Hence, for $m^2 \leqslant n < (m+1)^2$,

$$\frac{na_n}{A_n} \geqslant \frac{m^2 2^m}{(m-1)\,2^{m+3}+9} \to \infty,$$

and

$$\frac{a_{m^2}}{A_n} \geqslant \delta_m := \frac{m 2^{m+1}}{(m-1)\,2^{m+3}+9} \to \frac{1}{4}.$$

Now let $p > 1$, and define $x := \{x_k\} \in l_p$ by setting

$$x_k := \begin{cases} \dfrac{1}{m^{1/p}\log m} & \text{if } \ k = m^2, \quad m = 2, 3, ... \\ 0 & \text{otherwise,} \end{cases}$$

and let

$$y_n := \frac{1}{A_n} \sum_{k=0}^{n} a_k x_k.$$

Then, for $4 \leqslant m^2 \leqslant n < (m+1)^2$,

$$y_n \geqslant \frac{a_{m^2}}{A_n} x_{m^2} \geqslant \frac{\delta_m}{m^{1/p} \log m}.$$

Thus

$$\sum_{n=m^2}^{(m+1)^2-1} y_n^p \geqslant \frac{2m\delta_m^p}{m \log^p m}, \qquad \text{and so} \qquad \sum_{n=0}^{\infty} y_n^p \geqslant \sum_{m=2}^{\infty} \frac{2\delta_m^p}{\log^p m} = \infty.$$

Consequently $M_a \notin B(l_p)$, even though $\lim A_n/na_n = 0$.

## References


1. D. Borwein, Generalized Hausdorff matrices as bounded operators on $l_p$, *Math. Z.* **183** (1983), 483–487.
2. D. Borwein, F. P. Cass, and J. E. Sayre, On generalized Hausdorff matrices, *J. Approx. Theory* **48** (1986), 354–360.
3. D. Borwein and A. Jakimovski, Matrix operators on $l_p$, *Rocky Mountain J. Math.* **9** (1979), 463–476.
4. F. P. Cass and W. Kratz, Nörlund and weighted mean matrices as operators on $l_p$, *Rocky Mountain J. Math.* **20** (1990), 59–74.
5. F. Hausdorff, Summationsmethoden und Momentfolgen I, *Math. Z.* **9** (1921), 280–299.


JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **178**, 529–546 (1993)

# Intersection Theorems and Minimax Theorems Based on Connectedness

Jürgen Kindler

*Fachbereich Mathematik, Technische Hochschule Darmstadt, Schlossgartenstr. 7, D-64289 Darmstadt, Germany*

*Submitted by K. Fan*

Received April 1, 1992

## 1. Introduction

Throughout, let a system $\mathscr{C} = \{C_x : x \in X\}$ of nonvoid subsets of a set $Y$ be given ($X$ an index set). Then $\mathscr{C}^* := \{C_y^* : y \in Y\}$ with $C_y^* = \{x \in X : y \notin C_x\}$ is the system of *conjugate sets*. We write

$$C(A) := \bigcap \{C_x : x \in A\}, \quad A \subset X, \qquad \text{with} \quad C(\varnothing) = Y,$$

$$C^*(B) := \bigcap \{C_y^* : y \in B\} = \{x \in X : B \cap C_x = \varnothing\}, \quad B \subset Y,$$

$$\text{with} \quad C^*(\varnothing) = X,$$

and, for nonvoid $\mathscr{K} \subset 2^X$ and $\mathscr{L} \subset 2^Y$,

$$C(\mathscr{K}) = \{C(K) : K \in \mathscr{K}\} \qquad \text{and} \qquad C^*(\mathscr{L}) = \{C^*(L) : L \in \mathscr{L}\}.$$

Finally, we set $\mathscr{E}(X) = \{A \subset X : A \text{ finite nonvoid}\}$.

We are interested in the following

*Intersection Problems*

(a) When is $C(X) = \bigcap \{C_x : x \in X\}$ nonvoid, i.e., when does $\varnothing \notin C(2^X)$ hold?

(b) When does $\mathscr{C}$ possess the "finite intersection property" $C(A) = \bigcap \{C_x : x \in A\} \neq \varnothing$ for all $A \in \mathscr{E}(X)$, i.e., when does $\varnothing \notin C(\mathscr{E}(X))$ hold?

The present investigations arose out of the study of minimax theorems: Wu [19] was the first to observe that connectedness—rather than convexity—is essential in the proof of minimax theorems. By a refined method

Tuy [17], [18] derived a generalized version of Sion's classical minimax theorem [14]. Compare also [1] for a related result. Independently, inspired by Joó's paper [4], Stachó [15] established an intersection theorem which was used by Komornik [12] to derive a generalization of Ha's minimax theorem [2]. Both concepts were unified by Kindler-Trost [7].

All these techniques rely on the fact that the minimax relation

$$\sup_{x\in X}\inf_{y\in Y} a(x,y)=\inf_{y\in Y}\sup_{x\in X} a(x,y) \qquad \text{MM}$$

holds if and only if certain systems of level sets have nonvoid intersection (cf. Remark 10 below). The starting point of our investigations was the observation in [5] that some kind of *abstract connectedness* of certain level sets ("$\Gamma$-connectedness of $Y$") is *necessary* for MM to hold. Recently, abstract connectivity has been studied by Simons [13], König [9], and König-Zartman [10] in the same context.

In the following, it will be shown that the three properties

- $C(2^X)$ is connected.
- $C^*(2^Y)$ is connected.
- $\mathscr{C}$ is compact.

are sufficient for $\bigcap\{C_x : x\in X\}$ to be nonvoid. Moreover, the second condition can be slightly weakened such that the three properties become sufficient *and* necessary. Of course, our "abstract" intersection theorems can be specialized to the concrete situation of *topological connectedness*. We then obtain as special cases recent results of Horvath [3], Kindler's "topological intersection theorem" [6], and König's "minimax theorems based on connectedness" [9].

## 2. Abstract Intersection Theorems

If $S$ is a nonvoid set, then $2^S$ will denote its power set. A nonvoid $\mathscr{P}\subset 2^S$ will be called *paving in S*.

Let $\mathscr{A}$ and $\mathscr{B}$ be pavings in $S$. We say that

$\mathscr{A}$ *is connected for* $\mathscr{B}$ iff $A\in\mathscr{A}$, $B_1$, $B_2\in\mathscr{B}$, $A\subset B_1\cup B_2$, and $A\cap B_1\cap B_2=\varnothing$ imply $A\subset B_1$ or $A\subset B_2$.

In case $\mathscr{A}=\{A\}$ we say that $A$ *is connected for* $\mathscr{B}$, and in case $\mathscr{B}=\mathscr{A}$ the paving $\mathscr{A}$ is called *connected.*

$\mathscr{A}$ is *compact* iff for every nonvoid $\mathscr{R}\subset\mathscr{A}$ with $\bigcap\{B : B\in\mathscr{F}\}\neq\varnothing$ for all finite nonvoid $\mathscr{F}\subset 2^{\mathscr{R}}$ we have $\bigcap\{R : R\in\mathscr{R}\}\neq\varnothing$.



$\mathscr{A}$ is *finitely intersectional* iff $A_1$, $A_2\in\mathscr{A}$ implies $A_1\cap A_2\in\mathscr{A}$, and $\mathscr{A}$ is *intersectional* iff $\bigcap\{R : R\in\mathscr{R}\}\in\mathscr{A}$ for every nonvoid $\mathscr{R}\subset\mathscr{A}$. Finally, we set $\bar{\mathscr{A}} := \{S-A : A\in\mathscr{A}\}$ and $\mathscr{A}^d=\{\bigcap_{i=1}^n A_i : A_i\in\mathscr{A}, i\leqslant n\in\mathbb{N}\}$.

*Remark* 1. (a) If $\mathscr{B}$ is a chain in $S$ (i.e., $B_1$, $B_2\in\mathscr{B}$ implies $B_1\subset B_2$ or $B_2\subset B_1$) then every paving $\mathscr{A}$ in $S$ is connected for $\mathscr{B}$. In particular, every $\mathscr{A}$ is connected for $\{\varnothing\}$ or for $\{\varnothing, S\}$.

(b) Every finitely intersectional paving $\mathscr{A}$ in $S$ with $\varnothing\notin\mathscr{A}$ is connected.

(c) $\mathscr{A}$ is connected for $\mathscr{B}$ iff $\mathscr{A}$ is connected for $\bar{\mathscr{B}}$.

(d) Let $\mathscr{A}$ be upward directed and connected for $\mathscr{B}$. Then the set $\bigcup\{A : A\in\mathscr{A}\}$ is connected for $\mathscr{B}$.

(e) If the subsets of $\mathscr{A}$ are pairwise disjoint then $\mathscr{A}$ is connected.

Remark 2. *For a nonvoid subset* $A\subset S$ *and a paving* $\mathscr{B}$ *in* $S$ *the following are equivalent*:

(a) *$A$ is connected for $\mathscr{B}$.*

(b) *For all $(s_1, s_2)\in A\times A$ there exists a $C\subset S$ such that (i) $\{s_1, s_2\}\subset C\subset A$, and (ii) $C$ is connected for $\mathscr{B}$.*

*Proof.* (a) $\Rightarrow$ (b): Take $C=A$.

(b) $\Rightarrow$ (a): Let $B_1, B_2\in\mathscr{B}$ with $A\subset B_1\cup B_2$ and $A\cap B_1\cap B_2=\varnothing$. Suppose that for $k\in\{1,2\}$ there exist $s_k\in A\cap B_k$. Choose $C$ as in (b). Then $C\cap B_k\neq\varnothing$, $k\in\{1,2\}$, $C\subset B_1\cup B_2$, and from (ii) we infer $\varnothing\neq C\cap B_1\cap B_2\subset A\cap B_1\cap B_2$, a contradiction.

Remark 3. *Let* $\mathscr{C}=\{C_x : x\in X\}$ *as in Section* 1 *and* $\mathscr{B}$ *a paving in* $X$. *Then the following are equivalent*:

(a) *$C^*(2^Y)$ is connected for $\mathscr{B}$.*

(b) *$\{C^*(Y-(C_{x_1}\cup C_{x_2})) : (x_1, x_2)\in X\times X\}$ is connected for $\mathscr{B}$.*

*Proof.* (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (a): For $B\subset Y$ let $\{x_1, x_2\}\subset C^*(B)$. Then $D := C^*(Y-(C_{x_1}\cup C_{x_2}))$ is connected for $\mathscr{B}$ and $\{x_1, x_2\}\subset D\subset C^*(B)$. Now apply Remark 2.

Now we introduce a concept which will be basic for our further investigations:

In the following, let $T(\mathscr{C})$ denote the system of all triplets $\tau=(x_1, x_2, E)\in X\times X\times(\mathscr{E}(X)\cup\{\varnothing\})$ such that

$$C(E\cup\{x\})\neq\varnothing \quad \text{for all} \quad x\in X, \quad \text{and} \tag{1}$$

$$C(E\cup\{x_1, x_2\})=\varnothing. \tag{2}$$

*Remark* 4. (a) $\varnothing \notin C(\mathscr{E}(X)) \Leftrightarrow C^*(C(\mathscr{E}(X))) = \{\varnothing\} \Leftrightarrow T(\mathscr{C}) = \varnothing$.

(b) $\varnothing \notin C(2^X) \Leftrightarrow C(X) \neq \varnothing \Leftrightarrow C^*(C(2^X)) = \{\varnothing\} \Leftrightarrow \varnothing \notin C(\mathscr{E}(X))$ and $\mathscr{C}$ is compact.

For $\tau = (x_1, x_2, E) \in T(\mathscr{C})$ we define the sets

$$D_0(\tau) = C^*(C(E) - (C_{x_1} \cup C_{x_2})) \in C^*(2^Y) \text{ and}$$

$$D_k(\tau) = C^*(C(E \cup \{x_k\})) \in C^*(C(\mathscr{E}(X))),\ k \in \{1, 2\},$$

and we set

$$T_1(\mathscr{C}) = \{\tau \in T(\mathscr{C}) : D_0(\tau) \subset D_1(\tau) \cup D_2(\tau)\} \text{ and}$$

$$T_2(\mathscr{C}) = \{\tau \in T(\mathscr{C}) : D_0(\tau) \text{ is connected for } \{D_1(\tau), D_2(\tau)\}\}.$$

The proof of the following three lemmata is obvious:

LEMMA 1. *For $\tau = (x_1, x_2, E) \in T(\mathscr{C})$ we have*

$$D_0(\tau) = \{x \in X : C(E \cup \{x\}) \subset C_{x_1} \cup C_{x_2}\}, \tag{3}$$

$$x_{3-k} \in D_0(\tau) \cap D_k(\tau), \quad k \in \{1, 2\}, \quad \text{and} \tag{4}$$

$$D_0(\tau) \cap D_1(\tau) \cap D_2(\tau) = \varnothing. \tag{5}$$

LEMMA 2. $T_1(\mathscr{C}) \cap T_2(\mathscr{C}) = \varnothing$.

LEMMA 3. *For $\tau = (x_1, x_2, E) \in T(\mathscr{C})$ the following are equivalent*:

(a) $\tau \in T_1(\mathscr{C})$.

(b) $\{C(E \cup \{x\}) : x \in D_0(\tau)\}$ *is connected for* $\{C_{x_1}, C_{x_2}\}$.

(c) *There exist sets $F_k \supset C_{x_k}$, $k \in \{1, 2\}$, such that $C(E \cup \{x\}) \cap F_1 \cap F_2 = \varnothing$, $x \in D_0(\tau)$, and $\{C(E \cup \{x\}) : x \in D_0(\tau)\}$ is connected for $\{F_1, F_2\}$.*

The above observations lead to the following two abstract intersection theorems:

THEOREM 1. The following are equivalent:

(a) $\bigcap \{C_x : x \in A\} \neq \varnothing$ *for all* $A \in \mathscr{E}(X)$.

(b) *$C(\mathscr{E}(X))$ is connected for $\mathscr{C}$, and $C^*(2^Y)$ is connected for $C^*(C(\mathscr{E}(X)))$.*

(c) *There exist two pavings $\mathscr{K}_1 \supset C^*(\mathscr{L})$ and $\mathscr{K}_2 \supset \mathscr{C}^*$ such that $\mathscr{K}_2$ is intersectional and connected for $\mathscr{K}_1$, and a finitely intersectional connected paving $\mathscr{L} \supset \mathscr{C}$.*



*Proof.* (a) $\Rightarrow$ (c): $\mathscr{L} = C(\mathscr{E}(X))$ is finitely intersectional and connected, because $\varnothing \notin \mathscr{L}$. By Remark 4a) we have $C^*(\mathscr{L}) = \{\varnothing\}$, hence we may take $\mathscr{K}_1 = \{\varnothing\}$ and $\mathscr{K}_2 = 2^X$, say.

(c) $\Rightarrow$ (b): $\mathscr{C} \subset \mathscr{L}$, $\mathscr{L}$ finitely intersectional and connected, implies that $C(\mathscr{E}(X))$ is connected for $\mathscr{C}$, and from $C^*(C(\mathscr{E}(X))) \subset C^*(\mathscr{L}) \subset \mathscr{K}_1$ and $C^*(2^Y) \subset \mathscr{K}_2$ it follows that $C^*(2^Y)$ is connected for $C^*(C(\mathscr{E}(X)))$.

(b) $\Rightarrow$ (a): Suppose that $\varnothing \in C(\mathscr{E}(X))$. Then there exists a $\tau \in T(\mathscr{C})$. By Lemma 3 we have $\tau \in T_1(\mathscr{C})$ because $C(\mathscr{E}(X))$ is connected for $\mathscr{C}$. On the other hand, $D_0(\tau) \in C^*(2^Y)$ is connected for $\{D_1(\tau), D_2(\tau)\} \subset C^*(C(\mathscr{E}(X)))$ in contradiction to Lemma 2.

As an immediate consequence we obtain

THEOREM 2. *The following are equivalent*:

(a) $\bigcap \{C_x : x \in X\} \neq \varnothing$.

(b) *$C(\mathscr{E}(X))$ is connected for $\mathscr{C}$, $C^*(2^Y)$ is connected for $C^*(C(\mathscr{E}(X)))$, and $\mathscr{C}$ is compact.*

(c) *There exist two pavings $\mathscr{K}_1 \supset C^*(\mathscr{L})$ and $\mathscr{K}_2 \supset \mathscr{C}^*$ such that $\mathscr{K}_2$ is intersectional and connected for $\mathscr{K}_1$, and a finitely intersectional, compact and connected paving $\mathscr{L} \supset \mathscr{C}$.*

COROLLARY 1. *Suppose that there exist a finitely intersectional, compact and connected paving $\mathscr{L} \supset \mathscr{C}$ and an intersectional connected paving $\mathscr{K} \supset \mathscr{C}^*$. Then $\bigcap \{C_x : x \in X\} \neq \varnothing$.*

*Remark* 5. Theorem 2 remains true if in condition (b) we replace "$C(\mathscr{E}(X))$ is connected for $\mathscr{C}$" by the stronger property "$C(2^X)$ is connected" (compare Remark 1b)). One might conjecture that the assumption "$C^*(2^Y)$ is connected for $C^*(C(\mathscr{E}(X)))$" may also be replaced by the stronger property "$C^*(2^Y)$ is connected". The following example shows that this is false.

EXAMPLE 1. Let $X = \{1, 2\}$, $Y = \{1, 2, 3, 4\}$, $C_1 = \{1, 2\}$, and $C_2 = \{2, 3\}$. Here $C^*(2^Y) = 2^X$ is not connected.

The above results are of very simple structure. For applications in complex situations it seems worthwile to provide a more flexible version:

LEMMA 4. *Let $\tau = (x_1, x_2, E) \in T_1(\mathscr{C})$. Suppose that $Y$ has nonvoid subsets $G_x$, $x \in D_0(\tau)$, $H_x$, $x \in E$, and $F_1, F_2$ such that*

(i) $H(E) \cap G_x \neq \varnothing$, $x \in D_0(\tau)$,

(ii) $G_x \subset C_x$, $x \in D_0(\tau)$, *and* $H_x \subset C_x$, $x \in E$,

(iii) $C_{x_k} \subset F_k$, $k \in \{1, 2\}$,

(iv) $C(E \cup \{x\}) \cap F_1 \cap F_2 = \varnothing$, $x \in D_0(\tau)$.

*Then* $D_0(\tau) \cap D_k(\tau) = \{x \in D_0(\tau) : G_x \cap H(E) \cap F_k = \varnothing\}$, $k \in \{1, 2\}$.

*Proof.* For $k \in \{1, 2\}$ let $x \in L_k(\tau) := \{x \in D_0(\tau) : G_x \cap H(E) \cap F_k = \varnothing\}$. Then from (i), (ii), (iii) together with (3) we obtain $\varnothing \neq H(E) \cap G_x = (H(E) \cap G_x \cap C_{x_1}) \cup (H(E) \cap G_x \cap C_{x_2}) = H(E) \cap G_x \cap C_{x_{3-k}} \subset C(E \cup \{x, x_{3-k}\})$. Therefore, $x \notin D_{3-k}(\tau)$ so $x \in D_k(\tau)$ because $\tau \in T_1(\mathscr{C})$. Conversely, let $x \in D_0(\tau) \cap D_k(\tau)$. Then by (3) and (iii) we have $C(E \cup \{x\}) = C(E \cup \{x, x_{3-k}\}) \subset F_{3-k}$, and (iv) implies $C(E \cup \{x\}) \cap F_k = \varnothing$, hence $x \in L_k(\tau)$ by (ii).

Proposition 1. *For* $\mathscr{C} = \{C_x : x \in X\}$ *let* $\mathscr{H} = \{H_x : x \in X\}$, $\mathscr{G} = \{G_x : x \in X\}$ *and* $\mathscr{F} = \{F_x : x \in X\}$ *with*

$$\varnothing \neq H_x \subset G_x \subset C_x \subset F_x, \quad x \in X \tag{o}$$

*be given. Suppose that there exist pavings* $\mathscr{L}_1, \mathscr{L}_2$ *in* $Y$ *and* $\mathscr{K}_1, \mathscr{K}_2$ *in* $X$ *such that*

(i) $\mathscr{L}_1 \supset \mathscr{C}$, $\mathscr{L}_1$ *is finitely intersectional and connected for* $\mathscr{F}$,

(ii) $\mathscr{L}_2 \supset \mathscr{H} \cup \mathscr{F}$, $\mathscr{L}_2$ *is finitely intersectional,*

(iii) $\mathscr{K}_2 \supset \mathscr{C}^*$, $\mathscr{K}_2$ *is intersectional and connected for* $\mathscr{K}_1$,

(iv) *either* $G^*(\mathscr{L}_2) \subset \mathscr{K}_1$ *or* $H^*(\mathscr{L}_2) \subset \mathscr{K}_1$.

*Then* $T(\mathscr{F}) \cap T(\mathscr{H}) = \varnothing$.

*Proof.* Suppose that there exists a $\tau = (x_1, x_2, E) \in T(\mathscr{F}) \cap T(\mathscr{H})$. Then $\tau \in T_1(\mathscr{C})$ by (o), (i), and Lemma 3. Hence, if we set $L_k = G^*(H(E) \cap F_{x_k})$ in case $G^*(\mathscr{L}_2) \subset \mathscr{K}_1$ and $L_k = H^*(H(E) \cap F_{x_k})$ otherwise, then by Lemma 4 we have $D_0(\tau) \cap D_k(\tau) = D_0(\tau) \cap L_k$, $k \in \{1, 2\}$. With (4), (5), and $\tau \in T_1(\mathscr{C})$ we obtain $D_0(\tau) \cap L_k \neq \varnothing$, $k \in \{1, 2\}$, $D_0(\tau) \cap L_1 \cap L_2 = \varnothing$, and $D_0(\tau) \subset L_1 \cup L_2$. Thus, $D_0(\tau)$ is not connected for $\{L_1, L_2\}$. But $D_0(\tau)$ is connected for $\mathscr{K}_1$ by (iii), and $\{L_1, L_2\} \subset \mathscr{K}_1$ by (ii), (iv), which leads to a contradiction.

The above Proposition, together with Remark 4, can be used to establish generalized versions of Theorems 1 and 2. We shall confine ourselves to presenting applications in minimax theory (cf. Proposition 2 and Theorems 8, 9 and 10 below).

## 3. Topological Intersection Theorems

If $(S, \mathscr{T}_S)$ or $S$, for short, is a topological space then $\mathfrak{G}(S) (= \mathscr{T}_S)$, $\mathfrak{F}(S)$, $\mathfrak{K}(S)$ will denote the system of all open, closed, compact subsets of $S$. Of



course, a nonvoid subset $A \subset S$ is connected (in its relative topology) iff $A$ is connected for $\mathfrak{G}(S)$ (or for $\mathfrak{F}(S)$). We set $\mathfrak{C}(S) = \{A \subset S$: A nonvoid connected$\} \cup \{\varnothing\}$.

*Remark* 6. Let $S$ be a topological space. Then every paving $\mathscr{P} \subset \mathfrak{F}(S) \cap \mathfrak{K}(S)$ is compact.

We are now able to present generalized versions of the two main results in [6]:

Theorem 3. *The following are equivalent:*

(a) $\bigcap \{C_x : x \in A\} \neq \varnothing$ *for all* $A \in \mathscr{E}(X)$.

(b) *There exist topologies on* $X$ *and* $Y$ *such that* $C(\mathscr{E}(X)) \subset \mathfrak{G}(Y) \cap \mathfrak{C}(Y)$, $C^*(C(\mathscr{E}(X))) \subset \mathfrak{G}(X)$, *and* $C^*(\mathfrak{F}(Y)) \subset \mathfrak{C}(X)$.

(b)* *As* (b) *with* $\mathfrak{G}(X)$ *replaced by* $\mathfrak{F}(X)$.

(c) *There exist topologies on* $X$ *and* $Y$ *such that* $\mathfrak{C}(X)$ *is intersectional and contains* $\mathscr{C}^*$, $\mathfrak{G}(Y) \cap \mathfrak{C}(Y)$ *is finitely intersectional and contains* $\mathscr{C}$, *and* $C^*(\mathfrak{G}(Y)) \subset \mathfrak{G}(X)$.

(c)* *As* (c) *with* $\mathfrak{G}(X)$ *replaced by* $\mathfrak{F}(X)$.

(d) *There is a topology on* $Y$ *with* $\mathscr{C} \subset \mathfrak{G}(Y) \subset \mathfrak{C}(Y)$.

*Proof.* (a) $\Rightarrow$ (d): Take the topology $\mathscr{T}_Y = \{G \subset Y : \exists A \in \mathscr{E}(X)$ with $G \supset C(A)\} \cup \{\varnothing\}$.

(d) $\Rightarrow$ (c), (c)*: Take $\mathscr{T}_X = \{\varnothing, X\}$ and $\mathscr{T}_Y$ according to (d). Then $\mathscr{C}^* \subset 2^X = \mathfrak{C}(X)$, $\mathscr{C} \subset \mathfrak{G}(Y) \subset \mathfrak{C}(Y)$, and $C^*(\mathfrak{G}(Y) - \{\varnothing\}) = \{\varnothing\}$.

The implications (c) $\Rightarrow$ (b) and (c)* $\Rightarrow$ (b)* are obvious.

(b) $\Rightarrow$ (a), (b)* $\Rightarrow$ (a): From $C(\mathscr{E}(X)) \subset \mathfrak{G}(Y) \cap \mathfrak{C}(Y)$ it follows that $C(\mathscr{E}(X))$ is connected (for $\mathscr{C}$), and together with Remark 3 we infer that $C^*(2^Y)$ is connected for $C^*(C(\mathscr{E}(X)))$. Now apply Theorem 1 "(b) $\Rightarrow$ (a)".

Theorem 4. *The following are equivalent:*

(a) $\bigcap \{C_x : x \in X\} \neq \varnothing$.

(b) *There exist topologies on* $X$ *and* $Y$ *such that* $\mathscr{C} \cap \mathfrak{K}(Y) \neq \varnothing$, $C(\mathscr{E}(X)) \subset \mathfrak{F}(Y) \cap \mathfrak{C}(Y)$, $C^*(C(\mathscr{E}(X))) \subset \mathfrak{F}(X)$, *and* $C^*(\mathfrak{G}(Y)) \subset \mathfrak{C}(X)$.

(b)* *As* (b) *with* $\mathfrak{F}(X)$ *replaced by* $\mathfrak{G}(X)$.

(c) *There exist topologies on* $X$ *and* $Y$ *such that* $\mathfrak{C}(X)$ *is intersectional and contains* $\mathscr{C}^*$, $\mathfrak{F}(Y) \cap \mathfrak{C}(Y)$ *is finitely intersectional and contains* $\mathscr{C}$, $\mathscr{C} \cap \mathfrak{K}(Y) \neq \varnothing$, *and* $C^*(\mathfrak{F}(Y)) \subset \mathfrak{F}(X)$.

(c)* *As* (c) *with* $\mathfrak{F}(X)$ *replaced by* $\mathfrak{G}(X)$.

(d) *There is a compact topology on* $Y$ *with* $\mathscr{C} \subset \mathfrak{F}(Y) \subset \mathfrak{C}(Y)$.

*Proof.* (a)$\Rightarrow$(d): Take $\mathcal{T}_Y = \{G \subset Y: \hat{y} \notin G\} \cup \{Y\}$ for some $\hat{y} \in C(X)$.
(d)$\Rightarrow$(c), (c)*: Take $\mathcal{T}_X = \{\emptyset, X\}$ and $\mathcal{T}_Y$ according to (d).
(c)$\Rightarrow$(b) and (c)*$\Rightarrow$(b)* are obvious.
(b)$\Rightarrow$(a) and (b)*$\Rightarrow$(a): As in the proof of Theorem 3 it follows that $\emptyset \notin C(\mathscr{E}(X))$. Take $x_0 \in X$ with $C_{x_0} \in \mathfrak{K}(Y)$. Then $\emptyset \notin \mathscr{P} := \{C(A \cup \{x_0\}): A \in \mathscr{E}(X)\}$, hence $C(X) = \bigcap \{P: P \in \mathscr{P}\} \neq \emptyset$ according to Remark 6.

*Remark* 7. A topological space $S$ with $\mathfrak{G}(S) \subset \mathfrak{C}(S)$ or with $\mathfrak{F}(S) \subset \mathfrak{C}(S)$ (compare condition (d) in Theorems 3 and 4) is called *hyperconnected* resp. *ultraconnected.* Examples can be found in [16].

*Remark* 8. The set system $\mathscr{C} = \{C_x: x \in X\}$ gives rise to a *correspondence* $\Phi: X \to 2^Y - \{\emptyset\}$ according to $\Phi(x) = C_x$, $x \in X$. Here, $\Phi$ is upper semicontinuous or lower semicontinuous iff $C^*(\mathfrak{F}(Y)) \subset \mathfrak{G}(X)$ resp. $C^*(\mathfrak{G}(Y)) \subset \mathfrak{F}(X)$ holds. Compare [8] for details.

COROLLARY 2. [3, Theorem 6]. *Let $X$ and $Y$ be endowed with topologies such that $\mathscr{C} \cap \mathfrak{K}(Y) \neq \emptyset$, $C(\mathscr{E}(X)) \subset \mathfrak{F}(Y) \cap \mathfrak{C}(Y)$, and $C^*(2^Y) \subset \mathfrak{F}(X) \cap \mathfrak{C}(X)$. Then $\bigcap \{C_x: x \in X\} \neq \emptyset$.*

COROLLARY 3. *Let $X$ and $Y$ be endowed with topologies such that $C(\mathscr{E}(X)) \subset \mathfrak{K}(Y) \cap \mathfrak{C}(Y)$, $C^*(\mathfrak{G}(Y)) \subset \mathfrak{C}(X)$, and $C := \{(x, y) \in X \times Y: y \in C_x\} \in \mathfrak{F}(X \times Y)$. Then $\bigcap \{C_x: x \in X\} \neq \emptyset$.*

Here products of topological spaces are always endowed with the product topology.

*Proof.* Of course, $C \in \mathfrak{F}(X \times Y)$ implies $\mathscr{C} \subset \mathfrak{F}(Y)$. In view of Theorem 4 "(b)*$\Rightarrow$(a)" it is therefore sufficient to show that $C^*(\mathfrak{F}(Y) \cap \mathfrak{K}(Y)) \subset \mathfrak{G}(X)$. To see this, let $F \in \mathfrak{F}(Y) \cap \mathfrak{K}(Y)$ be given. Let $(x_n: n \in I)$ be a net in $X - C^*(F)$ which converges to some $x \in X$. For every $n \in I$ there is a $y_n \in C_{x_n} \cap F$. W.l.g. we may assume that the net $(y_n: n \in I)$ converges to some $y \in F$. From $(x_n, y_n) \in C$, $n \in I$, we infer $(x, y) \in C$, hence $y \in C_x \cap F$, so $x \in X - C^*(F)$. Therefore, $C^*(F)$ is open.

*Remark* 9. Corollary 3 is a slight generalization of Horvath's Theorem 7 in [3] which, in view of Remark 2, is equivalent to Stachó's Proposition 2 in [15].

## 4. ABSTRACT MINIMAX THEOREMS

For the rest of the paper let $X$ and $Y$ be nonvoid sets and $a: X \times Y \to \bar{\mathbb{R}}$ an extended real valued function on the cartesian product. We set

$$a_* = \sup_{x \in X} \inf_{y \in Y} a(x, y), \qquad a^* = \inf_{y \in Y} \sup_{x \in X} a(x, y),$$



and

$$\tilde{a}^* = \sup_{A \in \mathscr{E}(X)} \inf_{y \in Y} \max_{x \in A} a(x, y).$$

Then $a_* \leqslant \tilde{a}^* \leqslant a^*$ is always true. We say that $a$ fulfills

(the *minimax* relation) MM iff $a_* = a^*$,

(the *preminimax relation*) PMM iff $\tilde{a}^* = a_*$, and

(the *minimum minimax relation*) MMM iff $\sup_{x \in X} a(x, \hat{y}) = a_*$ for some $\hat{y} \in Y$.

A subset $\Lambda \subset (a_*, \infty)$ with $\inf \Lambda = a_*$ will be called a *border set* (of $a$). Observe that $\Lambda = \emptyset \Leftrightarrow a_* = \infty \Rightarrow$ MMM.

Pre/minimax problems and intersection problems are closely related:

REMARK 10. *Let* $\mathscr{C} = \{C_x: x \in X\}$ *with* $C_x = \{a(x, \cdot) \leqslant a_*\}$, $x \in X$. Furthermore, let a nonvoid border set $\Lambda$ and set systems $\mathscr{C}^\lambda = \{C_x^\lambda: x \in X\}$, $\lambda \in \Lambda$, be given with

$$\{a(x, \cdot) < \lambda\} \subset C_x^\lambda \subset \{a(x, \cdot) \leqslant \lambda\}, \qquad x \in X, \quad \lambda \in \Lambda.$$

Then the following equivalences hold:

(a) MMM $\Leftrightarrow \emptyset \notin C(2^X) \Leftrightarrow C(X) \neq \emptyset \Leftrightarrow$ There is a compact paving $\mathscr{L}$ in $Y$ such that $\emptyset \notin C^\lambda(\mathscr{E}(X)) \subset \mathscr{L}$, $\lambda \in \Lambda$.

(b) MM $\Leftrightarrow \emptyset \notin C^\lambda(2^X)$, $\lambda \in \Lambda$.

(c) PMM $\Leftrightarrow \emptyset \notin C^\lambda(\mathscr{E}(X))$, $\lambda \in \Lambda$.

*Proof.* (a) The first two equivalences are obvious. In case $C(X) \neq \emptyset$ the paving $\mathscr{L} = \{L: C(X) \subset L \subset Y\}$ has the desired property. Conversely, if the last condition holds then the paving $\mathscr{S} := \{C_x^\lambda: x \in X, \lambda \in \Lambda\}$ has the finite intersection property. Since $\mathscr{S}$ is contained in the compact paving $\mathscr{L}$, we obtain $\emptyset \neq \bigcap \{S: S \in \mathscr{S}\} = C(X)$.
The proof of (b) and (c) is similar.

We now formulate some *abstract minimax theorems* where the underlying sets $X$ and $Y$ do not carry any topological or algebraic structure.

THEOREM 5. *The following are equivalent*:

(a) PMM *holds.*

(b) *There exist pavings* $\mathscr{K}_1$, $\mathscr{K}_2$ *in* $X$ *and* $\mathscr{L}$ *in* $Y$ *such that*

(i) $\mathscr{K}_2$ *is intersectional and connected for* $\mathscr{K}_1$,

(ii) $\mathscr{L}$ *is finitely intersectional and connected, and for every* $\lambda > a_*$ *and every choice of* $\mathscr{C} = \{C_x : x \in X\}$ *with* $\{a(x,\cdot) < \lambda\} \subset C_x \subset \{a(x,\cdot) \leqslant \lambda\}$, $x \in X$, *one has*

(iii) $\mathscr{C} \subset \mathscr{L}$, $C^*(\mathscr{L}) \subset \mathscr{K}_1$ and $\mathscr{C}^* \subset \mathscr{K}_2$.

(c) *There is a border set* $\Lambda$ *such that for every* $\lambda \in \Lambda$ *there exist pavings* $\mathscr{K}_1, \mathscr{K}_2$ *in* $X$ *and* $\mathscr{L}$ *in* $Y$ *and a set system* $\mathscr{C} = \{C_x, : x \in X\}$ *such that* (i)–(iii) *of* (b) *are satisfied.*

*Proof.* (a) $\Rightarrow$ (b): Take $\mathscr{K}_1 = \{\varnothing\}$ and $\mathscr{K}_2 = 2^X$. In case $a_* = \infty$ let $\mathscr{L} = \{\varnothing\}$, say. Then condition (iii) holds vacuously. Otherwise, take $\mathscr{L} = \{L \subset Y : \exists A \in \mathscr{E}(X)\ \exists \mu \in (a_*, \infty)$ such that $\bigcap \{\{a(x,\cdot) < \mu\} : x \in A\} \subset L\}$.

(b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (a): Apply Theorem 1 "(c) $\Rightarrow$ (a)" and Remark 10c). (Observe that $\Lambda = \varnothing \Rightarrow$ MMM $\Rightarrow$ PMM.)

COROLLARY 4. *The following are equivalent*:

(a) PMM *holds.*

(b) *There exist pavings* $\mathscr{K}_1, \mathscr{K}_2$ *in* $X$ *and* $\mathscr{L}$ *in* $Y$ *and a border set* $\Lambda$ *such that*

(i) $\mathscr{K}_2$ *is intersectional and connected for* $\mathscr{K}_1$,

(ii) $\mathscr{L}$ *is finitely intersectional and connected,*

(iii) $\{a(x,\cdot) \leqslant \lambda\} \in \mathscr{L}$, $x \in X$. $\lambda \in \Lambda$;

(iv) $\bigcap_{y \in L} \{a(\cdot, y) > \lambda\} \in \mathscr{K}_1$, $L \in \mathscr{L}$, $\lambda \in \Lambda$, *and*

(v) $\{a(\cdot, y) > \lambda\} \in \mathscr{K}_2$, $y \in Y$, $\lambda \in \Lambda$.

(c) *As* (b) *with* $\leqslant$ *and* $>$ *replaced by* $<$ *and* $\geqslant$.

THEOREM 6. *The following are equivalent*:

(a) MM *holds.*

(b) *There exist pavings* $\mathscr{K}_1, \mathscr{K}_2$ *in* $X$, *a border set* $\Lambda$, *and pavings* $\mathscr{L}^\lambda$, $\lambda \in \Lambda$, *in* $Y$ *such that*

(i) $\mathscr{K}_2$ *is intersectional and connected for* $\mathscr{K}_1$,

(ii) *every* $\mathscr{L}^\lambda$, $\lambda \in \Lambda$, *is finitely intersectional, connected and compact,*

(iii) $\{a(x,\cdot) \leqslant \lambda\} \in \mathscr{L}^\lambda$, $x \in X$, $\lambda \in \Lambda$,

(iv) $\bigcap_{y \in L} \{a(\cdot, y) > \lambda\} \in \mathscr{K}_1$, $L \in \mathscr{L}^\lambda$, $\lambda \in \Lambda$, *and*

(v) $\{a(\cdot, y) > \lambda\} \in \mathscr{K}_2$, $y \in Y$, $\lambda \in \Lambda$.



*Proof.* (a) $\Rightarrow$ (b): Take $\mathscr{K}_1 = \{\varnothing\}$, $\mathscr{K}_2 = 2^X$, and $\Lambda = (a_*, \infty)$. In case $a_* = \infty$ conditions (ii)–(v) are vacuously satisfied. Otherwise, for $\lambda \in \Lambda$ choose $y_\lambda \in Y$ with $\sup_{x \in X} a(x, y_\lambda) < \lambda$ and take $\mathscr{L}^\lambda = \{L : y_\lambda \in L \subset Y\}$.

(b) $\Rightarrow$ (a): Since $\Lambda = \varnothing$ implies MM, we may assume that $\Lambda$ is nonvoid. Fix a $\lambda \in \Lambda$ and set $C_x = \{a(x,\cdot) \leqslant \lambda\}$, $x \in X$. Then Theorem 2(c) is satisfied, and together with Remark 10b) we obtain (a).

THEOREM 7. *The following are equivalent*:

(a) MMM *holds.*

(b) *There exist pavings* $\mathscr{K}_1, \mathscr{K}_2$ *in* $X$ *and* $\mathscr{L}$ *in* $Y$ *and a border set* $\Lambda$ *such that*

(i) $\mathscr{K}_2$ *is intersectional and connected for* $\mathscr{K}_1$,

(ii) $\mathscr{L}$ *is finitely intersectional, connected and compact,*

(iii) $\{a(x,\cdot) \leqslant \lambda\} \in \mathscr{L}$, $x \in X$, $\lambda \in \Lambda$,

(iv) $\bigcap_{y \in L} \{a(\cdot, y) > \lambda\} \in \mathscr{K}_1$, $L \in \mathscr{L}$, $\lambda \in \Lambda$,

(v) $\{a(\cdot, y) > \lambda\} \in \mathscr{K}_2$, $y \in Y$, $\lambda \in \Lambda$.

(c) *As* (b) *with* $\Lambda = \{a_*\}$ *and* $\{a(x,\cdot) \leqslant a_*\} \in \mathscr{L} - \{\varnothing\}$, $x \in X$.

*Proof.* (a) $\Rightarrow$ (b), (c): Take $\hat{y} \in Y$ with $\sup_{x \in X} a(x, \hat{y}) = a_*$, $\mathscr{L} = \{L : \hat{y} \in L \subset Y\}$, $\mathscr{K}_1 = \{\varnothing\}$, $\mathscr{K}_2 = 2^X$ and $\Lambda = (a_*, \infty)$ in case (b).

(b) $\Rightarrow$ (a), (c) $\Rightarrow$ (a): W.l.g. we may assume that $\Lambda$ is nonvoid. For fixed $\lambda \in \Lambda$ let $\mathscr{C} = \{C_x : x \in X\}$ with $C_x = \{a(x,\cdot) \leqslant \lambda\}$. Then Theorem 2(c) is satisfied. Together with Remark 10a) we obtain (a).

EXAMPLE 2. Suppose that there exist a connected intersectional paving $\mathscr{K}$ in $X$ and a compact, connected, finitely intersectional paving $\mathscr{L}$ in $Y$ such that $\{a(\cdot, y) > a_*\} \in \mathscr{K}$, $y \in Y$, and $\{a(x,\cdot) \leqslant a_*\} \in \mathscr{L} - \{\varnothing\}$, $x \in X$. Then MMM holds.

This is an "abstract version" of Example 4 in [6].

*Remark* 11. Let $a_* < \tilde{a}^*$. Then there exists an $A \in \mathscr{E}(X)$ such that

$$a^*(A, Y) := \max_{x \in A} \inf_{y \in Y} a(x, y) > a_* \quad \text{and} \tag{6}$$

$$a^*(M, Y) \leqslant a_* \text{ for all } M \in \mathscr{E}(X) \text{ with } card M < card A. \tag{7}$$

If for such an $A$ we take $\{x_1, x_2\} \subset A$ with $x_1 \neq x_2$ and set $E = A - \{x_1, x_2\}$, then we have $\tau := (x_1, x_2, E) \in T(\mathscr{C})$ for all set systems $\mathscr{C} = \{C_x : x \in X\}$ with

$$\{a(x,\cdot) < \beta\} \subset C_x \subset \{a(x,\cdot) \leqslant \gamma\}, \qquad x \in X,$$

where $a_* < \beta \leqslant \gamma < a^*(A, Y)$.

We now formulate a "preminimax version" of Proposition 1:

PROPOSITION 2. *The following are equivalent*:

(a) PMM *holds.*

(b) *There is a border set* $\Lambda$ *such that for every* $\nu \in \Lambda$ *there exist a* $\mu \in \Lambda$ *with* $\mu \leqslant \nu$, *set systems* $\mathscr{C} = \{C_x : x \in X\}$, $\mathscr{H} = \{H_x : x \in X\}$, $\mathscr{G} = \{G_x : x \in X\}$, $\mathscr{F} = \{F_x : x \in X\}$, *and pavings* $\mathscr{K}_1, \mathscr{K}_2$ *in X and* $\mathscr{L}_1, \mathscr{L}_2$ *in Y such that*

(o) $\{a(x, \cdot) < \mu\} \subset H_x \subset G_x \subset C_x \subset F_x \subset \{a(x, \cdot) \leqslant \nu\}$, $x \in X$,

(i) $\mathscr{L}_1 \supset \mathscr{C}$, $\mathscr{L}_1$ *is finitely intersectional and connected for* $\mathscr{F}$,

(ii) $\mathscr{L}_2 \supset \mathscr{F} \cup \mathscr{H}$, $\mathscr{L}_2$ *is finitely intersectional,*

(iii) $\mathscr{K}_2 \supset \mathscr{C}^*$, $\mathscr{K}_2$ *is intersectional and connected for* $\mathscr{K}_1$,

(iv) *either* $G^*(\mathscr{L}_2) \subset \mathscr{K}_1$ *or* $H^*(\mathscr{L}_2) \subset \mathscr{K}_1$.

*Proof.* (a)$\Rightarrow$(b): Take $\Lambda = (a_*, \infty)$. In case $a_* < \infty$ take $\mathscr{K}_1, \mathscr{K}_2$ and $\mathscr{L}$ as in Theorem 5(b), and set $\mathscr{L}_1 = \mathscr{L}_2 = \mathscr{L}$, $\mu = \nu$, and $C_x = H_x = G_x = F_x = \{a(x, \cdot) \leqslant \nu\}$.

(b)$\Rightarrow$(a): Assume that $a_* < \tilde{a}^*$. Choose $\Lambda$ and $\tau$ according to Remark 11. Fix a $\nu \in \Lambda$ with $a_* < \nu < a^*(A, Y)$ and choose $\mu, \mathscr{C}, \mathscr{H}, \mathscr{G}, \mathscr{F}, \mathscr{K}_1, \mathscr{K}_2, \mathscr{L}_1$, and $\mathscr{L}_2$ as in (b). Then $\tau \in T(\mathscr{F}) \cap T(\mathscr{H})$ by Remark 11 in contradiction to Proposition 1.

Now, with the help of Proposition 2, we are able to establish a more flexible version of our Theorem 5.

Let $\mathscr{P}$ and $\mathscr{Q}$ be pavings in X and Y, and let $\Lambda$ be a nonvoid subset of $\mathbb{R}$. Then we set

$$X(\mathscr{Q}, \Lambda, \geqslant) = \Big\{ \bigcap_{y \in Q} \{a(\cdot, y) \geqslant \lambda\}, Q \in \mathscr{Q}, \lambda \in \Lambda \Big\},$$

$$X(\mathscr{Q}, \Lambda, >) = \Big\{ \bigcap_{y \in Q} \{a(\cdot, y) > \lambda\}, Q \in \mathscr{Q}, \lambda \in \Lambda \Big\},$$

$$Y(\mathscr{P}, \Lambda, \leqslant) = \Big\{ \bigcap_{x \in P} \{a(x, \cdot) \leqslant \lambda\}, P \in \mathscr{P}, \lambda \in \Lambda \Big\}, \quad \text{and}$$

$$Y(\mathscr{P}, \Lambda, <) = \Big\{ \bigcap_{x \in P} \{a(x, \cdot) < \lambda\}, P \in \mathscr{P}, \lambda \in \Lambda \Big\}.$$

We shall consider the following conditions:

(A1) $\{a(x, \cdot) \leqslant \lambda\} \in \mathscr{Q}$ for all $x \in X$, $\lambda \in \Lambda$,

(A2) $\{a(x, \cdot) < \lambda\} \in \mathscr{Q}$ for all $x \in X$, $\lambda \in \Lambda$,

(B1) $X(\mathscr{Q}^d, \Lambda, \geqslant) \subset \mathscr{P}$,

(B2) $X(\mathscr{Q}^d, \Lambda, >) \subset \mathscr{P}$,



(C1) $X(2^Y, \Lambda, \geqslant)$ is connected for $\mathscr{P}$ and $Y(\mathscr{E}(X), \Lambda, <)$ is connected for $\mathscr{Q}$,

(C2) $X(2^Y, \Lambda, >)$ is connected for $\mathscr{P}$ and $Y(\mathscr{E}(X), \Lambda, \leqslant)$ is connected for $\mathscr{Q}$.

THEOREM 8. *Let* $a_* < \infty$. *Then the following are equivalent*:

(a) PMM *holds.*

(b) *There exist a border set* $\Lambda$ *and pavings* $\mathscr{P}$ *in X and* $\mathscr{Q}$ *in Y such that* (A1) *or* (A2), (B1) *or* (B2), *and* (C1) *or* (C2) *hold.*

(c) *There exist pavings* $\mathscr{P}$ *in X and* $\mathscr{Q}$ *in Y such that for* $\Lambda = (a_*, \infty)$ *all of* (A1), (A2), (B1), (B2), (C1), *and* (C2) *hold.*

*Proof.* (a)$\Rightarrow$(c): Take $\mathscr{P} = \{\varnothing\}$ and $\mathscr{Q} = \{Q : \bigcap_{x \in A} \{a(x, \cdot) < \lambda\} \subset Q \subset Y$ for some $A \in \mathscr{E}(X)$, $\lambda > a_*\}$.

(c)$\Rightarrow$(b) is obvious.

(b)$\Rightarrow$(a): We show that Proposition 2(b) holds:

For $\nu \in \Lambda$ take any $\mu \in \Lambda$ with $\mu < \nu$ and set $H_x = \{a(x, \cdot) \leqslant \mu\}$, $G_x = \{a(x, \cdot) < \nu\}$, and $F_x = \{a(x, \cdot) \leqslant \nu\}$ in case (A1), resp. $H_x = \{a(x, \cdot) < \mu\}$, $G_x = \{a(x, \cdot) \leqslant \mu\}$, and $F_x = \{a(x, \cdot) < \nu\}$ in case (A2). Then $\mathscr{F} \cup \mathscr{H} \subset \mathscr{Q}$, and we have $G^*(\mathscr{Q}^d) \subset \mathscr{P}$ in cases (A1), (B1) and (A2), (B2), resp. $H^*(\mathscr{Q}^d) \subset \mathscr{P}$ in cases (A1), (B2) and (A2), (B1). Now take $C_x = G_x$ in cases (A1), (C1) and (A2), (C2), resp. $C_x = F_x$ in cases (A1), (C2) and (A2), (C1). Then $C^*(2^Y)$ is connected for $\mathscr{P}$, and $C(\mathscr{E}(X))$ is connected for $\mathscr{Q}$ (hence for $\mathscr{F}$). Now apply Proposition 2 "(b)$\Rightarrow$(a)" with $\mathscr{L}_1 = C(\mathscr{E}(X))$, $\mathscr{L}_2 = \mathscr{Q}^d$, $\mathscr{K}_1 = \mathscr{P}$, and $\mathscr{K}_2 = C^*(2^Y)$.

The following example generalizes Komiya's minimax theorem in [11]:

EXAMPLE 3. *Suppose that there exist border sets* $\Lambda_1$ *and* $\Lambda_2$, $\Lambda_2$ *convex, a connected intersectional paving* $\mathscr{K}$ *in X and a compact, connected, finitely intersectional paving* $\mathscr{L}$ *in Y such that*

(i) $\{a(\cdot, y) \geqslant \lambda\} \in \mathscr{K}$, $y \in Y$, $\lambda \in \Lambda_1$, *and*

(ii) $\{a(x, \cdot) \leqslant \lambda\} \in \mathscr{L}$, $x \in X$, $\lambda \in \Lambda_2$.

*Then* MMM *holds.*

*Proof.* We may assume that the border set $\Lambda := \Lambda_1 \cap \Lambda_2$ is nonvoid. Then conditions (A1), (B1), and (C1) are satisfied for $\mathscr{P} = \mathscr{K}$ and $\mathscr{Q} = \mathscr{L}$. (To see that $Y(\mathscr{E}(X), \Lambda, <)$ is connected for $\mathscr{L}$, let $A \in \mathscr{E}(X)$ and $\lambda \in \Lambda$. Take $\lambda_n \in \Lambda_2$ with $\lambda_n < \lambda$, $n \in \mathbb{N}$, and $\lambda_n \to \lambda$. Since every set $B_n := \bigcap_{x \in A} \{a(x, \cdot) \leqslant \lambda_n\}$, $n \in \mathbb{N}$, is connected for $\mathscr{L}$, we infer from

$B_n \uparrow B := \bigcap_{x \in A} \{a(x, \cdot) < \lambda\}$ together with Remark 1d) that $B$ is connected for $\mathscr{L}$ as well.) Now from Theorem 8 "(b) $\Rightarrow$ (a)" together with Remark 10a) and c) the assertion follows.

## 5. Topological Minimax Theorems

We shall now make the abstract situation of Section 4 more precise and present some *purely topological minimax theorems.*

Theorem 9. *Let $a_* < \infty$. Then the following are equivalent:*

(a) PMM *holds.*

(b) *There are topologies on X and Y and a border set $\Lambda$ such that the sets $\{a(x, \cdot) < \lambda\}$, $x \in X$, $\lambda \in \Lambda$, are open and $X(\mathfrak{G}(Y), \Lambda, >) \subset \mathfrak{G}(X)$, $X(2^Y, \Lambda, \geqslant) \subset \mathfrak{C}(X)$, and $Y(\mathscr{E}(X), \Lambda, <) \subset \mathfrak{C}(Y)$ hold.*

(c) *As* (b) *with* $X(\mathfrak{G}(Y), \Lambda, >) \subset \mathfrak{G}(X)$, $X(2^Y, \Lambda, >) \subset \mathfrak{C}(X)$, *and* $Y(\mathscr{E}(X), \Lambda, \leqslant) \subset \mathfrak{C}(Y)$.

(d) *As* (b) *with* $X(\mathfrak{G}(Y), \Lambda, \geqslant) \subset \mathfrak{F}(X)$, $X(2^Y, \Lambda, \geqslant) \subset \mathfrak{C}(X)$, *and* $Y(\mathscr{E}(X), \Lambda, <) \subset \mathfrak{C}(Y)$.

(e) *As* (b) *with* $X(\mathfrak{G}(Y), \Lambda, \geqslant) \subset \mathfrak{F}(X)$, $X(2^Y, \Lambda, >) \subset \mathfrak{C}(X)$, *and* $Y(\mathscr{E}(X), \Lambda, \leqslant) \subset \mathfrak{C}(Y)$.

*Proof.* (a) implies condition (b)–(e) with $\Lambda = (a_*, \infty) = (\tilde{a}^*, \infty)$, $\mathscr{T}_X = \{\varnothing, X\}$, and $\mathscr{T}_Y = \{G \subset Y : \exists A \in \mathscr{E}(X), \lambda \in \Lambda$, such that $G \supset \bigcap_{x \in A} \{a(x, \cdot) < \lambda\}\} \cup \{\varnothing\}$.

(b) $\Rightarrow$ (a), (c) $\Rightarrow$ (a): Apply Theorem 8 with $\mathscr{P} = \mathfrak{G}(X)$ and $\mathscr{Q} = \mathfrak{G}(Y)$.

(d) $\Rightarrow$ (a), (e) $\Rightarrow$ (a): Apply Theorem 8 with $\mathscr{P} = \mathfrak{F}(X)$ and $\mathscr{Q} = \mathfrak{G}(Y)$.

Theorem 10. *Let $a_* < \infty$. Then the following are equivalent:*

(a) MMM *holds.*

(b) *There are topologies on X and Y such that for some $x_0 \in X$ the set $\{a(x_0, \cdot) \leqslant a_*\}$ is compact, all sets $\{a(x, \cdot) \leqslant a_*\}$, $x \in X$, are closed and nonvoid,* $X(\mathfrak{F}(Y), \{a_*\}, >) \subset \mathfrak{G}(X)$, $X(\mathfrak{G}(Y), \{a_*\}, >) \subset \mathfrak{C}(X)$, *and* $Y(\mathscr{E}(X), \{a_*\}, \leqslant) \subset \mathfrak{C}(Y)$.

(c) *There are topologies on X and Y and a border set $\Lambda$ such that for some pair $(x_0, \lambda_0) \in X \times \Lambda$ the set $\{a(x_0, \cdot) \leqslant \lambda_0\}$ is compact, the sets $\{a(x, \cdot) \leqslant \lambda\}$, $x \in X$, $\lambda \in \Lambda$, are closed, and* $X(\mathfrak{F}(Y), \Lambda, \geqslant) \subset \mathfrak{F}(X)$, $X(2^Y, \Lambda, >) \subset \mathfrak{C}(X)$, *and* $Y(\mathscr{E}(X), \Lambda, \leqslant) \subset \mathfrak{C}(Y)$ *hold.*

(d) *As* (c) *with* $X(\mathfrak{F}(Y), \Lambda, \geqslant) \subset \mathfrak{F}(X)$, $X(2^Y, \Lambda, \geqslant) \subset \mathfrak{C}(X)$, *and* $Y(\mathscr{E}(X), \Lambda, <) \subset \mathfrak{C}(Y)$.



(e) *As* (c) *with* $X(\mathfrak{F}(Y), \Lambda, >) \subset \mathfrak{G}(X)$, $X(2^Y, \Lambda, >) \subset \mathfrak{C}(X)$, *and* $Y(\mathscr{E}(X), \Lambda, \leqslant) \subset \mathfrak{C}(Y)$.

(f) *As* (c) *with* $X(\mathfrak{F}(Y), \Lambda, >) \subset \mathfrak{G}(X)$, $X(2^Y, \Lambda, \geqslant) \subset \mathfrak{C}(X)$, *and* $Y(\mathscr{E}(X), \Lambda, <) \subset \mathfrak{C}(Y)$.

*Proof.* (a) $\Rightarrow$ (b)–(f): Take $\hat{y} \in Y$ with $\sup_{x \in X} \tilde{a}(x, \hat{y}) = a_*$, $\mathscr{T}_X = \{\varnothing, X\}$, $\mathscr{T}_Y = \{G \subset Y : \hat{y} \notin G\} \cup \{Y\}$, and $\Lambda = (a_*, \infty)$.

(b) $\Rightarrow$ (a): Apply Theorem 4 "(b)* $\Rightarrow$ (a)" to $C_x = \{a(x, \cdot) \leqslant a_*\}$, $x \in X$.

(c) $\Rightarrow$ (a), (d) $\Rightarrow$ (a): Applying Theorem 8 with $\mathscr{P} = \mathfrak{F}(X)$ and $\mathscr{Q} = \mathfrak{F}(Y)$ we infer $a_* = \tilde{a}^*$. Hence, $\mathscr{R} := \{\{a(x_0, \cdot) \leqslant \lambda_0\} \cap \{a(x, \cdot) \leqslant \lambda\} : x \in X, \lambda \in \Lambda\}$ has the finite intersection property. By Remark 6 there exists a $\hat{y} \in \bigcap \{R : R \in \mathscr{R}\} = \{\sup_{x \in X} a(x, \cdot) \leqslant a_*\}$.

(e) $\Rightarrow$ (a), (f) $\Rightarrow$ (a): As above with $\mathscr{P} = \mathfrak{G}(X)$ and $\mathscr{Q} = \mathfrak{F}(Y)$.

*Remark* 12. An inspection of the proof of Theorem 10 shows the following:

(a) Conditions (c), (d), (e) or (f) of Theorem 10 without any compactness assumption imply PMM.

(b) Theorem 10 remains true if in (c) and (e) one replaces "$X(2^Y, \Lambda, >) \subset \mathfrak{C}(X)$" by the weaker condition "$X(\mathfrak{G}(Y), \Lambda, >) \subset \mathfrak{C}(X)$". This follows with Remark 3.

(c) If in Theorem 10(b)–(f) we make the stronger compactness assumption

- all sets $\{a(x, \cdot) \leqslant \lambda\}$, $x \in X$, $\lambda \in \Lambda$, are compact,

with $\Lambda = \{a_*\}$ in case (b), then we may replace $\mathfrak{F}(Y)$ by $\mathfrak{F}(Y) \cap \mathfrak{K}(Y) (=: \mathscr{Q})$.

(d) Theorem 10 remains true if one replaces "$X(\mathfrak{F}(Y), \Lambda, \geqslant) \subset \mathfrak{F}(X)$" in (c) and (d) resp. "$X(\mathfrak{F}(Y), \Lambda, >) \subset \mathfrak{G}(X)$" in (e) and (f) by the unconventional conditions "$X(\mathfrak{F}(Y), \Lambda, \geqslant) \subset \mathfrak{G}(X)$" and "$X(\mathfrak{F}(Y), \Lambda, >) \subset \mathfrak{F}(X)$".

For $x \in X$ and nonvoid $B \subset Y$ we set $a(x, B) = \inf_{y \in B} a(x, y)$.

Remark 13. *Let X and Y be topological spaces.*

(a) If $\mathscr{Q}$ is a paving in $Y$ then $X(\mathscr{Q}, \mathbb{R}, \geqslant) \subset \mathfrak{F}(X)$ iff every function $a(\cdot, Q)$, $Q \in \mathscr{Q} - \{\varnothing\}$, is upper semicontinuous. In particular, we have $X(2^Y, \mathbb{R}, \geqslant) \subset \mathfrak{F}(X)$ iff every $a(\cdot, y)$, $y \in Y$, is upper semicontinuous.

(b) If for $\varnothing \neq \Lambda \subset \mathbb{R}$ every set $\{a \leqslant \lambda\}$, $\lambda \in \Lambda$, is closed in $X \times Y$ then we have $X(\mathfrak{F}(Y) \cap \mathfrak{K}(Y), \Lambda, >) \subset \mathfrak{G}(X)$.

*Proof.* (a) is obvious and (b) follows as in the proof of Corollary 3.

The following two examples are slight generalizations of König's "minimax theorems based on connectedness" [9].

EXAMPLE 4. (cf. [9], Theorem 1.2). *Let $X$ and $Y$ be topological spaces and $\Lambda$ a nonvoid border set such that*

(i) $\{a(x_0, \cdot) \leqslant \lambda_0\}$ *is compact for some pair* $(x_0, \lambda_0) \in X \times \Lambda$,
(ii) *every set* $\{a(x, \cdot) \leqslant \lambda\}$, $x \in X$, $\lambda \in \Lambda$, *is closed,*
(iii) *every set* $\{a(\cdot, y) \geqslant \lambda\}$, $y \in Y$, $\lambda \in \Lambda$, *is closed.*

*Assume, moreover, that*

$$\begin{array}{llll} \text{either} & X(2^Y, \Lambda, \geqslant) \subset \mathfrak{C}(X) & \text{and} & Y(\mathscr{E}(X), \Lambda, <) \subset \mathfrak{C}(Y), \\ \text{or} & X(\mathfrak{G}(Y), \Lambda, >) \subset \mathfrak{C}(X) & \text{and} & Y(\mathscr{E}(X), \Lambda, \leqslant) \subset \mathfrak{C}(Y). \end{array} \tag{8}$$

*Then* MMM *holds.*

*Proof.* Condition (iii) implies $X(\mathfrak{F}(Y), \Lambda, \geqslant) \subset \mathfrak{F}(X)$. Hence, by Theorem 10 "(d) $\Rightarrow$ (a)", resp. "(c) $\Rightarrow$ (a)" and Remark 12b), the assertion follows.

EXAMPLE 5. (cf. [9], Theorem 1.3). *Let $X$ and $Y$ be topological spaces and $\Lambda$ a nonvoid border set such that*

(i) *every set* $\{a(x, \cdot) \leqslant \lambda\}$, $x \in X$, $\lambda \in \Lambda$, *is compact, and*
(ii) *every set* $\{a \leqslant \lambda\}$, $\lambda \in \Lambda$, *is closed in* $X \times Y$.

*Assume, moreover, that* (8) *is satisfied. Then* MMM *holds.*

*Proof.* This follows with Theorem 10 "(e) $\Rightarrow$ (a)" resp. "(f) $\Rightarrow$ (a)" together with Remarks 12b), c) and 13b).

Predecessors of our last two examples are Examples 4 and 5 in [6].

EXAMPLE 6. *Let $X$ and $Y$ be topological spaces such that for all pairs $(x_1, x_2) \in X \times X$ and $(y_1, y_2) \in Y \times Y$ there exist connected sets $\langle x_1, x_2 \rangle_X$ and $\langle y_1, y_2 \rangle_Y$ with*

$$\{x_1, x_2\} \subset \langle x_1, x_2 \rangle_X \subset \{x \in X: a(x, \cdot) \geqslant a(x_1, \cdot) \wedge a(x_2, \cdot)\} \quad \text{and}$$
$$\{y_1, y_2\} \subset \langle y_1, y_2 \rangle_Y \subset \{y \in Y: a(\cdot, y) \leqslant a(\cdot, y_1) \vee a(\cdot, y_2)\}.$$

*Then* PMM *holds if one of the following two conditions is satisfied:*

(i) *Every* $a(\cdot, G)$, $G \in \mathfrak{G}(Y) - \{\varnothing\}$, *and every* $a(x, \cdot)$, $x \in X$, *is upper semicontinuous.*



(ii) *Every* $a(\cdot, F)$, $F \in \mathfrak{F}(Y) - \{\varnothing\}$, *is upper semicontinuous, and every* $a(x, \cdot)$, $x \in X$, *is lower semicontinuous.*

*If $Y$ is compact then in case* (ii) *even* MMM *holds.*

*Proof.* For $B \subset Y$, $\lambda \in \mathbb{R}$, and $D = \bigcap_{y \in B} \{a(\cdot, y) \geqslant \lambda\}$ or $D = \bigcap_{y \in B} \{a(\cdot, y) > \lambda\}$ we have

$$\{x_1, x_2\} \subset D \Rightarrow \langle x_1, x_2 \rangle_X \subset D, \qquad (x_1, x_2) \in X \times X.$$

Hence $D$ is connected, so $X(2^Y, \mathbb{R}, \geqslant) \cup X(2^Y, \mathbb{R}, >) \subset \mathfrak{C}(X)$. Similarly, $Y(2^X, \mathbb{R}, \leqslant) \cup Y(2^X, \mathbb{R}, <) \subset \mathfrak{C}(Y)$.

Condition (i) implies $X(\mathfrak{G}(Y), \mathbb{R}, \geqslant) \subset \mathfrak{F}(X)$ and (ii) leads to $X(\mathfrak{F}(Y), \mathbb{R}, \geqslant) \subset \mathfrak{F}(X)$. Now apply Theorem 9 in case (i) and Theorem 10, together with Remark 12a), in case (ii).

EXAMPLE 7. *Let $X$ and $Y$ be topological spaces such that*

(i) *every set* $\{a(x, \cdot) \leqslant a_*\}$, $x \in X$, *is compact and nonvoid,*
(ii) *the set* $\{a \leqslant a_*\}$ *is closed in* $X \times Y$,
(iii) *every set* $\bigcap_{y \in G} \{a(\cdot, y) > a_*\}$, $G \in \mathfrak{G}(Y)$, *is connected or empty,*
(iv) *every set* $\bigcap_{x \in A} \{a(x, \cdot) \leqslant a_*\}$, $A \in \mathscr{E}(X)$, *is connected or empty.*

*Then* MMM *holds.*

*Proof.* Apply Theorem 10 "(b) $\Rightarrow$ (a)" together with Remarks 12c) and 13b).

*Remark* 14. Let $X$ and $Y$ be convex subsets of linear topological spaces and let the functions $a(\cdot, y)$, $y \in Y$, be quasiconcave and $a(x, \cdot)$, $x \in X$, quasiconvex.

Here we can take for $\langle x_1, x_2 \rangle_X$ and $\langle y_1, y_2 \rangle_Y$ the convex hull of $\{x_1, x_2\}$ resp. $\{y_1, y_2\}$ to satisfy the connectedness assumption of Example 6. In particular, Example 6(ii) generalizes Sion's minimax theorem [14].

Moreover, conditions (iii) and (iv) of Example 7 are satisfied because the involved sets are convex. Therefore, Example 7 contains Ha's minimax theorem [2].

Many of the minimax theorems mentioned in the introduction can be derived by this method.

## REFERENCES

1. A. GERAGHTY AND B.-L. LIN, Topological minimax theorems, *Proc. Amer. Math. Soc.* **91** (1984), 377–380.
2. CH.-W. HA, Minimax and fixed point theorems, *Math. Ann.* **248** (1980), 73–77.

3. CH. D. HORVATH, A connectivity approach to minimax inequalities, *Fund. Math.*, to appear.
4. I. JOÓ, A simple proof for von Neumann's minimax theorem, *Acta Sci. Math.* **42** (1980), 91–94.
5. J. KINDLER, On a minimax theorem of Terkelsen's, *Arch. Math.* **55** (1990), 573–583.
6. J. KINDLER, Topological intersection theorems, *Proc. Amer. Math. Soc.*, **117** (1993), 1003–1011.
7. J. KINDLER AND R. TROST, Minimax theorems for interval spaces, *Acta Math. Hung.* **54** (1989), 39–49.
8. E. KLEIN AND A. C. THOMSON, "Theory of correspondences," Wiley, New York, 1984.
9. H. KÖNIG, A general minimax theorem based on connectedness, *Arch. Math.* **59** (1992), 55–64.
10. H. KÖNIG AND F. ZARTMANN, New versions of the minimax theorem, preprint.
11. H. KOMIYA, A set theoretical approach to minimax theorems, preprint.
12. V. KOMORNIK, Minimax theorems for upper semincontinuous functions, *Acta Math. Acad. Sci. Hung.* **40** (1982), 159–163.
13. S. SIMONS, A flexible minimax theorem, *Acta Math. Hung.*, to appear.
14. M. SION, On general minimax theorems, *Pacific J. Math.* **8** (1958), 171–176.
15. L. L. STACHÓ, Minimax theorems beyond topological vector spaces, *Acta Sci. Math.* **42** (1980), 157–164.
16. L. A. STEEN AND J. A. SEEBACH, JR., "Counterexamples in Topology", Holt, Rinehart and Winston, New York, 1970.
17. H. TUY, On a general minimax theorem, *Sovjet Math. Dokl.* **15** (1974), 1689–1693.
18. H. TUY, On the general minimax theorem. *Colloq. Math.* **33** (1975), 145–158.
19. WU WEN-TSÜN, A remark on the fundamental theorem in the theory of games, *Sci. Rec. New Ser.* **3** (1959), 229–233.

JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **178**, 547–552 (1993)

# On the Solvability of an Operator Equation without the Landesman–Lazer Condition

CHUNG-WEI HA

*Department of Mathematics, Tsing Hua University, Hsin Chu, Taiwan*

*Submitted by Ky Fan*

Received June 11, 1992

Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$ be a bounded domain, $H = L^2(\Omega)$ and let $L: D(L) \subset H \to H$ be a linear operator. In this paper we study the solvability of operator equations of the form

$$Lu + Gu = h, \tag{1}$$

where $G: H \to H$, $(Gu)(x) = g(x, u(x))$ is the Nemytskii map defined by a Caratheodory function $g: \Omega \times \mathbb{R} \to \mathbb{R}$, that is, $g(x, u)$ is continuous in $u \in \mathbb{R}$ for a.e. $x \in \Omega$ and is measurable in $x \in \Omega$ for all $u \in \mathbb{R}$, $h \in H$ is a given function. We assume that

(H1) $L$ is closed, densely defined, with a closed range $R(L)$ and a finite dimensional nontrivial null space $N(L)$ such that $R(L) = N(L)^{\perp}$.

Here and in what follows, the real Hilbert space $H = L^2(\Omega)$ with the norm $|u|$ and the inner product $(u, v)$ is defined in the usual way. Properties such as orthogonality and selfadjointness are always referred to with respect to the inner product of $H$. Clearly the restriction of $L$ to $D(L) \cap R(L)$ is one-one onto $R(L)$ and so has an inverse denoted by $L^{-1}: R(L) \to R(L)$, which by the closed graph theorem is bounded. We assume also that

(H2) $L^{-1}: R(L) \to R(L)$ is a compact linear operator.

The main model and one of the most interesting examples of equations of the form (1) where $L$ satisfies (H1), (H2) is the Dirichlet boundary value problem for a symmetric uniformly elliptic operator on $\Omega$ of order $2m$, $m \geqslant 1$. Let

$$Au = \sum_{|\alpha|,\, |\beta| \leqslant m} (-1)^{|\beta|} D^{\beta}(a_{\alpha\beta} D^{\alpha} u),$$