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MINIMAX THEOREMS FOR INTERVAL SPACES

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1. Introduction

In his fundamental paper "Zur Theorie der Gesellschaftsspiele" v. Neumann established the following *minimax theorem.*

THEOREM A (v. Neumann [11]). *Let X and Y be two simplexes in Euclidean spaces, and let* $a: X\times Y\to\mathbf{R}$ *be a continuous function which is quasiconcave in the first and quasiconvex in the second variable. Then we have*

$$(*)\qquad \sup_{x\in X}\inf_{y\in Y} a(x, y) = \inf_{y\in Y}\sup_{x\in X} a(x, y).$$

This theorem was generalized by Sion as follows.

THEOREM B (Sion [12]). *Let X and Y be compact convex subsets of topological vector spaces, and let* $a: X\times Y\to\mathbf{R}$ *be upper semicontinuous and quasiconcave in the first and lower semicontinuous and quasiconvex in the second variable. Then* $(*)$ *holds.*

The standard proofs of such minimax theorems either use some form of separation of disjoint convex sets by a hyperplane, or they rely on some version of Brower's fixed point theorem. It is the contribution of Wu [22] to have observed that the only property of convex sets which is actually needed in the proof of the minimax theorem is connectedness. So Wu could establish a purely topological minimax theorem which contains Theorem A as a special case. By a skilful modification of Wu's method, Tuy [17], [18] could even derive topological minimax theorems which generalize Theorem B. Up to now several papers written in the same spirit have appeared [2], [4], [10], [14], [15].

Independently, inspired by Joó's [5] proof of Theorem A, Stachó [13] presented another generalization of Theorem B based on the concept of an *interval space* which generalizes the notion of a convex set.

In the present paper we want to demonstrate that Stachó's concept, in a slightly generalized form, is an adequate frame for the formulation of a fairly general minimax theorem which contains all the above mentioned minimax theorems as special cases. We hope that our exposition, which unifies ideas of Wu, Tuy, Joó and Stachó, can help to get a deeper insight into the nature of v. Neumann's minimax theorem.

2. Preliminaries

2.1. Systems of sets. For nonvoid sets S we shall use the following notation.

$\mathscr{P}(S)$ denotes the power set of S.

$\mathscr{E}(S)=\{T\subset S\colon T \text{ finite}\}$.

A subset $\mathscr{Q}\subset\mathscr{P}(S)$ will be called

chain, if $\{A\cup B, A\cap B\}=\{A, B\}$ for all $A\in\mathscr{Q}$, $B\in\mathscr{Q}$,

compact, if for every $\mathscr{R}\subset\mathscr{Q}$, $\bigcap\{R\colon R\in\mathscr{S}\}\neq\emptyset$ for all $\mathscr{S}\in\mathscr{E}(\mathscr{R})$ implies $\bigcap\{R\colon R\in\mathscr{R}\}\neq\emptyset$,

convexity, if $\{\emptyset, S\}\subset\mathscr{Q}$, $\bigcap\{R\colon R\in\mathscr{R}\}\in\mathscr{Q}$ for every $\mathscr{R}\subset\mathscr{Q}$, and $\bigcup\{R\colon R\in\mathscr{R}\}\in\mathscr{Q}$ for every chain $\mathscr{R}\subset\mathscr{Q}$,

topology, if $\{\emptyset, S\}\subset\mathscr{Q}$, $\bigcup\{R\colon R\in\mathscr{R}\}\in\mathscr{Q}$ for every $\mathscr{R}\subset\mathscr{Q}$, and $\bigcap\{R\colon R\in\mathscr{R}\}\in\mathscr{Q}$ for every $\mathscr{R}\in\mathscr{E}(\mathscr{Q})$. In this case, the pair $(S,\mathscr{Q})$ is called a *topological space.*

Now let $(S,\mathscr{T})$ be a topological space. Then we write

$\mathscr{G}(S)$ for the system $\mathscr{T}$ of open subsets of S,

$\mathscr{F}(S)$ for the system $\{S-G\colon G\in\mathscr{T}\}$ of closed subsets of S,

$\mathscr{K}(S)$ for the system of all compact $T\in\mathscr{F}(S)$, i.e.

$\mathscr{K}(S)=\{T\in\mathscr{F}(S)\colon T\cap\mathscr{F}(S) \text{ is compact}\}$,

$\mathscr{C}(S)$ for the system of connected subsets of S, and

$\mathscr{C}_0(S)=\mathscr{C}(S)\cup\{\emptyset\}$.

Subsets T will always be endowed with the relative topology $T\cap\mathscr{T}$. In particular, a nonvoid $T\subset S$ is called *connected* if $T\subset F_1\cup F_2$, $T\cap F_1\cap F_2=\emptyset$, $F_i\in\mathscr{F}(S)$ implies $T\subset F_1$ or $T\subset F_2$.

A function $f\colon S\to\overline{\mathbf{R}}$ is called *upper* (*lower*) *semicontinuous* if every level set $\{f\geqq\alpha\}:=\{s\in S\colon f(s)\geqq\alpha\}$ (every $\{f\leqq\alpha\}$), $\alpha\in\mathbf{R}$, is closed. Here $\overline{\mathbf{R}}=\mathbf{R}\cup\{-\infty,\infty\}$ denotes the set of extended reals.

Finally, a triplet $(S,\mathscr{T},h)$ will be called a *Wu space,* if $(S,\mathscr{T})$ is a topological space and $h\colon S\times S\times[0,1]\to S$ is a map such that for all pairs $(s,t)\in S\times S$ we have that $h(s,t,\cdot)$ is continuous, $h(s,t,0)=s$, and $h(s,t,1)=t$. The map h will be called *Wu map* (compare Wu [22]). Topological spaces which admit a Wu map are called *pathwise connected* [21].

2.2. Interval spaces. An *interval space* is a triplet $I=(S,\mathscr{T},\langle\cdot,\cdot\rangle)$ where $(S,\mathscr{T})$ is a topological space and $\langle\cdot,\cdot\rangle\colon S\times S\to\mathscr{C}(S)$ is a map such that $\langle s,t\rangle\supset\supset\{s,t\}$ for all $(s,t)\in S\times S$. If furthermore $\langle s,t\rangle=\langle t,s\rangle$ for all $(s,t)\in S\times S$, then I is called *symmetric.* Symmetric interval spaces were introduced by Stachó [13]. Subsets $C\subset S$ are called *convex* if $\{s,t\}\subset C$ implies $\langle s,t\rangle\subset C$. We set $\mathscr{C}_I(S)==\{C\subset S\colon C \text{ convex}\}$.

REMARK 2.1. (Stachó [13].) Let $I=(S,\mathscr{T},\langle\cdot,\cdot\rangle)$ be an interval space.

a) $\mathscr{C}_I(S)$ is a convexity and $\mathscr{C}_I(S)\subset\mathscr{C}_0(S)$.

b) For $f\colon S\to\overline{\mathbf{R}}$ the following are equivalent.

(1) $\{f\leqq\alpha\}\in\mathscr{C}_I(S)$ for all $\alpha\in\mathbf{R}$.

(2) $\{f<\alpha\}\in\mathscr{C}_I(S)$ for all $\alpha\in\mathbf{R}$.

(3) $f(s)\leqq\max\{f(s_1),f(s_2)\}$ for all $s\in\langle s_1,s_2\rangle$, $(s_1,s_2)\in S\times S$.

In this case, f is called *quasiconvex.* As usual, $g\colon S\to\overline{\mathbf{R}}$ is called *quasiconcave* iff $-g$ is quasiconvex.

Sometimes we shall write S for a topological space $(S, \mathcal{T})$ or for an interval space $(S, \mathcal{T}, \langle \cdot, \cdot \rangle)$ if there is no danger of confusion.

We now present examples of interval spaces, some of which will be used in the sequel.

EXAMPLE 2.2. (Wu interval spaces.) Every Wu spaces $(S, \mathcal{T}, h)$ gives rise to an interval space $I^h=(S, \mathcal{T}, \langle \cdot, \cdot \rangle^h)$ with $\langle s, t\rangle^h=\{h(s, t, \gamma)\colon 0\leqq\gamma\leqq 1\}$. In a *Wu interval space* I^h every interval is compact.

EXAMPLE 2.2.1. (Topological vector space.) Let S be a topological vector space. Then the Wu map $h(s, t, \gamma)=(1-\gamma)s+\gamma t$ defines the usual intervals $\langle s, t\rangle^h= =[s, t]:=\{(1-\gamma)s+\gamma t\colon 0\leqq\gamma\leqq 1\}$. Here our notions of convexity and quasiconvexity coincide with the usual ones.

EXAMPLE 2.2.2. $(\mathbf{R}^n, \mathcal{T}^n, [\cdot, \cdot])$, where $\mathcal{T}^n$ denotes the Euclidean topology, is the *n-dimensional Euclidean interval space*. Observe that in case $n=1$ $[a, b]$ stands for the convex hull of the points a and b and not for the order interval $\{x\in\mathbf{R}\colon a\leqq x\leqq b\}$.

EXAMPLE 2.2.3. For $n\in\mathbf{N}$ we consider the Wu spaces $(\mathbf{R}^n, \mathcal{T}^n, h_n)$ where the Wu maps h_n are defined inductively as follows.

For $s\in\mathbf{R}$, $t\in\mathbf{R}$ let $h_1(s, t, \gamma)=(1-\gamma)s+\gamma t$. If $h_1, \ldots, h_n$ $(n\geqq 1)$ are defined, then for $s=(s_1, \ldots, s_{n+1})\in\mathbf{R}^{n+1}$, $t=(t_1, \ldots, t_{n+1})\in\mathbf{R}^{n+1}$ we set

$$h_{n+1}(s, t, \gamma) = \big(s_1, \ldots, s_n, (1-2\gamma)s_{n+1}+2\gamma t_{n+1}\big)\cdot 1_{[0,1/2]}(\gamma)+$$
$$+\big(h_n((s_1, \ldots, s_n), (t_1, \ldots, t_n), 2\gamma-1), t_{n+1}\big)\cdot 1_{(1/2,1]}(\gamma).$$

Then the I^{h_n}, $n\in\mathbf{N}$, are interval spaces. Only I^{h_1} is symmetric.

EXAMPLE 2.2.4. (Joó—Stachó interval spaces.) Now we modify our definition in Example 2.2.3. We define h_1 as above. If $h_1, \ldots, h_n$ are defined, then we define $h_{n+1}(s, t, \gamma)$ as in Example 2.2.3 for all $s=(s_1, \ldots, s_{n+1})$ and $t=(t_1, \ldots, t_{n+1})$ with $t_{n+1}\geqq s_{n+1}$. For s and t with $t_{n+1}<s_{n+1}$ we set $h_{n+1}(s, t, \gamma)=h_{n+1}(t, s, 1-\gamma)$. Here all I^{h_n} are symmetric interval spaces. They coincide with the interval spaces introduced by Joó and Stachó in [7; § 3].

EXAMPLE 2.3 (pointed spaces). Let S be a set and $z\in S$. Then $(S, \mathcal{T}_z)$ with $\mathcal{T}_z=\{G\subset S\colon z\notin G\}\cup\{S\}$ is a compact topological space. $I=(S, \mathcal{T}_z, \langle \cdot, \cdot \rangle)$ is an interval space for every map $\langle \cdot, \cdot \rangle\colon S\times S\to\mathcal{P}(S)$ with $\langle s, t\rangle\supset\{s, t, z\}$ for all $(s, t)\in S\times S$.

EXAMPLE 2.3.1. $I_z=(S, \mathcal{T}_z, \langle \cdot, \cdot \rangle_z)$ with $\langle s, t\rangle_z=\{s, t, z\}$ will be called the *z-pointed interval space* in S.

Here we have $I_z=I^h$ for the Wu map $h(s, t, \gamma)=s, z, t$ for $\gamma<\frac{1}{2}$, $\gamma=\frac{1}{2}$, $\gamma>\frac{1}{2}$. A nonvoid $C\subset S$ is convex iff $z\in C$, i.e. a set is convex iff it is closed. A function $f\colon S\to\overline{\mathbf{R}}$ is quasiconvex iff $f(z)=\inf_{s\in S} f(s)$.

EXAMPLE 2.3.2 (starlike sets). Let $z\in\mathbf{R}^2$. Then $I=(\mathbf{R}^2, \mathcal{T}^2, \langle\cdot,\cdot\rangle)$ with $\langle s, t\rangle=[s, z]\cup[z, t]$ is an interval space. We have $I=I^h$ for the Wu map $h(s, t, \gamma)=$ $=[s+2\gamma(z-s)]\cdot 1_{[0,1/2]}(\gamma)+[2z(1-\gamma)+(2\gamma-1)t]\cdot 1_{(1/2,1]}(\gamma)$. Here, a nonvoid $C\subset\mathbf{R}^2$ is convex iff C is starlike with respect to z.

EXAMPLE 2.4 (chain spaces). Let $(S, \mathcal{T})$ be a topological space such that $\mathcal{T}$ is a chain. Then we have $\mathcal{C}_0(S)=\mathcal{P}(S)$. So $(S, \mathcal{T}, \langle\cdot,\cdot\rangle)$ is an interval space for every map $\langle\cdot,\cdot\rangle\colon S\times S\to\mathcal{P}(S)$ with $\langle\cdot,\cdot\rangle\supset\{\cdot,\cdot\}$. Especially, $(S, \mathcal{T}, \{\cdot,\cdot\})$ is an interval space with $\mathcal{C}_I(S)=\mathcal{P}(S)$.

EXAMPLE 2.4.1. $I=(S, \{\emptyset, S\}, \{\cdot,\cdot\})$ is the *indiscrete interval space* in S.

EXAMPLE 2.4.2. $I=(\mathbf{N}, \mathcal{T}, \langle\cdot,\cdot\rangle)$ with $\mathcal{T}=\{\{1, ..., n\}\colon n\in\mathbf{N}\}\cup\{\emptyset, \mathbf{N}\}$ and $\langle n, m\rangle=\{k\in\mathbf{N}\colon \min\{n, m\}\leqq k\leqq\max\{n, m\}\}$ is a symmetric interval space.

3. Construction and game theoretic interpretation of minimax theorems

In the following let a triplet $\Gamma=(X, Y, a)$ be given. Here X and Y denote nonvoid sets and a will be a function on the cartesian product $X\times Y$ into the extended reals. Sometimes it is helpful to interpret such a triplet as a *game*. Player 1 and Player 2 independently choose a strategy $x\in X$ and $y\in Y$, respectively. Afterwards Player 1 receives the (possibly negative) amount $a(x, y)$ from Player 2.

$$a_* = a_*(X, Y) := \sup_{x\in X}\inf_{y\in Y} a(x, y) \quad\text{and}\quad a^* = a^*(X, Y) := \inf_{y\in Y}\sup_{x\in X} a(x, y)$$

are called the *lower* and *upper value* of the game. If

$$\hat{X} := \{\hat{x}\in X\colon \inf_{y\in Y} a(\hat{x}, y) = a_*\} \quad\text{and}\quad \hat{Y} := \{\hat{y}\in Y\colon \sup_{x\in X} a(x, \hat{y}) = a^*\}$$

are nonvoid, then Player 1 can assure himself an amount of at least a_* by choosing a *minimax strategy* $\hat{x}\in\hat{X}$, whereas Player 2 can avoid to pay more than a^* by choosing a $\hat{y}\in\hat{Y}$. If the game is *strictly determined* which means that $a_*=a^*(=: v)$ holds, then we have

$$a(\hat{x}, y) \geqq v \geqq a(x, \hat{y}) \quad\text{for all}\quad x\in X,\quad y\in Y,$$

i.e. every pair $(\hat{x}, \hat{y})\in\hat{X}\times\hat{Y}$ is a *saddle point* of a. Hence in strictly determined games it is optimal for both players to choose minimax strategies $\hat{x}\in\hat{X}$ and $\hat{y}\in\hat{Y}$.

It is the aim of the present paper to give conditions which ensure the strict determinateness of a game $\Gamma=(X, Y, a)$. A standard method for proving such *minimax theorems* proceeds as follows. Suppose that Player 2 has to announce in advance a set $B\in\mathcal{E}(Y)$. Afterwards both players simultaneously choose strategies $x\in X$ and $y\in B$, respectively. In this case the guarantee value a_* of Player 1 improves to

$$\tilde{a}_* = \tilde{a}_*(X, Y) := \inf_{B\in\mathcal{E}(Y)} a_*(X, B).$$

We obviously have

REMARK 3.1. $a_* \leqq \tilde{a}_* \leqq a^*$.

So, as usual, $a_*=a^*$ will be proved in two steps: $a_*=\tilde{a}_*$ is shown by a standard compactness argument, whereas in the proof of $\tilde{a}_*=a^*$, in this paper the more difficult part, some convexity and connectedness properties are exploited.

In our theorems and proofs the following level sets play a crucial role. For $\alpha\in\mathbf{R}$ we set

$$X_\alpha(y) = \{x\in X:\ a(x,y) \geqq \alpha\},\quad y\in Y,\quad X_\alpha^*(y) = \{x\in X:\ a(x,y) > \alpha\},\quad y\in Y,$$

$$Y_\alpha(x) = \{y\in Y:\ a(x,y) \leqq \alpha\},\quad x\in X,\quad Y_\alpha^*(x) = \{y\in Y:\ a(x,y) < \alpha\},\quad x\in X.$$

Furthermore, for $\emptyset\neq B\subset Y$ we use the abbreviations

$$X_\alpha(B) = \bigcap\{X_\alpha(y):\ y\in B\},\quad X_\alpha^*(B) = \bigcap\{X_\alpha^*(y):\ y\in B\}.$$

Our further investigations rely on the following observation.

REMARK 3.2. Let $\beta\in\overline{\mathbf{R}}$.

a) The following are equivalent.

(1) $a_*\geqq\beta$.

(2) $X_\alpha(Y)\neq\emptyset$ for all real $\alpha<\beta$.

(3) $X_\alpha^*(Y)\neq\emptyset$ for all real $\alpha<\beta$.

b) The following are equivalent.

(1) $\tilde{a}_*\geqq\beta$.

(2) $X_\alpha(B)\neq\emptyset$ for all real $\alpha<\beta$ and all $B\in\mathscr{E}(Y)$.

(3) $X_\alpha^*(B)\neq\emptyset$ for all real $\alpha<\beta$ and all $B\in\mathscr{E}(Y)$.

This is well-known (compare [6; Theorem 2] and [9; Satz 5]) and easily established.

Now we state the announced "compactness arguments".

PROPOSITION 3.3. a) *In case* $\tilde{a}_*=-\infty$ *we have* $a_*=\tilde{a}_*$ *and* $\hat{X}=X$.

b) *In case* $\tilde{a}_*>-\infty$, *for* $\mathfrak{X}_\alpha=\{X_\alpha(y):\ y\in Y\}$, $\alpha\in\mathbf{R}$ *and* $\mathfrak{X}=\{X_\alpha(y):\ y\in Y,\ \alpha<\tilde{a}_*\}$ *the following are equivalent.*

(1) $a_*=\tilde{a}_*$ *and* $\hat{X}\neq\emptyset$.

(2) $X_{\tilde{a}_*}(Y)\neq\emptyset$.

(3) $\mathfrak{X}$ *is compact.*

(4) *There is a topology on X and a sequence of reals* $\alpha_n<\tilde{a}_*$ *with* $\lim_{n\to\infty}\alpha_n=\tilde{a}_*$ *such that* $\mathfrak{X}_{\alpha_n}\subset\mathscr{F}(X)$ *and* $\mathfrak{X}_{\alpha_1}\cap\mathscr{K}(X)\neq\emptyset$.

(5) *There is a topology on X such that* $\mathfrak{X}\subset\mathscr{F}(X)$ *and* $\mathfrak{X}\cap\mathscr{K}(X)\neq\emptyset$.

(6) *There is a compact topology on X such that* $\mathfrak{X}\subset\mathscr{F}(X)$.

The implication (5)$\Rightarrow$(1) is Lemma 2a) in [8].

PROOF. a) is obvious.

b) (1)$\Rightarrow$(6): Choose $\mathscr{T}_x$ for some $x\in\hat{X}$ (cf. Example 2.3).

(6)$\Rightarrow$(5)$\Rightarrow$(4) and (2)$\Rightarrow$(1) are obvious.

(4)$\Rightarrow$(2): W.l.g. we may assume $\alpha_1<\alpha_n$, $n\in\mathbf{N}$. Let $z\in Y$ with $X_{\alpha_1}(z)\in\mathscr{K}(X)$. From $\alpha_n<\tilde{a}_*$ we infer $X_{\alpha_n}(B)\cap X_{\alpha_1}(z)\supset X_{\alpha_n}(B\cup\{z\})\neq\emptyset$ for all $B\in\mathscr{E}(Y)$, $n\in\mathbf{N}$. So (4) yields $X_{\tilde{a}_*}(Y)=\bigcap\{X_{\alpha_n}(y)\cap X_{\alpha_1}(z):\ n\in\mathbf{N},\ y\in Y\}\neq\emptyset$.

(2)$\Leftrightarrow$(3) follows from $X_{\tilde{a}_*}(Y)=\bigcap\{T:\ T\in\mathfrak{X}\}$ and $\bigcap\{T:\ T\in\mathscr{S}\}\neq\emptyset$ for all $\mathscr{S}\in\mathscr{E}(\mathfrak{X})$.

4. Quasiconvex games

In this chapter we consider games $\Gamma=(X, Y, a)$ where Y is an interval space and where the level sets $Y_\alpha^*(x)$ are convex.

In the proofs of the following lemma and the subsequent theorem we use a concept due to Wu [22; Lemmas 1 and 2] and developed further by Tuy [17; Lemmas 2 and 3], combined with ideas of Joó [5] and Stachó [13].

LEMMA 4.1. *Let X be a topological space, $(Y, \mathcal{T}, \langle\cdot, \cdot\rangle)$ an interval space, $C\subset X$, $\alpha\in\mathbf{R}$, and $(y_1, y_2)\in Y\times Y$ such that*

(a1) $C\cap X_\alpha^*(z)\neq\emptyset$ *for all* $z\in\langle y_1, y_2\rangle$.

(a2) $C\cap X_\alpha(z)\in\mathscr{C}(X)\cap\mathscr{F}(X)$ *for all* $z\in\langle y_1, y_2\rangle$.

(b) $Y_\alpha^*(x)\in\mathscr{C}_I(Y)$ *for all* $x\in C$.

(c) $Y_\alpha(x)\cap\langle y_1, y_2\rangle\in\mathscr{F}(\langle y_1, y_2\rangle)$ *for all* $x\in C$.

Then $C\cap X_\alpha(\{y_1, y_2\})$ *is nonvoid.*

PROOF. Suppose that $C\cap X_\alpha(\{y_1, y_2\})=\emptyset$. Then for

$$M_i=\{z\in\langle y_1, y_2\rangle\colon C\cap X_\alpha(z)\subset X_\alpha(y_i)\}$$

we have $y_i\in M_i$, $i\in\{1, 2\}$, and $M_1\cap M_2=\emptyset$. From condition (b) we infer $C\cap X_\alpha(z)\subset(C\cap X_\alpha(y_1))\cup(C\cap X_\alpha(y_2))$, $z\in\langle y_1, y_2\rangle$. Together with (a2) we get $M_1\cup M_2=\langle y_1, y_2\rangle$. Now we show $M_i\in\mathscr{F}(\langle y_1, y_2\rangle)$, $i\in\{1, 2\}$, which is in contrast to the connectedness of $\langle y_1, y_2\rangle$. To this end, consider a net (z_i) in M_1, say, which converges to a $z^*\in\langle y_1, y_2\rangle$. $z_i\in M_1$ implies $C\cap X_\alpha(z_i)\cap X_\alpha(y_2)=\emptyset$. So for $x\in C\cap X_\alpha(y_2)$ we have $z_i\in Y_\alpha(x)\cap\langle y_1, y_2\rangle$, and with (c) we get $z^*\in Y_\alpha(x)$, hence $C\cap X_\alpha(y_2)\cap X_\alpha^*(z^*)=\emptyset$. In combination with (a1) we conclude $z^*\in\langle y_1, y_2\rangle-M_2=M_1$.

Now we are in the position to present our main result.

THEOREM 4.2. *Let X be a topological space, Y an interval space, and (α_n) a sequence of real numbers such that*

(a) $\alpha_n<a^*$ $(n\in\mathbf{N})$ *and* $\lim\limits_{n\to\infty}\alpha_n=a^*$.

(b) $X_{\alpha_n}(B)\in\mathscr{C}_0(X)\cap\mathscr{F}(X)$ *for all* $n\in\mathbf{N}$, $B\in\mathscr{E}(Y)$.

(c) $Y_{\alpha_n}^*(x)\in\mathscr{C}_I(Y)$ *for all* $n\in\mathbf{N}$, $x\in X$.

(d) $Y_{\alpha_n}(x)\cap\langle y_1, y_2\rangle\in\mathscr{F}(\langle y_1, y_2\rangle)$ *for all* $(n, x, y_1, y_2)\in\mathbf{N}\times X\times Y\times Y$.

Then we have $\tilde{a}_*=a^*$.

If, moreover, at least one of the sets $X_\alpha(y)$, $\alpha<a^*$, $y\in Y$ is compact, then we have $a_*=a^*$ and $\hat{X}\neq\emptyset$.

PROOF. 1. We first show that for all $k\in\mathbf{N}$ we have

(k) $X_{\alpha_n}(B)\neq\emptyset$ for all $n\in\mathbf{N}$, $B\in\mathscr{E}(Y)$, $|B|=k$.

The proof proceeds by induction on k. For $B=\{y\}$, $y\in Y$, $n\in\mathbf{N}$ we have $X_{\alpha_n}(B)\supset$

$\supset X^*_{\alpha_n}(y)\neq\emptyset$ in view of $a^*>\alpha_n$. (2) holds according to Lemma 4.1 (take $C=X$). Suppose that (k) is true for some $k\geqq 2$. For an arbitrary $D\in\mathscr{E}(Y)$ with $|D|=k+1$ choose $E\subset D$ with $|D-E|=2$. By (k) we have $C_n\cap X_{\alpha_n}(y)\neq\emptyset$ for $C_n=X_{\alpha_n}(E)$, $y\in Y$, $n\in\mathbf{N}$. For every $n\in\mathbf{N}$ there exists an $m\in\mathbf{N}$ with $\alpha_m>\alpha_n$, hence $C_n\cap X^*_{\alpha_n}(y)\supset$ $\supset C_m\cap X_{\alpha_m}(y)\neq\emptyset$.

Now Lemma 4.1 yields $X_{\alpha_n}(D)=C_n\cap X_{\alpha_n}(D-E)\neq\emptyset$.

2. From 1 and Remark 3.2b) we obtain $\tilde{a}_*\geqq\lim_{n\to\infty}\alpha_n=a^*$. So we have shown $\tilde{a}_*=a^*$.

3. An application of Proposition 3.3 completes the proof.

COROLLARY 4.3. (Tuy [17], Theorem 1.) *Let X be a compact topological space, $(Y,\mathscr{T},h)$ a Wu space, and (α_n) a sequence of reals such that*

(a) $\alpha_n<a^*$, $n\in\mathbf{N}$, *and* $\lim_{n\to\infty}\alpha_n=a.^*$

(b1) $X_{\alpha_n}(B)\in\mathscr{C}_0(X)$ *for all* $n\in\mathbf{N}$, $B\in\mathscr{E}(Y)$.

(b2) $a(\cdot,y)$ *is upper semicontinuous for every* $y\in Y$.

(c) $\{a(x,h(y_1,y_2,\cdot))<\alpha_n\}$ *is convex for all* $(n,x,y_1,y_2)\in\mathbf{N}\times X\times Y\times Y$.

(d) $a(x,\cdot)$ *is lower semicontinuous for every* $x\in X$.

Then we have $a_*=a^*$ *and* $\hat{X}\neq\emptyset$.

PROOF. Condition 4.3 (c) implies condition 4.2 (c) with respect to the interval space $I^h=(Y,\mathscr{T},\langle\cdot,\cdot\rangle^h)$ (compare Example 2.2). Hence, Theorem 4.2 can be applied.

REMARK 4.4. It has been shown by Tuy [17] that Corollary 4.3 remains true, if condition 4.3 (d) is replaced by

(d)* $a(x,\cdot)$ *is upper semicontinuous for every* $x\in X$.

Related results have been obtained by Cuóng [2], Geraghty—Lin [4], Komornik [10], and by the second author in his unpublished master thesis [16].

QUESTION 4.5. Does Theorem 4.2 remain true if Condition 4.2 (d) is replaced by

(d)* $Y^*_{\alpha_n}(x)\cap\langle y_1,y_2\rangle\in\mathscr{G}(\langle y_1,y_2\rangle)$ for all $(n,x,y_1,y_2)\in\mathbf{N}\times X\times Y\times Y$?

COROLLARY 4.6. *Let X be a compact Hausdorff space and Y an interval space. For $J=(-\infty,a^*)$ let the following assumptions be satisfied.*

(a) $X_\alpha(y)\in\mathscr{F}(X)$ *for all* $\alpha\in J$, $y\in Y$.

(b) $X^*_\alpha(B)\in\mathscr{C}_0(X)$ *for all* $\alpha\in J$, $B\in\mathscr{E}(Y)$.

(c) $Y^*_\alpha(x)\in\mathscr{C}_I(Y)$ *for all* $\alpha\in J$, $x\in X$.

(d) $Y_\alpha(x)\cap\langle y_1,y_2\rangle\in\mathscr{F}(\langle y_1,y_2\rangle)$ *for all* $(\alpha,x,y_1,y_2)\in J\times X\times Y\times Y$.

Then we have $a_*=a^*$ *and* $\hat{X}\neq\emptyset$.

PROOF. W.l.g. we may assume $a^*>-\infty$. For $B\in\mathscr{E}(Y)$, $\alpha\in J$, $\alpha>\gamma\in\mathbf{R}$ let F_γ denote the closure of $X^*_\gamma(B)$. If $X_\alpha(B)\neq\emptyset$, then the chain $\{F_\gamma\colon\gamma<\alpha\}$ is a collection of continua, hence $X_\alpha(B)=\bigcap_{\gamma<\alpha}F_\gamma$ is connected [21; 28.2]. So Theorem 4.2 can be applied.

COROLLARY 4.7. *Let X be a compact Hausdorff space and $(Y, \mathcal{T}, h)$ a Wu space. Let the following assumptions be satisfied.*

(a) *$a(\cdot, y)$ is upper semicontinuous for every $y \in Y$.*
(b) *$X_\alpha^*(B) \in \mathcal{C}_0(X)$ for all $\alpha \in \mathbf{R}$, $B \in \mathcal{E}(Y)$.*
(c) *$a(x, h(y_1, y_2, \cdot))$ is quasiconvex for all $(x, y_1, y_2) \in X \times Y \times Y$.*
(d) *$a(x, \cdot)$ is lower semicontinuous for every $x \in X$.*

Then we have $a_* = a^*$ and $\hat{X} \neq \emptyset$.

This is Tuy's [17] generalization of Wu's minimax theorem [22].

PROOF. Endow Y with the Wu interval structure I^h and apply Corollary 4.6.

5. Quasiconcave — quasiconvex games

Now we study games $\Gamma = (X, Y, a)$ where both X and Y are interval spaces. We shall show that Sion's minimax theorem carries over to this more general situation.

THEOREM 5.1. *Let X and Y be interval spaces, and let the following conditions be satisfied.*

(a) *$X_\alpha(y) \in \mathcal{C}_I(X) \cap \mathcal{F}(X)$ for all $a^* > \alpha \in \mathbf{R}$, $y \in Y$.*

(b *$Y_\beta(x) \in \mathcal{C}_I(Y)$ and $Y_\beta(x) \cap \langle y_1, y_2 \rangle \in \mathcal{F}(\langle y_1, y_2 \rangle)$ for all $\tilde{a}_* < \beta \in \mathbf{R}$,*

$(x, y_1, y_2) \in X \times Y \times Y$.

Then we have $\tilde{a}_* = a^*$.

If in addition

(c) *$X_\alpha(z)$ is compact for some real $\alpha < a^*$ and some $z \in Y$, then we have $a_* = a^*$ and $\hat{X} \neq \emptyset$.*

PROOF. 1. Suppose that $\tilde{a}_* < a^*$. Choose a sequence of reals α_n with $\tilde{a}_* < \alpha_n < a^*$ and $\lim_{n \to \infty} \alpha_n = a^*$. Then we have $Y_\alpha^*(x) = \bigcup \{Y_\beta(x) : \tilde{a}_* < \beta < \alpha_n\} \in \mathcal{C}_I(Y)$, because $\mathcal{C}_I(Y)$ is a convexity. But now $\tilde{a}_* < a^*$ is in contradiction to Theorem 4.2. Hence we have $\tilde{a}_* = a^*$.

2. If the additional compactness property holds, then we can apply Proposition 3.3.

COROLLARY 5.2. *Let X and Y be interval spaces such that*

(a) *For each $y \in Y$, $a(\cdot, y)$ is quasiconcave and upper semicontinuous on X.*

(b) *For each $x \in X$, $a(x, \cdot)$ is quasiconvex on Y and lower semicontinuous on any interval of Y.*

Then we have $\tilde{a}_* = a^*$.

If in addition

(c) *$X_\alpha(z)$ is compact for some real $\alpha < a^*$ and some $z \in Y$, then we have $a_* = a^*$ and $\hat{X} \neq \emptyset$.*

Special cases are Stachó's Theorem 2 in [13] and Proposition 1 of Brézis—Nirenberg—Stampacchia [1] which generalizes Sions minimax theorem.

The following example is classical [20; p. 32].

EXAMPLE 5.3. We consider the game $\Gamma=(X, Y, a)$ with $X=Y=\mathbf{N}$ and $a(x,y)=1, 0, -1$ for $x>y$, $x=y$, $x<y$. If X and Y are endowed with the interval structure as defined in Example 2.4.2, then Conditions 5.2 (a) and (b) are satisfied, hence $a^*=\tilde{a}_*$. Of course we have $a_*=-1\neq 1=a^*$: Condition 5.2 (c) is violated.

But if instead of $(\mathbf{N}, \mathscr{T})$ we take the Alexandroff compactification $(\mathbf{N}\cup\{\infty\}, \mathscr{T}\cup\{\mathbf{N}\cup\{\infty\}\})$ and extend a and $\langle\cdot,\cdot\rangle$ in the obvious manner, then the new game satisfies (5.2) (a), (b), (c) and we have $a_*=a^*$. Certainly now (∞, ∞) is a saddle point.

The following observation shows that Corollary 5.2 is quite general.

COROLLARY 5.4. *For a game $\Gamma=(X, Y, a)$ the following are equivalent.*

(1) *Γ has a saddle point.*

(2) *X and Y can be endowed with interval structures such that*

(a) *X is compact and $X_\alpha(y)\in\mathscr{C}_I(X)\cap\mathscr{F}(X)$ for all $a^*\geqq\alpha\in\mathbf{R}$, $y\in Y$.*

(b) *Y is compact and $Y_\beta(x)\in\mathscr{C}_I(Y)\cap\mathscr{F}(Y)$ for all $a_*\leqq\beta\in\mathbf{R}$, $x\in X$.*

The implication (2)⇒(1) generalizes Stachó's Theorem 1 in [13].

PROOF. (1)⇒(2): If $(\hat{x},\hat{y})$ is a saddle point of Γ, then the pointed interval structures $(X, \mathscr{T}_{\hat{x}}, \langle\cdot,\cdot\rangle_{\hat{x}})$ and $(Y, \mathscr{T}_{\hat{y}}, \langle\cdot,\cdot\rangle_{\hat{y}})$ satisfy (2).

(2)⇒(1): From Theorem 5.1 we infer $a_*=a^*$ and $\hat{X}\neq\emptyset$. In case $a^*=\infty$ we have $\hat{Y}=Y$; otherwise for each real sequence $\beta_n\downarrow a^*$ we have $Y_{\beta_n}(X)\neq\emptyset$ and $Y_{\beta_n}(X)\downarrow\hat{Y}$, hence $\hat{Y}\neq\emptyset$ by (2) b).

REMARK 5.5. In almost all papers on minimax theorems cited in this text the underlying topological spaces are assumed to be Hausdorff. As we have seen, this is not necessary in general. (Our only exceptions are Corollaries 4.6 and 4.7.) The Hausdorff axiom is a serious restriction in this context. Neither pointed spaces (Example 2.3) nor chain spaces (Example 2.4) with more than one element are Hausdorff. The following simple example shows that Corollary 5.4 turns wrong if the underlying spaces are required to be Hausdorff.

EXAMPLE 5.6. Let $\Gamma=(X, Y, a)$ and $X=Y=\{1, 2\}$. If $(X, \mathscr{T}, \langle\cdot,\cdot\rangle)$ is an interval space, then the topology $\mathscr{T}$ must be a chain, hence cannot be Hausdorff.

EXAMPLE 5.6.1. Let $(a(i,j))=\begin{pmatrix}2 & 3\\ 1 & 2\end{pmatrix}$. Then $(1, 1)$ is the only saddle point of Γ. If X and Y are endowed with the interval structure $(\{1,2\}, \{\emptyset, \{2\}, \{1, 2\}\}, \{\cdot,\cdot\})$, then the assumptions (a), (b), (c) of Corollary 5.2 are satisfied.

EXAMPLE 5.6.2. Let $(a(i,j))=\begin{pmatrix}2 & 3\\ 1 & 4\end{pmatrix}$. Again, $(1, 1)$ is the only saddle point of Γ, and Condition 5.4 (2) is satisfied if we take the same interval structures as in Example 5.6.1. However here there exists no interval structure on X such that condition (a) in Corollary 5.2 is satisfied.

EXAMPLE 5.7. (Wald [19].) Let $\Gamma=(X, Y, a)$ be a game, X or Y a finite set, and a real valued. Let $P_X(P_Y)$ denote the set of all probability measures on $X(Y)$ with finite support. We extend a on $P_X \times P_Y$ according to

$$\mathbf{a}(p, q) = \int_X \int_Y a(x, y) q(dy) p(dx).$$

Then for the *discrete mixed extension* $\boldsymbol{\Gamma}=(P_X, P_Y, \mathbf{a})$ we have $\mathbf{a}_*=\mathbf{a}^*$.

PROOF. P_X and P_Y are convex subsets of the dual of $\mathbf{R}^X$ and $\mathbf{R}^Y$, respectively. Finiteness of X, say, implies compactness of P_X. Hence we can apply Corollary 5.2 to $\boldsymbol{\Gamma}$.

EXAMPLE 5.8. (Fan [3].) Let X be a compact topological space such that every $a(\cdot, y)$, $y \in Y$ is upper semicontinuous. Assume that

(a) $\forall (x_1, x_2, \alpha) \in X \times X \times [0, 1]\ \exists x_0 \in X\ \forall y \in Y\colon a(x_0, y) \geqq \alpha a(x_1, y)+(1-\alpha) a(x_2, y)$.

(b) $\forall (y_1, y_2, \alpha) \in Y \times Y \times [0, 1]\ \exists y_0 \in Y\ \forall x \in X\colon a(x, y_0) \leqq \alpha a(x, y_1)+(1-\alpha) a(x, y_2)$

Then we have $a_*=a^*$ and $\hat{X} \neq \emptyset$.

As Sion mentioned in [12; § 4], the above result is an easy consequence of Theorem A or B. Let us carry out the details.

PROOF. From (a) one easily obtains by induction

$$\forall p \in P_X\ \exists x_0 \in X\ \forall y \in Y\colon\ a(x_0, y) \geqq \mathbf{a}(p, y).$$

This implies $\mathbf{a}_*(P_X, P_B)=a_*(X, B)$ for all $B \in \mathscr{E}(Y)$. Similarly (b) implies $\mathbf{a}^*(P_X, P_Y)=a^*(X, Y)$, and from Example 5.7 we infer $\mathbf{a}_*(P_X, P_B)=\mathbf{a}^*(P_X, P_B)$. Hence we have $a^*(X, Y)=\mathbf{a}^*(P_X, P_Y) \leqq \mathbf{a}^*(P_X, P_B)=\mathbf{a}_*(P_X, P_B)=a_*(X, B)$ for all $B \in \mathscr{E}(Y)$ which yields $a^*=\tilde{a}_*$. But $\tilde{a}_*=a_*$ and $\hat{X} \neq \emptyset$ follows from Proposition 3.3.

REMARK 5.9. It would be interesting to know whether Fan's minimax theorem can be derived directly from Corollary 5.2 by endowing X and Y with appropriate "intrinsic" interval structures.

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