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A FLEXIBLE MINIMAX THEOREM

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Dedicated to Professor Heinz König

Introduction

The purpose of this paper is to unify a number of minimax theorems that use hypotheses that are superficially very different.

The important role of *connectedness* in minimax theorems was first noted by Wu [29], followed by Tuy [27,28], who was able to generalize Sion's minimax theorem [24]. Based on Joó's result [8], Stachó [25] and Komornik [16] proved minimax theorems for "interval spaces". These results were unified by Kindler–Trost [12].

Minimax conditions that use *algebraic* conditions were considered by Fan [1], König [17], Neumann [19], Irle [7], Lin–Quan [18], Kindler [11] and Simons [20].

Minimax theorems that *mix* both connectedness and algebraic conditions were considered by Terkelsen [26], Geraghty–Lin [2,4,5], Kindler [11] and Simons [21].

Kindler [11] was the first to observe that the algebraic conditions *force* conditions akin to connectedness.

In this paper, we give results that unify all the ideas mentioned above, as well as other ideas due to Ha [6] and Simons [22,23].

The basic minimax theorem is Theorem 1 which has a simple proof using a *compactness* condition (1.1), a *condition on* Y, (1.2) and a *condition on* X, (1.3).

There are obvious *topological* situations in which (1.2) holds — see (8.2). Lemma 2 gives a *set-theoretic* situation in which (1.2) holds — in Remarks 3, we show that, to within ε, Lemma 2 encompasses all the *algebraic* situations mentioned above.

Lemmas 4 and 5 give *topological* situations (which will require that X be an interval space) in which (1.3) holds. Lemma 6 gives a *set-theoretic* situation in which (1.3) holds — in Remarks 7, we show that, to within ε again, Lemma 6 encompasses all the *algebraic* situations mentioned above.

The reader will undoubtedly notice the similarity between the hypotheses (2.2) and (6.1). In Remarks 7, we give a common result from which both

Lemma 2 and Lemma 6 can be derived. (We have not used this in the text for clarity of exposition.)

Let X and Y be nonempty sets and $f : X \times Y \to \mathbf{R}$. If $\gamma \in \mathbf{R}$ we define multifunctions $\underline{\gamma}\rfloor$ from X into 2^Y and $\lceil\overline{\gamma}$ from Y into 2^X by

$$\forall x \in X, \quad \underline{\gamma}\rfloor x := \{y : y \in Y,\ f(x,y) \leqq \gamma\}$$

and

$$\forall y \in Y, \quad \lceil\overline{\gamma}\, y := \{x : x \in X,\ f(x,y) > \gamma\}.$$

For convenience, we write $LE(W,\gamma)$ for $\bigcap_{w \in W} \underline{\gamma}\rfloor w$.

The author would like to thank Professor Jürgen Kindler for an interesting discussion on minimax theorems and for suggesting that he incorporate [12] into an earlier version of this work.

The joining of sets and pseudoconnectedness

We say that sets H_0 and H_1 are *joined* by a set H if

$$H \subset H_0 \cup H_1, \ H \cap H_0 \neq \emptyset \quad \text{and} \quad H \cap H_1 \neq \emptyset.$$

We say that a family $\mathcal{H}$ of sets is *pseudoconnected* if,

$$(0.1) \qquad H_0, H_1, H \in \mathcal{H} \text{ and } H_0 \text{ and } H_1 \text{ joined by } H \Rightarrow H_0 \cap H_1 \neq \emptyset.$$

Any family of closed connected subsets of a topological space is pseudoconnected. So also is any family of open connected subsets. In Lemma 2 we give a situation related to minimax theorems in which a certain family of sets is *automatically* pseudoconnected.

THEOREM 1. *Let Y be a topological space, and $\mathcal{B}$ be a nonempty subset of* $\mathbf{R}$ *such that* $\inf \mathcal{B} = \sup_X \inf_Y f$. *Suppose that,* $\forall \beta \in \mathcal{B}$ *and finite subsets W of X (with the convention $LE(\emptyset,\beta) = Y$),*

$$(1.1) \qquad \forall x \in X, \quad \underline{\beta}\rfloor x \quad \textit{is closed and compact,}$$

$$(1.2) \qquad \left\{\underline{\beta}\rfloor x \cap LE(W,\beta)\right\}_{x \in X} \quad \textit{is pseudoconnected}$$

and,

$$(1.3) \qquad \begin{cases} \forall x_0,\ x_1 \in X,\ \exists x \in X \quad \textit{such that} \\ \underline{\beta}\rfloor x_0 \textit{ and } \underline{\beta}\rfloor x_1 \ \textit{are joined by } \underline{\beta}\rfloor x \cap LE(W,\beta). \end{cases}$$

Then

$$\min_Y \sup_X f = \sup_X \inf_Y f.$$

PROOF. Let $\beta \in \mathcal{B}$. Let V be a nonempty finite subset of X. We can write $V = \{x_0, x_1\} \cup W$. Let x be as in (1.3). It follows that $\underline{\beta\rfloor} x_0 \cap \cap LE(W,\beta)$ and $\underline{\beta\rfloor} x_1 \cap LE(W,\beta)$ are joined by $\underline{\beta\rfloor} x \cap LE(W,\beta)$. From (1.2) and (0.1), $LE(V,\beta) \neq \emptyset$. The result follows from (1.1) and the finite intersection property.

Sufficient conditions for (1.2)

In our next result, W does not necessarily have to be finite.

LEMMA 2. *Let $W \subset X$ and $\beta \in \mathbf{R}$. Suppose that,*

(2.1) $\forall \gamma > \beta$ *and* $x \in X$, $\underline{\gamma\rfloor} x \cap LE(W,\beta)$ *is closed and compact,*

and, whenever $\delta > \gamma$, $\exists N \geqq 1$ and $\gamma_0, \ldots, \gamma_N \in \mathbf{R}$ such that

$$(2.2)\quad \begin{cases} & \gamma_0 = \delta,\ \gamma_N = \gamma \ \text{ and,} \\ & \forall y_0, y_1 \in Y,\ \exists\, y \in Y \ \text{ such that, } \forall n \in \{1, \ldots, N\}, \\ (2.2.1) & \overline{\lceil \gamma_n} \, y \subset \overline{\lceil \gamma_{n-1}} \, y_0 \cup \overline{\lceil \beta} \, y_1, \\ (2.2.2) & \overline{\lceil \gamma_n} \, y \subset \overline{\lceil \beta} \, y_o \cup \overline{\lceil \gamma_{n-1}} \, y_1, \\ (2.2.3) & \overline{\lceil \beta} \, y \subset \overline{\lceil \beta} \, y_0 \cup \overline{\lceil \beta} \, y_1, \\ (2.2.4) & \overline{\lceil \delta} \, y \subset \overline{\lceil \delta} \, y_0 \cup \overline{\lceil \delta} \, y_1. \end{cases}$$

Then

(1.2) $\left\{ \underline{\beta\rfloor} x \cap LE(W,\beta) \right\}_{x \in X}$ *is pseudoconnected.*

PROOF. Suppose that the result fails. Then $\exists\, x_0, x_1, x \in X$ such that, writing $T := \underline{\beta\rfloor} x \cap LE(W,\beta)$,

$$(2.3)\qquad T \subset \underline{\beta\rfloor} x_0 \cup \underline{\beta\rfloor} x_1,$$

$$(2.4)\qquad \underline{\beta\rfloor} x_0 \cap \underline{\beta\rfloor} x_1 \cap T = \emptyset,$$

and, for $i = 0, 1$,

$$(2.5)\qquad u_i \in \underline{\beta\rfloor} x_i \cap T.$$

From (2.1) and (2.4), $\exists\, \gamma > \beta$ such that

$$\underline{\gamma|}x_0 \cap \underline{\gamma|}x_1 \cap T = \emptyset. \tag{2.6}$$

From (2.5) and (2.6), $u_0 \notin \underline{\gamma|}x_1$. Let $\delta := f(x_1, u_0) \vee f(x_0, u_1) > \gamma$,

$$U_0 := \underline{\beta|}x_0 \cap \underline{\delta|}x_1 \cap T \ni u_0 \quad \text{and} \quad U_1 := \underline{\delta|}x_0 \cap \underline{\beta|}x_1 \cap T \ni u_1.$$

Choose N and $\gamma_0, \dots, \gamma_N$ as in (2.2). Then, from (2.6),

$$U_0 \subset \underline{\delta|}x_1 = \underline{\gamma_0|}x_1 \quad \text{and} \quad U_0 \cap \underline{\gamma_N|}x_1 = U_0 \cap \underline{\gamma|}x_1 = \emptyset.$$

Thus, $\forall\, t \in U_0$, $\exists\,! g_0(t) \in \{1, \dots, N\}$ such that

$$g_0(t) \leqq n \leqq N \Rightarrow t \notin \underline{\gamma_n|}x_1 \quad \text{and} \quad n = g_0(t) \Rightarrow t \in \underline{\gamma_{n-1}|}x_1. \tag{2.7}$$

Similarly, $\forall\, t \in U_1$, $\exists\,! g_1(t) \in \{1, \dots, N\}$ such that

$$g_1(t) \leqq n \leqq N \Rightarrow t \notin \underline{\gamma_n|}x_0 \quad \text{and} \quad n = g_1(t) \Rightarrow t \in \underline{\gamma_{n-1}|}x_0.$$

We fix $y_i \in U_i$ to maximize $g_i(y_i)$ and choose $y \in Y$ as in (2.2). From (2.2.3), $y \in T$. From (2.3), we can suppose without loss of generality that $y \in \underline{\beta|}x_0$. From (2.2.4) since $y_i \in \underline{\delta|}x_1$, $y \in \underline{\delta|}x_1$. Thus $y \in U_0$. Let $n := g_0(y_0)$. From (2.7), $y_0 \in \underline{\gamma_{n-1}|}x_1$. Since $y_1 \in U_1$, $y_1 \in \underline{\beta|}x_1$. From (2.2.1), $y \in \underline{\gamma_n|}x_1$. From (2.7), $n < g_0(y)$. This contradiction of the maximality of $g_0(y_0)$ completes the proof of the Lemma.

REMARKS 3. In the context of minimax theorems, various authors have introduced conditions that imply (2.2).

Inspired by a result of Fan [1], König [17] introduced the condition:

$$\begin{cases} \forall\, y_0, y_1 \in Y, \exists\, y \in Y \quad \textit{such that,} \\ x \in X \Rightarrow f(x,y) \leqq [f(x,y_0) + f(x,y_1)]/2. \end{cases} \tag{3.1}$$

(3.1) was weakened by Neumann [19], who also showed that it sufficed that his condition hold "to within ε". (See the discussion on Irle's theorem below.)

Neumann's condition was further weakened by Geraghty–Lin [2,4,5] and Lin–Quan [18], who introduced the condition:

(3.2)

$$\begin{cases} \exists\, s \in (0,1) \ \textit{such that,} \ \forall\, y_0, y_1 \in Y, \ \exists\, y \in Y \ \textit{such that,} \\ x \in X \Rightarrow f(x,y) \leqq (1-s)[f(x,y_0) \vee f(x,y_1)] + s[f(x,y_0) \wedge f(x,y))]. \end{cases}$$

(To see this take $s := 1/2$).

Simons [20] weakened (3.2) to the "penalty condition":

(3.3)

$$\begin{cases} \exists \ \textit{a nondecreasing function} \ \pi : \mathbf{R}^+ \to \mathbf{R}^+ \ \textit{such that} \\ \lambda > 0 \Rightarrow \pi(\lambda) > 0 \ \textit{and} \ \forall y_0, y_1 \in Y, \ \exists y \in Y \ \textit{such that,} \\ x \in X \Rightarrow f(x,y) \leqq f(x,y_0) \vee f(x,y_1) - \pi\left(|f(x,y_0) - f(x,y_1)|\right). \end{cases}$$

(To see this take $\pi(\lambda) := s\lambda$. Much smaller choices of π are possible, for instance, $\pi(\lambda) := e^{-1/\lambda^2}$).

Simons [20] weakened (3.3) to the "upward condition":

(3.4)

$$\begin{cases} \forall \varepsilon > 0, \ \exists \eta > 0 \ \textit{such that,} \ \forall y_0, y_1 \in Y, \exists y \in Y \ \textit{such that,} \\ x \in X \ \textit{and} \ |f(x,y_0) - f(x,y_1)| \geqq \varepsilon \Rightarrow f(x,y) \leqq f(x,y_0) \vee f(x,y_1) - \eta \\ \textit{and} \ \ x \in X \Rightarrow f(x,y) \leqq f(x,y_0) \vee f(x,y_1). \end{cases}$$

(To see this take $\eta := \pi(\varepsilon)$.)

We now show that *if* $\beta < \gamma < \delta$ *then* (3.4) *implies* (2.2): We set $\varepsilon := \gamma - \beta$, choose η as in (3.4) and $\gamma_0, \dots, \gamma_N \in [\gamma, \delta]$ with $\gamma_0 = \delta$, $\gamma_N = \gamma$ and, $\forall n \in \{1, \dots, N\}$, $\gamma_{n-1} - \gamma_n \leqq \eta$. Let $y_0, y_1 \in Y$ and choose $y \in Y$ as in (3.4). Suppose that $f(x,y_0) \leqq \gamma_{n-1}$ and $f(x,y_1) \leqq \beta$. We distinguish two cases:

Case 1: $f(x,y_0) \leqq \gamma$. Then $f(x,y) \leqq \gamma \vee \beta = \gamma \leqq \gamma_n$.

Case 2: $f(x,y_0) > \gamma$. Then $f(x,y_0) - f(x,y_1) \geqq \varepsilon$ hence, from (3.4),

$$f(x,y) \leqq \gamma_{n-1} \vee \beta - \eta = \gamma_{n-1} - \eta \leqq \gamma_n.$$

Thus $f(x,y_0) \leqq \gamma_{n-1}$ and $f(x,y_1) \leqq \beta \Rightarrow f(x,y) \leqq \gamma_n$, from which (2.2.1) follows. We can prove similarly that (2.2.2) holds. Finally, $f(\cdot,y) \leqq$ $\leqq f(\cdot,y_0) \vee f(\cdot,y_1))$ gives (2.2.3) and (2.2.4).

Irle [7] introduced the concept of an *averaging function* φ (a suitable real function defined on a suitable subset of $\mathbf{R} \times \mathbf{R}$) and considered a condition of the form:

$$\begin{cases} \forall \varepsilon > 0 \ \textit{and} \ y_0, y_1 \in Y, \ \exists y \in Y \ \textit{such that,} \\ x \in X \Rightarrow f(x,y) \leqq \varphi(f(x,y_0), \ f(x,y_1)) + \varepsilon. \end{cases}$$

We see that, in common with the situation already described for Neumann's result, it suffices that Irle's condition hold "to within ε". However, if φ is a suitable averaging function or, more generally, *mean function* in the sense of Kindler [11] then

$$\text{(3.5)} \qquad \begin{cases} \forall y_0, y_1 \in Y, \ \exists y \in Y \ \textit{such that,} \\ x \in X \Rightarrow f(x,y) \leqq \varphi(f(x,y_0), \ f(x,y_1)) \end{cases}$$

implies that (2.2) holds if $\beta < \gamma < \delta$.

Irle's minimax theorem was generalized by Simons [22], however it complicates the proof immensely to have to deal with "to within ε" conditions. In this paper, we shall follow the philosophy of Kindler [11] and not consider "to within ε" conditions. We hope that this simplification will show the underlying structures more clearly.

Using the same method of proof as that used in Lemma 2, one can establish the following more general result:

LEMMA 2′. *Let $T \subset Y$ and $\beta, \gamma \in \mathbf{R}$ with $\beta \leqq \gamma$. Suppose that, $\forall\, \delta > \gamma$, $\exists\, N \geqq 1$ and $\gamma_0, \ldots, \gamma_N \in \mathbf{R}$ such that $\gamma_0 = \delta$, $\gamma_N = \gamma$ and $\forall\, y_0, y_1 \in T$, $\exists\, y \in T$ such that, $\forall\, n \in \{1, \ldots, N\}$, (2.2.1), (2.2.2) and (2.2.4) hold. Let $x_0, x_1 \in X$ and $\underline{\beta|}x_0$ and $\underline{\beta|}x_1$ be joined by T. Then*

$$\underline{\gamma|}x_0 \cap \underline{\gamma|}x_1 \cap T \neq 0.$$

Kindler [11] was the first to observe that there are conditions resembling connectedness that are automatic in certain minimax theorems. He defines two concepts, *φ-connectedness* and *Γ-connectedness* and uses φ- connectedness to establish a general minimax theorem. We will not discuss φ-connectedness further since it involves a *mean function* φ, and the philosophy of this paper is to work as much as possible with the intrinsic properties of X, Y and f and avoid additional functions. The precise definition of Γ-connectedness is: *if* $\sup_X \inf_Y f < \beta < \gamma < \infty$, *$W$ is a finite subset of X, $x_0, x_1 \in X$, and $\underline{\beta|}x_0$* and *$\underline{\beta|}x_1$ are joined by $LE(W, \beta)$, then $\underline{\gamma|}x_0 \cap \cap\, \underline{\gamma|}x_1 \cap LE(W, \gamma) \neq 0$.* Thus Lemma 2′ can be used to give a sufficient condition for Γ-connectedness and, in fact, for a more general concept in which W is not restricted to be finite.

Sufficient conditions for (1.3)

We suppose throughout this section that $Z \subset Y$.

LEMMA 4. *Let X be a topological space, $\beta \in \mathbf{R}$, $x_0, x_1 \in X$, and C be a connected subset of X such that*

$$\text{(4.1)} \qquad C \ni x_0,\ x_1 \ \textit{and},\ \forall\, x \in C,\ \underline{\beta|}x \subset \underline{\beta|}x_0 \cup \underline{\beta|}x_1.$$

Suppose that

$$\text{(4.2)} \qquad \forall\, y \in Z,\ \{x : x \in C,\ f(x, y) < \beta\} \ \textit{is open in} \ C$$

and

$$\text{(4.3)} \qquad \forall\, x \in C,\ \exists\, y \in Z \ \textit{such that} \ f(x, y) < \beta.$$

Then $\exists x \in X$ *such that*

$$\underline{\beta|}x_0 \text{ and } \underline{\beta|}x_1 \text{ are joined by } \underline{\beta|}x \cap Z. \tag{4.4}$$

PROOF. We can suppose that

$$\underline{\beta|}x_0 \cap \underline{\beta|}x_1 \cap Z = \emptyset, \tag{4.5}$$

for otherwise (4.4) follows with $x := x_0$. For $i = 0, 1$, let

$$C_i := \left\{x : x \in C,\ \underline{\beta|}x \cap Z \subset \underline{\beta|}x_i\right\} \ni x_i. \tag{4.6}$$

From (4.1) and (4.5),

$$C_i = \left\{x : x \in C,\ \underline{\beta|}x \cap \underline{\beta|}x_{1-i} \cap Z = \emptyset\right\}. \tag{4.7}$$

From (4.3), (4.5) and (4.6),

$$C_0 \cap C_1 = \emptyset. \tag{4.8}$$

We can suppose that

$$C_0 \cup C_1 = C, \tag{4.9}$$

for if $x \in C \setminus (C_0 \cup C_1)$ then (4.4) follows from (4.1) and (4.7). Let $x \in C$. We now prove that

$$x \in C_0 \Leftrightarrow \exists\, y \in \underline{\beta|}x_0 \cap Z \text{ such that } f(x,y) < \beta. \tag{4.10}$$

($\Rightarrow$) If $x \in C_0$ and y is as in (4.3) then $y \in \underline{\beta|}x \cap Z$. From (4.6), $y \in \underline{\beta|}x_0 \cap Z$, as required. ($\Leftarrow$) If y is as in the right-hand side of (4.10) then $y \in \underline{\beta|}x \cap \underline{\beta|}x_0 \cap Z$. From (4.7), $x \notin C_1$. From (4.9) $x \in C_0$. This completes the proof of (4.10). From (4.2) and (4.10), C_0 is open in C. Similarly, C_1 is open in C. Then (4.8) and (4.9) contradict the connectedness of C. This contradiction completes the proof of the Lemma.

LEMMA 5. *Let X be a topological space, $\beta \in \mathbf{R}$, $x_0, x_1 \in X$, and C be a connected subset of X such that*

$$C \ni x_0, x_1 \quad \text{and,} \quad \forall x \in C,\ \underline{\beta|}x \subset \underline{\beta|}x_0 \cup \underline{\beta|}x_1. \tag{4.1}$$

Let Y be a compact topological space,

$$\{(x,y) : x \in C,\ y \in Z,\ f(x,y) \leqq \beta\} \text{ be closed in } C \times Y, \tag{5.1}$$

and

$$(5.2) \qquad \forall\, x \in C,\ \underline{\beta|}x \cap Z \neq \emptyset.$$

Then $\exists x \in X$ *such that*

$$(4.4) \qquad \underline{\beta|}x_0 \text{ and } \underline{\beta|}x_1 \text{ are joined by } \underline{\beta|}x \cap Z.$$

PROOF. Even though (5.2) is weaker than (4.3), we can proceed as in the proof of Lemma 4 up to (4.9). Instead of (4.10), we have: $\forall\, x \in C$,

$$(5.3) \qquad x \in C_0 \Leftrightarrow \exists\, y \in \underline{\beta|}x_0 \cap Z \quad \text{such that} \quad f(x,y) \leqq \beta.$$

Let x_λ be a net of elements of $C_0, x \in C$ and $x_\lambda \to x$. From (5.3),

$$\exists\, y_\lambda \in \underline{\beta|}x_0 \cap Z \quad \text{such that} \quad f(x_\lambda, y_\lambda) \leqq \beta.$$

Since Y is compact, by passing to an appropriate subnet, we can suppose that $\exists\, y \in Y$ such that $y_\lambda \to y$. Then $(x_\lambda, y_\lambda) \to (x,y)$ and $(x_0, y_\lambda) \to$ $\to (x_0, y)$. From (5.1), $y \in Z$, $f(x,y) \leqq \beta$ and $f(x_0,y) \leqq \beta$. From (5.3), $x \in C_0$. Thus C_0 is closed in C. Similarly, C_1 is closed in C. Then (4.8) and (4.9) contradict the connectedness of C. This contradiction completes the proof of the Lemma.

LEMMA 6. *Let* $\alpha, \beta \in \mathbf{R}$ *and* $\alpha < \beta$. *Suppose that,* $\forall\, \zeta < \alpha$, $\exists\, N \geqq 1$ *and* $\alpha_0, \dots, \alpha_n \leqq \beta$ *such that*

$$(6.1) \quad \begin{cases} & \alpha_0 = \zeta,\ \alpha_N = \alpha \text{ and,} \\ & \forall\, t_0, t_1 \in X,\ \exists\, x \in X \text{ such that}, \forall\, n \in \{1, \dots, N\}, \\ (6.1.1) & \underline{\alpha_n|}x \subset \underline{\alpha_{n-1}|}t_0 \cup \underline{\beta|}t_1, \\ (6.1.2) & \underline{\alpha_n|}x \subset \underline{\beta|}t_0 \cup \underline{\alpha_{n-1}|}t_1, \\ (6.1.3) & \underline{\beta|}x \subset \underline{\beta|}t_0 \cup \underline{\beta|}t_1, \\ (6.1.4) & \underline{\zeta|}x \subset \underline{\zeta|}t_0 \cup \underline{\zeta|}t_1. \end{cases}$$

Suppose that

$$(6.2) \qquad \forall\, x \in X,\ \underline{\alpha|}x \cap Z \neq \emptyset,$$

Let

$$(6.3) \qquad x_0, x_1 \in X,\ \inf f(x_0, Z) > -\infty \text{ and } \inf f(x_1, Z) > -\infty.$$

Then $\exists x \in X$ *such that*

$$(4.4) \qquad \underline{\beta|}x_0 \text{ and } \underline{\beta|}x_1 \text{ are joined by } \underline{\beta|}x \cap Z.$$

PROOF. From (6.3), we can choose $\zeta \in \mathbf{R}$ such that $\underline{\zeta}|x_0 \cap Z = \underline{\zeta}|x_1 \cap$ $\cap\, Z = \emptyset$. From (6.2), $\zeta < \alpha$. Let $N \geqq 1$ and $\alpha_0 \dots, \alpha_N$ satisfy (6.1). If $t \in X$ and $\underline{\zeta}|t \cap Z = \emptyset$ then, from (6.2),

$$\underline{\alpha_0}|t \cap Z = \underline{\zeta}|t \cap Z = \emptyset \text{ and } \underline{\alpha_N}|t \cap Z = \underline{\alpha}|t \cap Z \neq \emptyset.$$

Thus $\exists\,!\, g(t) \in \{1, \dots, N\}$ such that

$$(6.4) \quad g(t) \leqq n \leqq N \Rightarrow \underline{\alpha_n}|t \cap Z \neq \emptyset \text{ and } n = g(t) \Rightarrow \underline{\alpha_{n-1}}|t \cap Z = \emptyset.$$

For $i = 0, 1$ let $U_i := \left\{ t : t \in X,\ \underline{\zeta}|t \cap Z = \emptyset,\ \underline{\beta}|t \cap Z \subset \underline{\beta}|x_i \right\} \ni x_i$.
We fix $t_i \in U_i$ to maximize $g(t_i)$ and choose $x \in X$ to satisfy (6.1.1)–(6.1.4). From (6.1.4),

$$(6.5) \qquad \underline{\zeta}|x \cap Z = \emptyset.$$

From (6.1.3), $\underline{\beta}|x \cap Z \subset \left(\underline{\beta}|t_0 \cap Z\right) \cup \left(\underline{\beta}|t_1 \cap Z\right)$. Since $t_i \in U_i$,

$$(6.6) \qquad \underline{\beta}|x \cap Z \subset \underline{\beta}|x_0 \cup \underline{\beta}|x_1.$$

We next prove that

$$(6.7) \qquad \underline{\beta}|x \cap \underline{\beta}|x_1 \cap Z \neq \emptyset.$$

If $x \notin U_0$ then, from (6.5), $\underline{\beta}|x \cap Z \not\subset \underline{\beta}|x_0$ and (6.7) follows from (6.6). If, on the other hand, $x \in U_0$ we set $n := g(t_0)$. From the assumed maximality of $g(t_0)$, $g(x) \leqq n$. From (6.4),

$$\underline{\alpha_n}|x \cap Z \neq \emptyset \text{ and } \underline{\alpha_{n-1}}|t_0 \cap Z = \emptyset.$$

From (6.1.1), $\underline{\alpha_n}|x \cap \underline{\beta}|t_1 \cap Z \neq \emptyset$. (6.7) follows since $\alpha_n \leqq \beta$ and $t_1 \in U_1$. This completes the proof of (6.7). We can prove similarly that $\underline{\beta}|x \cap \underline{\beta}|x_0 \cap$ $\cap\, Z \neq \emptyset$. The result follows from (6.6).

REMARKS 7. The numbering of the statements in these remarks is chosen to correspond with the numbering of the statements in Remarks 3. The credits are identical.

$$(7.1) \quad \begin{cases} \forall\, t_0, t_1 \in X,\ \exists\, x \in X \quad \textit{such that,} \\ y \in Y \Rightarrow f(x, y) \geqq [f(t_0, y) + f(t_1, y)]/2 \end{cases}$$

implies

$$(7.2) \quad \begin{cases} \exists\, s \in (0,1) \quad \textit{such that,} \quad \forall\, t_0, t_1 \in X,\ \exists\, x \in X \quad \textit{such that,} \\ y \in Y \Rightarrow f(x, y) \geqq (1 - s)[(t_0, y) \vee f(t_1, y)] + s[f(t_0, y) \wedge f(t_1, y)] \end{cases}$$

which implies

$$(7.3)\quad \begin{cases} \exists \text{ a nondecreasing function } \pi : \mathbf{R}^+ \to \mathbf{R}^+ \quad \text{such that} \\ \lambda > 0 \Rightarrow \pi(\lambda) > 0 \\ \text{and} \quad \forall\, t_0, t_1 \in X,\ \exists\, x \in X \quad \text{such that,} \\ y \in Y \Rightarrow f(x,y) \geqq f(t_0,y) \wedge f(t_1,y) + \pi(|f(t_0,y) - f(t_1,y)|) \end{cases}$$

which implies

$$(7.4)\quad \begin{cases} \forall\, \varepsilon > 0,\ \exists\, \eta > 0 \text{ such that, } \forall\, t_0, t_1 \in X,\ \exists\, x \in X \text{ such that,} \\ y \in Y \text{ and } |f(t_0,y) - f(t_1,y)| \geqq \varepsilon \Rightarrow f(x,y) \geqq f(t_0,y) \wedge f(t_1,y) + \eta \\ \text{and} \quad y \in Y \Rightarrow f(x,y) \geqq f(t_0,y) \wedge f(t_1,y) \end{cases}$$

which implies that (6.1) holds if $\zeta < \alpha < \beta$. If φ is a suitable averaging or mean function

$$(7.5)\quad \begin{cases} \forall\, t_0, t_1 \in X,\ \exists\, x \in X \quad \text{such that,} \\ y \in Y \Rightarrow f(x,y) \geqq \varphi(f(t_0,y), f(t_1,y)) \end{cases}$$

also implies that (6.1) holds if $\zeta < \alpha < \beta$.

The following more abstract result can be used to prove both Lemma 2 and Lemma 6. *Let U and V be nonempty sets, $B : U \to 2^V$, and $\forall\, n \in \{1, \dots, N\}$, $D_n : U \to 2^V$. Let $D_0 = \emptyset$. Suppose that,*

$$\begin{aligned} &\forall\, t_0, t_1 \in U,\ \exists\, u \in U \quad \text{such that,} \quad \forall\, n \in \{1, \dots, N\}, \\ &\quad D_{n-1}t_0 = \emptyset \quad \text{and} \quad Bu \cap Bt_1 = \emptyset \Rightarrow D_n u = \emptyset, \\ &\quad D_{n-1}t_1 = \emptyset \quad \text{and} \quad Bu \cap Bt_0 = \emptyset \Rightarrow D_n u = \emptyset, \end{aligned}$$

and

$$Bu \subset Bt_0 \cup Bt_1.$$

Suppose also that $\{Bu\}_{u \in U}$ is pseudoconnected and, $\forall\, u \in U$, $D_N u \neq \emptyset$. Then $\forall\, u_0, u_1 \in U$, $Bu_0 \cap Bu_1 \neq \emptyset$.

We note, finally, that (4.1) automatically holds if, $\forall\, y \in Y$, $f(\cdot, y)$ is quasiconcave in the sense of interval spaces.

Applications of Theorem 1

For Theorems 8 and 9, we suppose that Y is a topological space, $\mathcal{B}$ is a nonempty subset of $\mathbf{R}$, $\inf \mathcal{B} = \sup_X \inf_Y f$ and, $\forall\, \beta \in \mathcal{B}$,

$$(8.1)\quad \forall\, x \in X,\ \underline{\beta|}x \quad \text{is nonempty, closed and compact,}$$

and *either*

$$\text{(8.2)} \qquad \forall \text{ nonempty finite subsets } V \text{ of } X, \ LE(V,\beta) \text{ is connected}$$

or

$$\text{(8.3)} \qquad \begin{cases} \forall\, \delta > \gamma > \beta \quad \text{and} \quad x \in X, \ \underline{\gamma}\rfloor x \ \text{ is closed and} \\ \exists\, N \geqq 1 \quad \text{and} \quad \gamma_0, \ldots, \gamma_N \in \mathbf{R} \ \text{ such that (2.2) holds.} \end{cases}$$

(The choice can depend on β.) We point out that the "nonempty" assumption in (8.1) automatically holds if *either*, $\forall\, \beta \in \mathcal{B}$, $\beta > \sup_X \inf_Y f$ *or*, $\forall\, x \in X$, $\min f(x,Y)$ exists.

THEOREM 8. *Let Y be compact, X be a topological space and, $\forall\, \beta \in \mathcal{B}$ and $x_0, x_1 \in X, \exists$ a connected subset C of X such that*

$$\text{(4.1)} \qquad C \ni x_0,\ x_1 \ \text{and, } \forall\, x \in C,\ \underline{\beta}\rfloor x \subset \underline{\beta}\rfloor x_0 \cup \underline{\beta}\rfloor x_1$$

and

$$\{(x,y) : x \in C,\ y \in Y,\ f(x,y) \leqq \beta\} \ \textit{is closed in}\ C \times Y.$$

Then

$$\min_Y \sup_X f = \sup_X \inf_Y f.$$

PROOF. Let $\beta \in \mathcal{B}$. By assumption, (1.1) holds and, from Lemma 2 if necessary, if W is finite then (1.2) holds. From Lemma 5 with $Z := Y$,

$$\text{if } W = \emptyset \text{ then (1.3) holds.}$$

Now suppose that $n \geqq 1$ and

$$\text{if card } W \leqq n - 1 \text{ then (1.3) holds.}$$

From the proof of Theorem 1, if card $V \leqq n + 1$ then $LE(V,\beta) \neq \emptyset$. Thus

$$\text{if card } W \leqq n \text{ and } Z = LE(W,\beta) \text{ then (5.2) holds.}$$

From Lemma 5,

$$\text{if card } W \leqq n \text{ then (1.3) holds.}$$

Thus we have proved by induction that

$$\text{if } W \text{ is finite then (1.3) holds.}$$

The result follows from Theorem 1.

THEOREM 9. *Suppose that either*

$$(9.1)\quad \begin{cases} \forall\,\beta \in \mathcal{B},\ \beta > \sup_X \inf_Y f,\ X \textit{ is a topological space and,} \\ \forall\, x_0, x_1 \in X,\ \exists \textit{ a connected subset } C \textit{ of } X \\ \textit{such that } (4.1) \textit{ holds and} \\ \forall\, y \in Y,\ \{x : x \in C,\ f(x,y) < \beta\} \textit{ is open in } C. \end{cases}$$

or,

$$(9.2)\quad \begin{cases} \forall\,\beta \in \mathcal{B},\ \beta > \sup_X \inf_Y f, \\ \forall\,\zeta < \alpha < \beta,\ \exists\, N \geqq 1 \textit{ and } \alpha_0, \dots, \alpha_N \leqq \beta \textit{ such that } (6.1) \textit{ holds} \\ \textit{and } \forall\, x \in X,\ \inf f(x,Y) > -\infty. \end{cases}$$

Then

$$\min_Y \sup_X f = \sup_X \inf_Y f.$$

PROOF. By assumption, $\forall\,\beta \in \mathcal{B}$, (1.1) holds and, from Lemma 2 if necessary, if W is finite then (1.2) holds. From Lemma 4 or Lemma 6 with $Z := Y$,

$$\text{if } \beta \in \mathcal{B} \text{ and } W = \emptyset \text{ then (1.3) holds.}$$

Now suppose that $n \geqq 1$ and

$$\text{if } \beta \in \mathcal{B} \text{ and card } W < n - 1 \text{ then (1.3) holds.}$$

If $\beta \in \mathcal{B}$, we choose $\alpha \in \mathcal{B}$ such that $\alpha < \beta$. From the proof of Theorem 1 with β replaced by α, if card $V \leqq n + 1$ then $LE(V, \alpha) \neq \emptyset$. Thus

$$\text{if } \beta \in \mathcal{B},\ \text{card } W \leqq n \text{ and } Z = LE(W, \beta) \text{ then (4.3) and (6.2) hold.}$$

From Lemma 4 or Lemma 6,

$$\text{if } \beta \in \mathcal{B} \text{ and card } W \leqq n \text{ then (1.3) holds.}$$

Thus we have proved by induction that

$$\text{if } \beta \in \mathcal{B} \text{ and } W \text{ is finite then (1.3) holds.}$$

The result follows from Theorem 1.

REMARKS 10. The minimax theorems referred to in the introduction that depend only on *connectedness* follow from either Theorem 8–(8.2) or Theorem 9–(8.2, 9.1). Those that depend on *algebraic* conditions, and their *set-theoretic* generalizations follow from Theorem 9–(8.3, 9.2). Those that *mix* algebraic conditions and connectedness follow from Theorem 9–(8.2, 9.2). Theorem 8–(8.3) and Theorem 9–(8.3, 9.1) give new results. We remark, finally, that in Theorem 8 and Theorem 9–(9.1), C can depend on β.

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