On a Schwarz Lemma for Bounded Symmetric Domains

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ABSTRACT.

1. Introduction

It is well known that the classical Schwarz Lemma allows the following higher dimensional extension: Let $E, F$ be complex Banach spaces with open unit balls $B \subset E, D \subset F$ and let $f: B \to D$ be a holomorphic mapping with $f(0) = 0$. Then $\|f(z)\| \leq \|z\|$ for all $z \in B$ and $\|L\| \leq 1$ hold, where the linear operator $L: E \to F$ is the complex derivative at 0 of $f$. But in contrast to the classical case $E = F = \mathbb{C}$, the condition $\|f(z)\| = \|z\|$ for some $z \neq 0$ and also the condition $\|L\| = 1$ does in general not imply that $f$ is linear (more precisely the restriction to $B$ of a linear map – necessarily the derivative $L = df(0)$). To have a short notation we call the ordered pair of complex Banach spaces $(E, F)$ rigid if every holomorphic mapping $f: B \to D$ with $f(0) = 0$ is linear provided that the derivative $df(0): E \to F$ is a (not necessarily surjective) isometry. In case this conclusion already follows without the assumption $f(0) = 0$ we call the pair strictly rigid. For instance, $(E, F)$ is rigid if every unit vector in $F$ is a complex extremal boundary point of $D$ and this condition is also necessary if $E = \mathbb{C}$, compare [1]. Also, $(E, E)$ is strictly rigid for every complex Banach space $E$ of finite dimension as a consequence of Cartan’s uniqueness theorem, compare [5] and [1]. The rigidity condition for $(E, F)$ is not symmetric in $E, F$. In particular, $(E, F)$ trivially is rigid if there is no linear isometry $E \to F$.

Suppose that $\mathcal{K}$ is a class of complex Banach spaces and that $\varphi: \mathcal{K} \to \mathbb{N} \cup \{\infty\}$ is a function. We will consider the following property for $\varphi$.

**Property A:** For all $E, F \in \mathcal{K}$ with $\varphi(F) \leq \varphi(E) < \infty$ the pair $(E, F)$ is rigid.

Since for spaces with $\varphi$-value $\infty$ nothing is claimed in this property we always may assume without loss of generality that $\mathcal{K}$ is the class $B$ of all complex Banach spaces (simply by extending $\varphi$ using the value $\infty$). For instance, on $B$ the function $\varphi = \dim$ satisfies Property A. But also the following function $\psi$ satisfies Property A: For every complex Hilbert space $E$ put $\psi(E) = 1$. In case $E$ is not a Hilbert space but any unit vector is an extreme point of its unit ball put $\psi(E) = 2$. In the remaining cases put $\psi(E) = \infty$. Clearly, this would be more interesting if some of the values $\infty$ could be changed to a finite one while keeping Property A.

In the present paper we consider certain rank functions with Property A on the class of complex Banach spaces associated with bonded symmetric domains. It is known that every bounded symmetric domain in a complex Banach space can be realized as the open unit ball of another complex Banach space $E$ uniquely determined up to linear isometry [7]. These Banach spaces are called $JB^*$-triples since they may be algebraically characterized by a certain ternary structure, the Jordan triple product.

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2. The rank function

Fix the field $\mathbb{K}$ in the following which is either $\mathbb{R}$ or $\mathbb{C}$. Denote by $B$ the category of all $\mathbb{K}$-Banach spaces with the bounded $\mathbb{K}$-linear mappings as morphisms. Throughout, $E$ and $F$ are Banach spaces with open unit balls $B \subset E$ and $D \subset F$. The notation $E \subset F$ means that $E$ carries the induced norm from $F$, i.e. $B = D \cap F$. Also we write $E \lesssim F$ to indicate that there exists a (not necessarily surjective) linear isometry $E \to F$. By $\mathcal{L}(E,F)$ we denote the Banach space of all bounded linear operators $E \to F$. Furthermore $\mathcal{L}(E):= \mathcal{L}(E,E)$ is the Banach algebra of all continuous endomorphisms and $E^*: = \mathcal{L}(E, \mathbb{C})$ is the dual of $E$. The group of all invertible operators in $\mathcal{L}(E)$ is denoted by $\text{GL}(E)$. The vector space dimension of $E$ over $\mathbb{K}$ is denoted by $\text{dim}(E)$ and will be considered as an element of $\overline{\mathbb{N}}:= \mathbb{N} \cup \{\infty\}$.

The boundary of $B$ (the unit sphere in $E$) is denoted by $\partial B$. The subset of all extreme boundary points of $B$ is denoted by $\partial_B B$, that is the set of all $a \in \partial B$ with the property: $\|a \pm v\| = 1$ implies $v = 0$ for all $v \in E$. In the complex case (i.e. $\mathbb{K} = \mathbb{C}$) the point $a \in \partial B$ is called complex extreme if $\|a + tv\| = 1$ for all $t \in \Delta$ always implies $v = 0$, where $\Delta \subset \mathbb{C}$ is the open unit disc. With $\partial_B B \subset \partial B$ we denote the subset of all complex extreme boundary points. Also we denote for every complex Banach space $E$ by $E^\mathbb{R}$ the underlying real Banach space. Clearly, $E$ and $E^\mathbb{R}$ have to be distinguished, for instance $\dim(E^\mathbb{R}) = 2 \dim(E)$ holds in our notation.

We are interested in functions $\varphi : B \to \overline{\mathbb{N}}$ satisfying

**Property B**: $\varphi(E) \leq \varphi(F)$ for all $E, F \in B$ with $E \lesssim F$.

It is clear that $\varphi = \text{dim}$ satisfies this property. Further examples can be obtained in the following way: Let $\varphi$ satisfy Property B. For every Banach space $E$ and every $a \in E$ let $\Theta_a$ be the closed linear span of

$$\{v \in E : \|a + tv\| = \|a\| \text{ for all } t \in \mathbb{K} \text{ with } |t| \leq 1\}$$

in $E$ (this notion coincides except for $a = 0$ with the one in [1]). Then $\varphi'(E) := \sup_{a \in E} \varphi(\Theta_a)$ defines a function $\varphi' : K \to \overline{\mathbb{N}}$, and it is clear that every linear isometry $L : E \to F$ maps $\Theta_a$ into $\Theta_{La}$, therefore, with $\varphi$ also $\varphi'$ satisfies Property A. We call $\varphi'$ the derived function of $\varphi$. Then $\varphi' \leq \varphi$ is easily seen and by iteration we also get $\varphi''$ and so on. As an example, $\dim'(E) = 0$ holds if and only if $\partial B = \partial_B B$ in case $\mathbb{K} = \mathbb{R}$ and $\partial B = \partial_B B$ in case $\mathbb{K} = \mathbb{C}$. Also, if $E = \mathcal{L}(H,K)$ for Hilbert spaces $H, K$ with $\dim(H) = 2$ and $\dim(K) = n \geq 1$ we have $\dim(E) = 2n$, $\dim'(E) = n - 1$ and $\dim''(E) = 0$ (even if $n$ is infinite).

For every $E$ put furthermore $\tau(E) := \inf\{n \in \mathbb{N} : \varphi^{(n)}(E) = 0\}$, where $\varphi^{(n)}$ is the $n$-th derivative of $\varphi$ and $\inf \emptyset = \infty$. In case of $\varphi = \text{dim}$ we also write $\tau(E) = \tau_{\dim}(E)$ and call it the rank of the Banach space $E$. The following statement is easily verified.

2.1 Lemma. With $\varphi$ all derivatives $\varphi^{(n)}$ and also $\tau_\varphi$ satisfies Property B.

In particular, the rank $\tau$ satisfies Property B. All elements of $\overline{\mathbb{N}}$ occur as a rank: Consider for example the Banach space $E = C_0(S, \mathbb{K})$ of all $\mathbb{K}$-valued continuous functions vanishing at infinity on the locally compact topological space $S$. Then it is not difficult to see that $\tau(E) = \dim(E) = |S|$ where $|S| \in \overline{\mathbb{N}}$ is the number of elements in $S$. Actually, we can show a little bit more. Denote by $\ell^p \oplus \oplus F$ the $\ell^p$-sum of $E$ and $F$, that is $E \oplus F$ with norm satisfying $
(x, w)\| = \max(\|x\|, \|w\|)$ if $p = \infty$ and $
(x, w)\| \| = \|x\| + \|w\| p$ if $1 \leq p < \infty$. Instead of $E \oplus F$ we also write $E \times F$ since then the open unit ball is $B \times D$.

2.2 Proposition. For all Banach spaces $E, F$ the following statements hold.

(i) $\tau(E) \leq \dim(E)$ and $\tau(E) = 0$ if and only if $E = \{0\}$.

(ii) $\sup_{a \in E} \tau(\Theta_a) = \tau(E) - 1$ if $E \neq \{0\}$.

(iii) $\tau(E \times F) = \tau(E) + \tau(F)$.

**Proof**. (i) is obvious.

(ii) We may assume that $k := \sup_{a \in E} \tau(\Theta_a) < \infty$ since $k \leq \tau(E)$. For $\varphi := \dim$ this means

$$\varphi^{(k+1)}(E) = \sup_{a \in E} \varphi^{(k)}(\Theta_a) = 0,$$
i.e. \( r(E) \leq k + 1 \). But \( r(E) \leq k \) would contradict the definition of \( k \).

(iii) We assume that \( 0 < r(E) \leq r(F) \) holds and use induction on \( n = r(E) + r(F) \). The case \( n = 0 \) is trivial and for \( n = \infty \) the statement follows from \( r(E \times F) \geq r(F) = \infty \). Therefore we only have to consider the case \( 0 < n < \infty \). For all \((a, b) \in E \times F\)

\[
\Theta_{(a,b)} = \begin{cases} 
\Theta_c \times F & ||a|| > ||b|| \\
\Theta_a \times \Theta_b & ||a|| = ||b|| \\
E \times \Theta_b & ||a|| < ||b|| 
\end{cases}
\]

is easily seen. Then by induction hypothesis we have

\[
\sup_{(a,b) \in E \times F} r(\Theta_{(a,b)}) = n - 1 \quad \text{and hence} \quad r(E \times F) = n \quad \text{by (ii)}. \quad \square
\]

Property (ii) implies that the \( n \)-th derivative \( r^{(n)} \) of the rank function does not give further information since

\[
r^{(n)} = \max(r - n, 0) \quad \text{for all} \quad n \in \mathbb{N}.
\]

For the rest of the paper let \( \mathbb{K} \) be the complex field. For every complex Banach space \( E \) then let \( \rho(E) = r(E^{\mathbb{R}}) \) be the \textit{real rank} of \( E \). Then it is clear that also the function \( \rho \) on \( B \) satisfies Property B. Also, by induction it can be shown that always \( r(E) \leq \rho(E) \) holds. The question arises: To what extent do the rank functions \( r \) and \( \rho \) satisfy Property A? In the next section we prove this for the class of \( \mathbb{J}B^* \)-triples.

For certain complex Banach spaces \( E \) of finite dimension Vigué [12] has defined a rank \( r(B) \) of the open unit ball \( B \subset E \). Since in case that \( B \) is a bounded symmetric domain this rank in general is not the usual one, we prefer to write \( r_V(E) \) instead of \( r(B) \) here. Let \( V \) be the class of all complex Banach spaces of finite dimension such that the set

\[
\{ x \in E : \dim \Theta_x = \sup_{a \in E} \dim \Theta_a = \dim'(E) \}
\]

is dense in \( E \). Then \( r_V(E) = 1 + \dim'(E) \) in our language and the result in [12], Théorème 5.2, can be expressed in the following way: \textit{The function} \( r_V \) \textit{on} \( V \) \textit{satisifies Property A}.

3. \( \mathbb{J}B^* \)-triples

For complex Banach spaces \( E, F \) with open unit balls \( B, D \) a mapping \( f : B \to D \) is called \textit{holomorphic} if for every \( a \in B \) the Fréchet derivative \( df(0) \in \mathcal{L}(E, F) \) exists. The holomorphic mapping \( f \) is called \textit{biholomorphic} if the inverse mapping \( D \to B \) exists and is holomorphic. Cartan’s uniqueness theorem states that for every \( a \in B \) every biholomorphic map \( f : B \to D \) is uniquely determined within the space of all holomorphic mappings \( B \to D \) by \( f(a) \) and \( df(a) \) (compare i.e. [4] p. 75). With \( \text{Aut}(B) \) we denote the group of all biholomorphic mappings \( g : B \to B \), also called \textit{biholomorphic automorphisms} of \( B \).

The complex Banach space \( E \) is called a \( \mathbb{J}B^* \)-\textit{triple} if the group \( \text{Aut}(B) \) acts transitively on the open unit ball \( B \). To every \( a \in B \) then there is a unique automorphism \( s_a \in \text{Aut}(B) \) with \( s_a = s_a^{-1}, s_a(a) = a \) and \( ds_a(a) = -id \), i.e. \( D \) is a bounded symmetric domain. Denote by \( \mathbb{J}B \) the category of all \( \mathbb{J}B^* \)-triples. By definition a linear map \( L : E \to F \) is a morphism in \( \mathbb{J}B \) if \( L \circ s_B = s_F \circ L \) holds for all \( a \in B \) and \( c = L(a) \in D \). It is clear that with \( E, F \) also the \( \ell^\infty \)-sum \( E \times F \) is in \( \mathbb{J}B \) and that the canonical projections are triple morphisms. \( \mathbb{J}B^* \)-triples can also be introduced without any reference to holomorphy by the existence of a Jordan triple product \( (a, b, c) \mapsto \{abc\} \) from \( E^3 \) to \( E \) that is symmetric complex bilinear in the outer variables \( a, c \) and conjugate linear in the middle variable \( b \) together with some other properties, compare [7]. For instance, for every pair \( H, K \) of complex Hilbert spaces every closed linear subspace \( E \subset \mathcal{L}(H, K) \) stable under the triple product \( \{abc\} = (ab^*c + cb^*a)/2 \) is a \( \mathbb{J}B^* \)-triple. Therefore, every \( C^* \) algebra and also every complex Hilbert space is in \( \mathbb{J}B \), whereas in the latter case \( \{aba\} = (a|b|a) \) holds. The morphisms in \( \mathbb{J}B \) can also been characterized algebraically by the triple product:
The linear map $L: E \to F$ is a triple morphisms if and only if $L(abc) = L(a)L(b)L(c)$ holds for all $a, b, c \in E$. Triple morphisms always have closed range and are automatically continuous (the induced map $E/\ker(L) \to F$ is an isometry). On the other hand, every surjective linear isometry in $JB^*$ is a triple isomorphism.

Let $E, F$ always be $JB^*$-triples in the following. For every $a, b \in E$ denote the linear operator $z \mapsto \{abz\}$ by $a \circ b$. Then $\|ab\| \leq \|a\|^4 \|b\|$ holds and $\square$ may be considered as an operator-valued inner product on $E$. We write $a \perp b$ and call $a, b$ orthogonal if $\|a \circ b\| = 0$ or equivalently if $\|a \circ a\| = 0$ holds. For every $a \in E$ and $n \in \mathbb{N}$ the odd powers are defined by $a^{2n+1} = (a \circ a)^n a$. These always satisfy $\|a^{2n+1}\| = \|a\|^{2n+1}$. It is clear that the triple product on $E$ is uniquely determined by the cube mapping $a \mapsto a^3 = \{aaa\}$. The fixed points of the cube mapping are called tripotents. The set $M \subseteq E$ of all tripotents is a real analytic submanifold of $E$ and every non-zero tripotent $e \in E$ has norm 1. Suppose $e_1, \ldots, e_r$ are pairwise orthogonal tripotents in $E$. Then for every $i, j \in \{0, 1, \ldots, r\}$ the Peirce space

$$E_{ij} := E_{ij}(e_1, \ldots, e_r) := \{ z \in E : 2\{e_k e_k z\} = (\delta_{ik} + \delta_{jk})z \text{ for all } k \}$$

is a subtriple with $E_{ij} = E_{ji}$ and

$$E = \bigoplus_{0 \leq i \leq r} E_{ij}$$

is called the corresponding Peirce decomposition, compare [10]. The Peirce spaces multiply according to the rules

$$\{E_{ij}E_{jk}E_{kl}\} \subseteq E_{il} \quad \text{and} \quad E_{ij}E_{pq} = 0 \quad \text{if} \quad i, j \notin \{p, q\}.$$ 

In particular, we have the Peirce decomposition $E = E_{11}(e) \oplus E_{10}(e) \oplus E_{00}(e)$ for every single tripotent $e \in E$. The tripotent $e$ is called minimal in $E$ if $\dim(E_{11}(e)) = 1$ holds.

For every $a \in E$ denote by $E_a \subseteq E$ the smallest closed subtriple of $E$ that contains $a$ and put $d(a) := \dim(E_a) \in \mathbb{N}$. It is known that $E_a$ is isometrically isomorphic to $C_0(S) := C_0(S, \mathbb{C})$ for some locally compact topological space $S$. In particular, also $d(a) = r(E_a)$ holds where $r(E_a)$ is the Banach space rank as defined in the previous section. By definition, the triple rank of $E$ is the supremum in $\mathbb{N}$ of all $d(a)$ with $a \in E$.

3.1 Proposition. For every $JB^*$-triple $E$ the triple rank and the Banach space rank $r(E)$ coincide.

Proof. Denote for a while the triple rank of $E$ by $\tilde{r}(E)$. We have to show $\tilde{r}(E) = r(E)$. For every $a \in E$ we have $d(a) = r(E_a) \leq r(E)$ and hence $\tilde{r}(E) \leq r(E)$. Therefore we may assume that $n = \tilde{r}(E) < \infty$ holds. In case $n = 0$ we have $E = 0$, i.e. in addition we may assume $n > 0$. For every $a \in E$ with $a \neq 0$ there exists a unique representation

$$a = \lambda_1 e_1 + \cdots + \lambda_s e_s \quad \text{with} \quad \lambda_1 > \lambda_2 > \cdots > \lambda_s > 0,$$

where $e_1, e_2, \ldots, e_s$ are pairwise orthogonal non-zero tripotents in $E$, compare [8]. By [1] Lemma 7.8 we know that $\Theta_a = E_{00}(e_1)$ is a subtriple of $E$. Since $\Theta_a$ has triple rank $\tilde{r}(\Theta_a) < n$ we get by induction hypothesis $r(\Theta_a) = \tilde{r}(\Theta_a) \leq n - 1$, i.e. $r(E) \leq n = \tilde{r}(E)$ by 2.2.ii. \hfill $\square$

$JB^*$-triples of finite rank can be characterized in many ways, compare also [8].

3.2 Proposition. For every $JB^*$-triple $E$ the following conditions are equivalent.

(i) $E$ has finite rank.

(ii) Every finite subset of $E$ is contained in a subtriple of finite dimension.

(iii) Every $a \in \partial B$ has a (unique) representation $a = e + u$ with $u \in B$, $e$ a tripotent and $e \perp u$.

(iv) For every $a \in E$ the operator $a \circ a \in \mathcal{L}(E)$ is algebraic (i.e. satisfies a nontrivial polynomial equation).

(v) $E$ is reflexive.

$JB^*$-triples $E$ of finite rank behave essentially like those of finite dimension, compare [10] for the following discussion. A tuple $(e_1, \ldots, e_r)$ of pairwise orthogonal minimal tripotents in $E$ is called
a frame in $E$ if $E_0(e_1, \ldots, e_r) = 0$. All frames have the same length $r = r(E)$. The tripotent $e(\alpha) = e$ in 3.2.iv can be obtained by $e = \lim n^{-1}$. The fibres of the mapping $e: D \to M$ are the holomorphic arc components of $D$, i.e. the smallest non-empty subsets $A \subset D$ with the property: $f(A) \subset A$ for every holomorphic mapping $f: \Delta \to D$ with $f(A) \cap A \neq \emptyset$. For every $a \in D$ and $e = e(a)$ the holomorphic arc component of $a$ is $e^{-1}(e) = e + (D \cap E_0(e))$.

The $n$-dimensional Banach space $F = \ell^n$ is a JB*-triple with open unit ball $\Delta^n$. Let $f_1 = (1, 0, \ldots, 0, \ldots, f_n = (0, \ldots, 0, 1)$ be the standard basis of $F$. Suppose, $E$ has finite rank and $L: F \to E$ is a linear isometry. Let $a_k = L(f_k)$ and write $e_k = e_k + u_k$ with $e_k = e(a_k)$ for all $k$. Then for all $j \neq k$ we have $a_j + \Delta a_k \subset e^{-1}(a_j) = e^{-1}(e_j)$. This implies $u_j + \Delta a_k \subset E_0(e_j)$ and hence $a_k \in E_0(e_j)$. The closed subtriple $E_0(e_j)$ contains with $a_k$ also all odd powers of $a_k$ and hence also the limit $e_k$, i.e. $e_j \perp e_k$ and $u_j \perp e_k$ for all $j \neq k$. This implies $u_j \perp e = e_1 + \ldots + e_n$ and also $n \leq r(E)$. In case of equality all $u_j$ vanish and $L(F)$ is a subspace of $E$. This implies that then $L$ is a triple homomorphism. Since for every $1 \leq k < n$ there exist linear isometries $\ell^n \to \ell^n$ that are not triple homomorphisms we have thus proved

3.3 Lemma. For every JB*-triple $E$ and every integer $r \geq 1$ the following conditions are equivalent.

(i) $E$ has finite rank $r$.

(ii) $n \leq r$ if there exists a linear isometry $\ell^n \to E$.

(iii) Every linear isometry $\ell^n \to E$ is a triple homomorphism.

As a consequence, for every JB*-triple of finite rank, $r(E)$ is the maximal $n$ such that there exists a linear isometry $\ell^n \to E$.

We are now ready to prove the main result of this section.

3.4 Theorem. Let $E, F$ be JB*-triples with $r(F) \leq r(E) < \infty$ and open unit balls $B, D$. Suppose $f: B \to D$ is a holomorphic mapping such that the derivative $L = df(0) \in L(E, F)$ is an isometry. Then $r(F) = r(E)$, $f = L|B$ and $L$ is a triple homomorphism. In particular, the rank function $r$ on JB satisfies Property A.

Proof. Fix $a \in E$ and put $r = r(E)$. Then there exists a frame $(e_1, \ldots, e_r)$ in $E$ and a spectral decomposition $a = \lambda_1 e_1 + \cdots + \lambda_r e_r$ with coefficients $\lambda_i \geq 0$ for all $i$. Since $L$ is an isometry $t \mapsto \sum_{i=1}^r t_i L(e_i)$ defines an isometry $R: \ell^r \to F$. From 3.3.i we derive $r(E) = r(F)$ and also that $R$ is a triple homomorphism. This implies $L(a)^3 = L(a^3)$ for all $a \in E$, i.e. also $L$ is a triple homomorphism. The set $\partial E \subset \partial B$ of all extreme boundary points of $B$ coincides with $\{ e \in M : E_0(e) = 0 \}$ and is a set of determinacy in $E$ in the sense of [1]. Because of $L(\partial E) \subset \partial D$ we derive $f = L|B$ as a consequence of Corollary 3.3 in [1].

Suppose, $E$ with open unit ball $B$ is a JB*-triple of finite rank $r$. In [8] all equivalent norms $\Phi$ on $E$ have been determined which are invariant under the group $GL(B) \subset GL(E)$. Among these are all $p$-norms for $1 \leq p \leq \infty$ on $E$ defined as follows: Write every $a \in E$ as linear combination $a = \lambda_1 e_1 + \cdots + \lambda_r e_r$ for some frame $(e_1, e_2, \ldots, e_r)$ in $E$ and put $\|a\|_p = \|(\lambda_1, \lambda_2, \ldots, \lambda_r)\|_p$. Then $\partial E \subset \{ a \in B : \|a\|_p = 1 \}$, the original norm of $E$ coincides with $\|\cdot\|_e$ and $\|\cdot\|_2$ is a Hilbert norm. This implies $E$ is isomorphic to a complex Hilbert space. Now suppose that $F$ is another JB*-triple of finite rank and $L: E \to F$ is a linear map with $\|L\| \leq 1$ and $\|L(a)^p\| = \|a\|_p$ for all $a \in E$ (i.e an isometry with respect to the $p$-norm on both spaces). Since $B$ is the closed convex hull of $\partial E \subset E$ it is clear that then $L$ is also an isometry of JB*-triples. Thus as a consequence of our main Theorem ?? we get.

3.5 Proposition. Let $E, F$ with open unit balls $B, D$ be JB*-triples of rank $r(F) \leq r(E) < \infty$ and let $f: B \to D$ be a holomorphic mapping such that $L = df(0)$ is an isometry with respect to the $p$-norm for some $1 \leq p \leq \infty$. Then $r(F) = r(E)$ and $f = L|B$ is linear.

For $p = 2$ and finite dimensions this result is already contained in [13].

References