$, then $\overline{F}(y) = \{t \in [0,1] \mid tg_1(y) + (1-t)g_2(y) \geq 0\}$. ($\overline{F}(y)$ is the closure in $[0,1]$ of $F(y)$.) Denote by $E(y) = \overline{F}(y) \setminus F(y)$, and by $A_i = \{y \in Y \mid g_i(y) \leq 0\}$, $i = 1, 2$.

THEOREM 4.3. *Let* Y *be connected and let* g_1, g_2 *be continuous. Assume that the sets* A_i, $i = 1, 2$, *are compact in* Y *and* (12) *holds. If*

$$E(A_1) \cap E(A_2) = \emptyset \tag{16}$$

then there exists a dyadic number $t \in [0,1]$ *such that* (14) *holds.*

Remark 5. One can easily see that (16) is verified if and only if $\mathcal{G} = \{g_1, g_2\}$ is almost 0-affine connected on Y.

Proof. Note first that from (12) it follows that $A_1 \cap A_2 = \emptyset$. If $y \in Y \setminus (A_1 \cup A_2)$, then $\overline{F}(y) = [0,1]$; if $y \in A_1$, then $\overline{F}(y) = [0, a(y)]$; and if $y \in A_2$, then $\overline{F}(y) = [a(y), 1]$, where $a(y) = g_2(y)/(g_2(y) - g_1(y))$.

The proof of the theorem consists of four steps.

1. $\overline{F}$ is l.s.c. on Y. It suffices to show that F is l.s.c.

Let G be open in $[0, 1]$ and assume that $F(y_0) \cap G \neq \emptyset$, for some $y_0 \in Y$. Pick a $t_0 \in F(y_0) \cap G$. Then $\delta > 0$, where $\delta = t_0 g_1(y_0) + (1 - t_0) g_2(y_0)$. There exists a neighborhood $V(y_0)$ of y_0 such that $g_i(y) \geq g_i(y_0) - \delta/2$, for $i = 1, 2$.

Then, for $y \in V(y_0)$ one has

$$t_0 g_1(y) + (1 - t_0) g_2(y) \geq t_0 g_1(y_0) + (1 - t_0) g_2(y_0) - \delta/2 = \delta/2 > 0;$$

i.e., $\overline{F}(y) \cap G \neq \emptyset$.

2. $\overline{F}$ is u.s.c. on Y. We will show that $\overline{F}$ is closed.

Pick a $(y_0, t_0) \in Y \times [0, 1]$ such that $t_0 g_1(y_0) + (1 - t_0) g_2(y_0) < 0$. Denote by $-\delta$ the left-hand member of this inequality. Obviously, there exists a neighborhood $V(t_0)$ of t_0 such that $t g_1(y_0) + (1 - t) g_2(y_0) \leq -\delta/2$ for all $t \in V(t_0)$. On the other hand, one can find a neighborhood $V(y_0)$ of y_0 such that $g_i(y) \leq g_i(y_0) + \delta/4$, whenever $y \in V(y_0)$. Then, if $(y, t) \in V(y_0) \times V(t_0)$, it results that

$$t g_1(y) + (1 - t) g_2(y) \leq t g_1(y_0) + (1 - t) g_2(y_0) + \delta/4 \leq -\delta/4 < 0;$$

i.e., (y, t) does not belong to the graph of $\overline{F}$.

3. $\overline{F}(y) \cap \overline{F}(y') \neq \emptyset$ for any $y, y' \in Y$. Pick a $y_0 \in Y$. It suffices to show that $\overline{F}(y_0) \cap \overline{F}(y) \neq \emptyset$, for all $y \in Y$; i.e., $Y_0 = Y$, where $Y_0 = \{y \in Y \mid \overline{F}(y_0) \cap \overline{F}(y) \neq \emptyset\}$.

Obviously, the non-trivial case is where $\min_{i=1,2} g_i(y_0) < 0$. (Otherwise, $\overline{F}(y_0) = [0, 1]$.)

Assume that $g_2(y_0) < 0$. Then, $F(y_0) = (a(y_0), 1]$ and $a(y_0) \in (0, 1)$. If $y \in Y \setminus A_1$, then it is obvious that $F(y_0) \cap \overline{F}(y) \neq \emptyset$. If $y \in A_1$, then it follows by (15) that $\overline{F}(y_0) \cap \overline{F}(y) \neq \emptyset$ if and only if $F(y_0) \cap \overline{F}(y) \neq \emptyset$.

Now, since $\overline{F}$ is u.s.c. it follows that $Y_0 = \overline{F}^{-l}(\overline{F}(y_0))$ is closed (obviously, nonempty) in Y. On the other hand, $Y_0 = \{y \in Y \mid \overline{F}(y) \cap F(y_0) \neq \emptyset\} = \overline{F}^{-l}(F(y_0))$, and since $\overline{F}$ is l.s.c. it follows that Y_0 is open in Y. Hence, the connectedness of Y implies that $Y_0 = Y$.

4. $\bigcap_{y \in Y} \overline{F}(y)$ contains at least a dyadic number.

If one of A_1 or A_2 is empty (say $A_1 = \emptyset$), the statement is trivial ($0 \in \bigcap_{y \in Y} \overline{F}(y)$).

Assume now that both A_1 and A_2 are nonempty. By the compactness of A_i and the continuity of g_i, $i = 1, 2$, it follows that there exist $y_i \in A_i$, $i = 1, 2$, such that $a(y_1) = \min_{y \in A_1} a(y)$ (denoted by $\bar{a}$) and $a(y_2) = \max_{y \in A_2} a(y)$ (denoted by $\underline{a}$). One has $[0, \bar{a}] = \overline{F}(y_1) \subseteq \bigcap_{y \in Y \setminus A_2} \overline{F}(y)$ and $[\underline{a}, 1] = \overline{F}(y_2) \subseteq \bigcap_{y \in Y \setminus A_1} \overline{F}(y)$. By Step 3 it follows that $\underline{a} \leq \bar{a}$. By (16) it follows that $\underline{a} < \bar{a}$. ∎

5. PROOFS OF THE MINIMAX THEOREMS

The proofs of Theorems 3.1–3.3 follow the same line. The second step uses only the compactness assumptions. At the crucial point of the first step one invokes the alternative theorems of Section 4. Thus the proof of Theorem 3.1 below holds for Theorem 3.2 (respectively, Theorem 3.3), invoking Theorem 4.3 (respectively, Theorem 4.2), instead of Theorem 4.1.

Step 1. Show that if $\inf_{y\in Y}\max_{x\in X'} f(x,y) > \alpha$ for some $X' \in \mathcal{F}(X)$, then $\sup_{x\in X}\min_{y\in Y} f(x,y) \geq \alpha$.

We will prove this statement by induction to $n = |X'|$.

If $n = 2$, let $X' = \{x^1, x^2\}$ and invoke Theorem 4.1 for $g_i(.) = f(x^i, .) - \alpha$, $i = 1, 2$. Thus, $\inf_{y\in Y}[tf(x^1, y) + (1-t)f(x^2, y)] \geq \alpha$, for some $t \in [0,1]$. Then since $(f(., y))_{y\in Y}$ is weakly concavelike, it follows that $\sup_{x\in X}\min_{y\in Y} f(x,y) \geq \alpha$.

Consider now $X' = \{x^1, \ldots, x^n\}$, $n \geq 3$, and let us discuss two situations which can occur:

(i) $\min_{y\in Y}\max_{3\leq i\leq n} f(x^i, y) > \alpha$

(ii) $\min_{y\in Y}\max_{3\leq i\leq n} f(x^i, y) \leq \alpha$.

In case (i) the inductive assumption implies that $\sup_{x\in X}\min_{y\in Y} f(x,y) \geq \alpha$.

In case (ii), denote by $Y' = Y_\alpha(\{x^3, \ldots, x^n\})$. Obviously, Y' is compact and $\min_{y\in Y'}\max_{i=1,2} f(x^i, y) > \alpha$, so that Theorem 4.1 can be used for Y', $g_i(.) = f(x^i, .) - \alpha$, $i = 1, 2$. Therefore, there exists $t \in [0,1]$ such that $\inf_{y\in Y'}[tf(x^1, y) + (1-t)f(x^2, y)] \geq \alpha$. Then, one has $\sup_{x\in X}\min_{y\in Y'} f(x,y) \geq \alpha$.

Now, for every $\epsilon > 0$, there exists $x_\epsilon \in X$ such that $\min_{y\in Y'} f(x_\epsilon, y) > \alpha - \epsilon$. Put $X'' = \{x_\epsilon, x^3, \ldots, x^n\}$. Obviously, $\inf_{y\in Y}\max_{x\in X''} f(x,y) > \alpha - \epsilon$. Since $|X''| = n-1$ it follows that $\sup_{x\in X}\inf_{y\in Y} f(x,y) \geq \alpha - \epsilon$. Since ϵ is arbitrarily positive, one finally obtains that $\sup_{x\in X}\inf_{y\in Y} f(x,y) \geq \alpha$.

Step 2. Show that $\inf_{y\in Y}\sup_{x\in X} f(x,y) > \alpha$ implies $\sup_{x\in X}\inf_{y\in Y} f(x,y) \geq \alpha$. Hence, (1) holds.

By standard arguments of compactness one can show that $\inf_{y\in Y}\sup_{x\in X} f(x,y) > \alpha$ implies $\inf_{y\in Y}\sup_{x\in X'} f(x,y) > \alpha$, for some finite subset X' of X. The conclusion follows now from Step 1. ∎

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