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CONVEXITY, MINIMAX THEOREMS AND THEIR APPLICATIONS

By

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Dedicated to Professor P. ERDŐS for his 80-th birthday

0. Introduction

The results concerning minimax theorems have many applications in several fields of pure and applied mathematics. Our purpose is to study the following two problems: let X and Y be nonempty sets, and let $f, g : X \times Y \to \mathbb{R}$, $\varphi : X \to Y$ be given functions such that $f \leq g$ on $X \times Y$. Under which hypothesis on X, Y, f, g and φ holds

(A)
$$\inf_{y\in Y} \sup_{x\in X} f(x,y) \leq \sup_{x\in X} \inf_{y\in Y} g(x,y)$$

or

(B)
$$\inf_{y\in Y} \sup_{x\in X} f(x,y) \leq \sup_{x\in X} g(x,\varphi(x)).$$

Results related to problem (A) (called minimax theorems, since in case $f = g$ we have equality) have been obtained by several authors. These can be applied in game theory, mathematical economics and optimization theory. Problem (B) has been studied just in case $X = Y$ and $\varphi(x) = x$. These results (called minimax inequalities) are useful for studying variational inequalities, differential equations, potential theory, etc. It is easy to see that (A) implies (B), therefore (B) can be usually stated under weaker hypothesis.

The aim of this note is to state theorems which extend some of the most important known results concerning problems (A) and (B), to study the

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connection between them and to give some applications. The first section is concerned on problem (B). Using a convexity concept introduced by JOÓ [14], Theorem 1.1 contains, in particular KY FAN [4], while Theorem 1.3 those of BRÉZIS–NIRENBERG–STAMPACCHIA [1], SIMONS [25] and KY FAN [5]. Here we use a method related to KNASTER–KURATOWSKI–MAZURKIEWICZ (KKM in short) theorem which first appeared in KASSAY–KOLUMBÁN [17]. In section 2 we apply our results from section 1 and deduce Brouwer's fixed point theorem for this kind of convexity (pseudoconvex spaces). Further, we show that Brouwer's fixed point theorem fails in a more general convexity structure (interval spaces, introduced by STACHÓ [28]). Although problem (A) can be treated in interval spaces (see the results of JOÓ [14], [15], STACHÓ [28], KOMORNIK [21]) our counterexample shows that problem (B) cannot be attached in interval spaces using the argument of section 1 (which uses Brouwer's fixed point theorem). This question remains open (i.e., problem (B) in interval spaces).

In section 3 we are concerned on problem (A). It is shown how one can prove a general minimax theorem (Theorem 3.1) using three methods: the *KKM-method* used in section 1, the *level set method* discovered by JOÓ [11] and the *mixture* of the *level set* and *cone method* (discovered in the strongest form by JOÓ [13]) given by KASSAY [19].

Theorem 4.1 (section 4) is deduced using Theorem 1.3 and extends a result on variational inequalities due to BROUWER [2].

Recently, SIMONS [27] introduced the concept of *upward-downward* function in order to establish minimax theorems. He asked whether his result remains true for two functions. In section 5 we answer in the negative to both of Simons conjectures. On the other hand, we prove these conjectures under an additional hypothesis.

Finally, in section 6 we state generalized Kuhn–Tucker theorems for cone constrained and inequality constrained (nonlinear) optimization problems.

For other results concerning problems (A) and (B) see also [3], [7], [8], [9], [10], [16], [18], [29].

The results of this paper were first announced in [15].

1. Minimax inequalities on pseudoconvex spaces

The aim of this section is to give a minimax inequality without linear structure which, in particular implies KY FAN [5], BRÉZIS–NIRENBERG–STAMPACCHIA [1] and SIMONS [25]. Instead of the usual convexity in linear spaces, we consider a convexity concept introduced by JOÓ [14] which doesn't need the linear structure. First we recall the following definitions:

DEFINITION 1.1 (KOMIYA [20]). Let X be a nonempty set. A mapping $h : 2^X \to 2^X$ is said to be *convex hull operation* if it satisfies the following conditions: $h(\emptyset) = \emptyset$, $h(\{x\}) = \{x\}$, $h(A) = \bigcap\{h(F) : F \subset A \text{ is a finite set}\}$, $h(h(A)) = A (A \subset X)$. One says that $A \subset X$ is convex, if $h(A) = A$.

DEFINITION 1.2 (JOÓ [14]). A triple $(X, h, \mathcal{F})$ is called *pseudoconvex space* if

(1) X is a topological space and h is a convex hull operation on it;

(2) $\mathcal{F} = \{\psi_F : F \subset X \text{ is finite}\}$, $\psi_F : \Delta^n \to h(F)$, $(n = \operatorname{card} F - 1)$ is a continuous mapping of Δ^n onto $h(F)$, where Δ^n denotes the standard simplex of $\mathbb{R}^n$;

(3) For each finite $F \subset X$, ψ_F is convex hull preserving, i.e., if $\Delta^n = (e_0, e_1, \ldots, e_n)$ and $F = \{x_0, x_1, \ldots, x_n\}$ for each subsimplex $(e_{i_0}, e_{i_1}, \ldots, e_{i_k}) \subset (e_0, e_1, \ldots, e_n)$ we have $\psi_F((e_{i_0}, \ldots, e_{i_k})) = h(x_{i_0}, \ldots, x_{i_k})$.

In [12] can be found an example for pseudoconvex space (see the next section too), where $X = \mathbb{R}^n$ and which differs from the usual convexity. We also mention that pseudoconvex spaces include convex spaces in the sense of KOMIYA [20].

Now we state a result which extend the well known Ky Fan's intersection theorem [4] (often called the Ky Fan's lemma). For, we first need the following definition:

DEFINITION 1.3. Let $(X, h, \mathcal{F})$ be a pseudoconvex space and let $X_0, X_1, \ldots, X_n$ be a family of subsets of X. The set $\{x_0, x_1, \ldots, x_n\} \subset X$ is said to be *KKM-selection* for $X_0, \ldots, X_n$ if for any subset $j \subset \{0, 1, \ldots, n\}$ we have

$$h(\{x_j : j \in J\}) \subset \bigcup_{j \in J} X_j.$$

Note that this concept differs from that in Ky Fan's lemma even in case when the convexity above reduces to the usual one, since the elements $x_0, \ldots, x_n$ are not necessarily distinct. For instance we could have $x_0 = x_1 = \ldots = x_n$; then $\bigcap_{i=0}^{n} X_i \neq \emptyset$.

Moreover, we have the following property: $\bigcap_{i=0}^{n} X_i \neq \emptyset$ if and only if there exists a KKM-selection for $X_1, \ldots, X_n$ which contains one element. The following theorem gives a sufficient condition for the existence of such a KKM-selection.

THEOREM 1.1. *Let $(X, h, \mathcal{F})$ be a pseudoconvex space, I a nonempty set and $(X_i)_{i \in I}$ a family of compact subsets of X. Then $\bigcap_{i \in I} X \neq \emptyset$ if and* *only if for each finite subset $i_0, i_1, \ldots, i_n \in I$, the family $(X_{i_k})_{0\leq k\leq n}$ admits a KKM-selection.*

PROOF. The necessity is obvious. We prove the sufficiency. By compactness, it is enough to show the "finite intersection property" for $(X_i)_{i\in I}$. Let $X_{i_0}, X_{i_1}, \ldots, X_{i_n}$ $(i_0, i_1, \ldots i_n \in I)$ be given and let $\{x_0, x_1, \ldots x_n\}$ be a KKM-selection of them. Without loss of generality we may suppose that $x_0, x_1, \ldots, x_n$ are distinct. Otherwise, if, say $x_l = x_p$ for $l \neq p$ then the corresponding sets X_{i_l} and X_{i_p} could be changed with their (nonvoid) intersection. Let $E_k = \psi_F^{-1}(X_{i_k} \cap h(F))$, $0 \leq k \leq n$, where $F = \{x_0, x_1, \ldots, x_n\}$. Then for $\{e_0, e_1, \ldots, e_n\} = \Delta^n$ we have $\text{co}\{e_j : j \in J\} \subset \bigcup_{j\in J} E_j$ for each $J \subset \{0, 1, \ldots, n\}$. Indeed, let $z \in co\{e_j : j \in J\}$. Then $\psi_F(z) \in h(\{x_j : j \in J\}) \subset \bigcup_{j\in J} X_{i_j}$. Let $j_0 \in J$ such that $\psi_F(z) \in X_{i_{j_0}}$. Therefore, $z \in \psi_F^{-1}(X_{i_{j_0}} \cap h(F)) = E_{j_0}$. By a variant of the classical KKM theorem (see [17], Lemma 2.1) it follows that $\bigcap_{k=0}^{n} E_k \neq \emptyset$; thus $\bigcap_{k=0}^{n} X_{i_k} \neq \emptyset$. This completes the proof.

In the following we give an extension to an intersection theorem due to BRÉZIS–NIRENBERG–STAMPACCHIA ([1], Lemma 1).

THEOREM 1.2. *Let $(X, h, \mathcal{F})$ be a pseudoconvex space, I a nonempty set and $\varphi : I \to X$ a given function. Let $(X_i)_{i\in I}$ be a family of subsets of X for which*

(4) *there is an index $i_0 \in I$ such that $\overline{X_{i_0}}$ (the closure of X_{i_0}) is compact;*

(5) *for each $i \in I$ and for each finite subset F of X, $X_i \cap h(F)$ is closed;*

(6) *for each finite subset F of X we have*

$$\overline{\left(\bigcap_{i\in I_F} X_i\right)} \cap h(F) = \left(\bigcap_{i\in I_F} X_i \cap h(F)\right),$$

where $I_F := \varphi^{-1}(h(F))$ (may be empty);

(7) *for each finite subset J of I we have*

$$h(\{\varphi(j) : j \in J\}) \subset \bigcap_{j\in J} X_j.$$

Then $\bigcap_{i\in I} X_i \neq \emptyset$.

PROOF. Let $\Phi = \{F : F \subset X \text{ is a finite set}, \ \varphi(i_0) \in F\}$ and $\mathcal{H} = \{h(F) : F \in \Phi\}$. For simplicity, denote the family $\mathcal{H}$ by $(H_s)_{s\in S}$. On S introduce the ordering relation as follows: $s \leq t$ $(s, t \in S)$ iff $H_s \subset H_t$. Let $s \in S$ be fixed and $I_s := \varphi^{-1}(H_s)$ (Since $i_0 \in I_s$ for each $s \in S$, I_s is nonempty). Define

$$E_i^s := X_i \cap H_s, \qquad i \in I_s$$

Then we have

(8) E_i^s is nonempty and closed for each $i \in I_s$;

(9) $E_{i_0}^s$ is compact;

(10) For each finite subset J of I_s we have

$$h(\{\varphi(j) : j \in J\}) \subset \bigcup_{j \in J} E_j^s.$$

Properties (8) and (9) are obvious. For (10) let J be a finite subset of I_s. It is clear by (7) that $h(\{\varphi(j) : j \in J\}) \subset \bigcup_{j \in J} X_j$. On the other hand, using the axioms of h we have $h(\{\varphi(j) : j \in J\}) \subset H_s$. Applying Theorem 1.1 (without loss of generality we may assume that E_i^s are compact, $i \in I_s$), $\bigcap_{i \in I_s} E_i^s \neq \emptyset$. For each $s \in S$, choose an element $u_s \in \bigcap_{i \in I_s} E_i^s$ and let $K_s := \bigcup_{t \geq s} \{u_t\}$. We have $K_s \subset \overline{X_{i_0}}$; further for $s_1, s_2 \in S$ there exists $s_3 \in S$ with $K_{s_3} \subset K_{s_1} \cap K_{s_2}$. Using compactness of $\overline{X_{i_0}}$ it follows that $\bigcap_{s \in S} \overline{K_s} \neq \emptyset$. Let $\overline{x} \in \bigcap_{s \in S} \overline{K_s}$. We show that $\overline{x} \in \bigcap_{i \in I} X_i$. By definition, $K_s \subset \bigcap_{i \in I_s} X_i$. Choose an index $s_0 \in S$ such that $\overline{x} \in H_{s_0}$. Let $i \in I$ be arbitrarily chosen and $s \geq s_0$ such that $\varphi(i) \in H_s$. Then by (6)

$$\overline{x} \in \overline{K_s} \cap H_s \subset \overline{\left(\bigcap_{i \in I_s} X_i \right)} \cap H_s = \left(\bigcap_{i \in I_s} X_i \right) \cap H_s.$$

Thus $\overline{x} \in X_i$. This completes the proof.

Our next purpose is to establish a minimax inequality for pseudoconvex spaces. First we need the following definition.

A pair (X, h) is said to be *convex space* of X is a point set and $h : 2^X \to 2^X$ is a convex hull operation (see Definition 1.1).

THEOREM 1.3 *Let (X, h_1) be a convex space and $(Y, h_2, \mathcal{F})$ be a pseudoconvex space. Let $f, g : X \times Y \to \mathbb{R}$ such that $f \leq g$ on $X \times Y$, let $\varphi : X \to Y$ be a function and $a := \sup_{x \in X} g(x, \varphi(x))$. Suppose*

(11) *for each finite subset X_0 of X, $h_2(\{\varphi(x) : x \in X_0\}) \subset \varphi(h_1(X_0))$;*

(12) *f is l.s.c. in its second variable on $h_2(Y_0)$, for each finite $Y_0 \subset Y$;*

(13) *g is quasiconcave in its first variable, i.e. the set $\{x \in X : g(x,y) > \beta\}$ is convex for each $y \in Y$ and $\beta \in \mathbb{R}$;*

(14) *for each finite subset $Y_0 \subset Y$ and for each filter $y_\alpha \in Y$ converging to $y \in h_2(Y_0)$, we have: $f(x,y_\alpha) \le a$ for each $x \in I_0$ implies $f(x,y) \le a$ for each $x \in I_0$, where $I_0 := \varphi^{-1}(h_2(Y_0))$;*

(15) *there is a compact subset D of Y and $x_0 \in X$ such that $f(x_0,y) > a$ for each $y \in Y \setminus D$.*

Then there exists $y_0 \in Y \cap D$ such that $f(x,y_0) \le a$ for all $x \in X$. In particular, $\inf_{y \in Y} \sup_{x \in X} f(x,y) \le \sup_{x \in X} g(x,\varphi(x))$.

REMARK. Assumptions (14) and (15) are clearly satisfied if Y is compact and f is l.s.c. in its second variable on Y.

PROOF. For each $x \in X$ define the sets

$$Y_x := \{y \in Y : f(x,y) \le a\}.$$

Using Theorem 1.2, we show that $\bigcap_{x \in X} Y_x \neq \emptyset$. It is easy to see that properties (4), (5) and (6) follow by (15), (12) and (14) respectively, with $I := X$, $Y_x := X_i$. We have to verify that (11) and (13) imply (7). Let $x_1, x_2, \dots \dots, x_n \in X$ and suppose $h_2(\{\varphi(x_i) : i \in \{1,\dots,n\}\}) \not\subset \bigcup_{i=1}^{n} Y_{x_i}$. Then for some $y \in h_2(\{\varphi(x_i) : i \in \{1,\dots,n\}\})$ we have $f(x_i,y) > a$ for each $i \in \{1,2,\dots,n\}$. By (11), $y \in \varphi(h_1(\{x_1,\dots,x_n\}))$, hence there exists $z \in h_1(\{x_1,\dots,x_n\})$ such that $y = \varphi(z)$. Since $g(x_i,y) > a$ for each $i \in \{1,\dots,n\}$, by (13), $g(z,y) > a$, i.e., $g(z,\varphi(z)) > a$, which contradicts the definition of a. ∎

This completes the proof.

COROLLARY 1.1 (BRÉZIS–NIRENBERG–STAMPACCHIA [1]). *Let Z be a closed convex subset of a Hausdorff topological vector space E and let $f : Z \times Z \to \mathbb{R}$ such that*

(16) $$f(x,x) \le 0 \quad \text{for each } x \in Z;$$

(17) *f is l.s.c. in its second variable on the intersection of Z with any finite dimensional subspace of E;*

(18) *f is quasiconcave in its first variable;*

(19) *Whenever C is a convex subset of Z and y_α is a filter converging to $y \in C$, then $f(x,y_\alpha) \le 0$ for every $x \in C$ implies $f(x,y) \le 0$ for every $x \in C$;*

(20) *There is a compact subset D of E and $x_0 \in D \cap Z$ such that $f(x_0,y) > 0$ for $y \in Z \setminus D$.*

Then there exists $y_0 \in D \cap Z$ *such that* $f(x,y_0) \leq 0$ *for all* $x \in Z$. *In particular,* $\inf_{y \in Z} \sup_{x \in Z} f(x,y) \leq 0$.

PROOF. Apply Theorem 1.3 with $X = Y = Z, f = g, \varphi(x) = x$ for each $x \in X$ and $h_1(A) = h_2(A) = \operatorname{co} A$ if $A \subset X$.

COROLLARY 1.2 (SIMONS, [25]). *Let* Z *be a nonempty convex subset of a topological vector space, let* $f : Z \times Z \to \mathbb{R}$ *be l.s.c. in its second variable,* $g : Z \times Z \to \mathbb{R}$ *quasiconcave in its first variable and* $f \leq g$ *on* $Z \times Z$. *Then* $\min_{y \in Z} \sup_{x \in Z} f(x,y) \leq \sup_{x \in Z} g(x,x)$.

COROLLARY 1.3 (KY FAN [5]). *Let* Z *be a nonempty compact convex subset of a topological vektor space and* $f : Z \times Z \to \mathbb{R}$ *be quasiconcave in its first variable and l.s.c. in its second variable. Then* $\min_{y \in Z} \sup_{x \in Z} f(x,y) \leq$
$\leq \sup_{x \in Z} f(x,x)$.

2. Application in fixed point theory

Using the results above, in this section we prove that Brouwer's fixed point theorem holds under convexity introduced by I. LOÓ and L. L. STACHÓ [12]. This is a kind of pseudoconvexity defined in $\mathbb{R}^n$.

Let $x = (x_0, \dots, x_n)$, $y = (y_0, \dots, y_n) \in \mathbb{R}^{n+1}$. The interval joining x and y will be defined as a polygon with at most $n+1$ pairwise orthogonal segments as follows (see also [12] and [14]). If $x_n \geq y_n$ then let $I_n = \{(x_0, \dots, x_{n-1}, t) : y_n \leq t \leq x_n\}$. If $x_n \leq y_n$ then let $I_n = \{(y_0, \dots, y_{n-1}, t) : x_n \leq t \leq y_n\}$. In the first case we get I_{n-1} analogously to I_n; if, for example $x_{n-1} \leq y_{n-1}$ then $I_{n-1} = \{(y_0, \dots, y_{n-2}, t, y_n) : x_{n-1} \leq t \leq y_{n-1}\}$; if $x_{n-1} > y_{n-1}$ then $I_{n-1} = \{(x_0, \dots, x_{n-2}, t, y_n) : y_{n-1} \leq t \leq x_{n-1}\}$. In the second case ($x_n \leq y_n$) we construct analogously I_{n-1} and in the third step I_{n-2}, etc. The segments $I_0, I_1, \dots, I_n$ parallel to the axis will join x and y. A set $A \subset \mathbb{R}^n$ is *convex* if $x, y \in A$ implies that the interval joining x and y is contained in A. It can be shown that $\mathbb{R}^n$ with this convexity is a pseudoconvex space in the sense of Definition 1.2 (see JOÓ [14]).

In which follows, we use the notions of convex set, quasiconvex (quasiconcave) function in the sense of JOÓ–STACHÓ [12], described above.

LEMMA 2.1. *The Chebyshev norm in* $\mathbb{R}^n$, *i.e.* $\|x\|_C = \max\{|x_1|, \dots, |x_n|\}$ *for* $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ *is a quasiconvex function.*

PROOF. It is easy to see that each Chebyshev sphere centered at the origin is a convex set. Let $x,y \in \mathbb{R}^n$ and $z \in [x,y]$ (the interval joining x and y). Let $R = \max\{\|x\|_C, \|y\|_C\}$. Then the sphere centered at the origin with radius R contains z. Hence $\|z\|_C \leq R = \max\{\|x\|_C, \|y\|_C\}$.

We mention that this statement is false for the Euclidean norm. For instance let $x = (-2,0)$, $y = (1,-1) \in \mathbb{R}^2$. Then $z = (-2,-1) \in [x,y]$ and $\|z\| > \max\{\|x\|, \|y\|\}$ where $\|\cdot\|$ denotes the Euclidean norm.

THEOREM 2.1 (See also KY FAN [6]). *Let X be a compact convex subset of $\mathbb{R}^n$ and $\bar{f}, \hat{f} : X \to \mathbb{R}^n$ be two continuous mappings. Suppose*

$$\|x - \hat{f}(x)\|_C \geq \|x - \bar{f}(x)\|_C \quad \textit{for each} \quad x \in X.$$

Then there exists a point $x_0 \in X$ such that

$$\|x - \hat{f}(x)\|_C \geq \|x_0 - \bar{f}(x_0)\|_C \quad \textit{for each} \quad x \in X. \tag{21}$$

PROOF. Take $f(x,y) = \|y - \bar{f}(y)\|_C - \|x - \hat{f}(y)\|_C$. It is clear that f is continuous in its second variable and (using Lemma 2.1) quasiconcave in its first variable. Applying Theorem 1.3 with $X = Y$, $h_1 = h_2$, $f = g$, $\varphi(x) = x$ for each $x \in X$ we obtain (21).

REMARK. If in addition X is invariant under $\hat{f}$, i.e. $\hat{f}(X) \subset X$, then (21) is equivalent with $\bar{f}(x_0) = x_0$.

In [28] L. L. STACHÓ introduced the following convexity structure: the pair $(X, [\cdot,\cdot])$ is called an *interval space* if X is a topological space and $[\cdot,\cdot] : X \times X \to 2^X$ is a mapping such that x_0, $x_1 \in [x_0,x_1]$ and $[x_0,x_1]$ is a connected set for each x_0, $x_1 \in X$. A set $A \subset X$ is called *convex* if x_0, $x_1 \in A$ implies $[x_0,x_1] \subset A$. We can define the concept of quasiconvex (quasiconcave) function $f : X \to \mathbb{R}$ in a similar way as above. It is easy to see that every pseudoconvex space $(X, h, \mathcal{F})$ is an interval space, where the interval $[x_0,x_1]$ is defined to be $\psi_F(\Delta^1)$, with $F = \{x_0,x_1\} \subset X$. It is natural to ask whether Brouwer's fixed point theorem remains true for interval spaces. We answer to this question in the negative. Let $X = [0,1] \subset \mathbb{R}$. Introduce the following topology in X : $A \subset X$ is closed iff A is finite or $A = X$. It is clear that X is a compact topological space with the topology above. For each x_1, $x_2 \in X$ let the interval joining x_1 and x_2 be the whole space X. Then every interval is closed, connected and contains its endpoints. Now consider the function $f : X \to X$ defined by $f(x) = x + 1/2$ for $x \in [0,1/2]$, $f(x) = x - 1/2$ for $x \in]1/2,1[$ and $f(1) = 0$. Then f is bijective, without fixed points. It remains to prove that f is continuous. Let $x \in [0,1]$ be an arbitrary element and let V be a neighbourhood of $f(x)$. By definition V must contain each element of X except a finite number of them. Let $V = [0,1] \setminus \{y_1, y_2, \ldots, y_n\}$

and $x_i := f^{-1}(y_i)$, $(1 \leq i \leq n)$. Then the set $U := [1,0] \setminus \{x_1, x_2, \ldots, x_n\}$ is a neighborhood of x for which $f(U) \subset V$. Thus f is continuous at x.

In section one we have seen that results concerning minimax inequalities (problem (B)) can be extended for pseudoconvex spaces. The proof is based on KKM methods which use Brouwer's fixed point theorem. As it could be seen, these theorem fails in interval spaces. Therefore, in these spaces, the results in section one can't be proved using the argument above. However, it would be interesting to attack problem (B) in case of interval spaces.

3. Minimax theorems

In this section we are concerned on problem (A). We start with the following definition.

DEFINITION 3.1. Let $(X, h, \mathcal{F})$ be a pseudoconvex space. A function $f : X \to \mathbb{R}$ is said to be *convex* if $f \circ \psi_F$ is convex (in the usual sense) for each finite $F \subset X$. f is said to be *concave* if $-f$ is convex. Recall that f is *quasiconcave* if $-f$ is quasiconvex. It is clear that if f is convex (concave) then it is also quasiconvex (quasiconcave).

THEOREM 3.1. *Let $(X, h_1, \mathcal{F}_1)$ be a pseudoconvex space, $(Y, h_2, \mathcal{F}_2)$ be a compact pseudoconvex space. Let $f : X \times Y \to \mathbb{R}$ be* u.s.c. *and concave in its first variable and* l.s.c. *and convex in its second variable. Then*

$$\min_{y \in Y} \sup_{x \in X} f(x,y) = \sup_{x \in X} \inf_{y \in Y} f(x,y).$$

We shall deduce Theorem 3.1 in three ways: the first one uses the results above concerning minimax inequalities (therefore, the proof is based on KKM methods and its generalizations); the second one uses the Hahn–Banach method (separation of convex sets), while the third one will be the method of "level sets" dicovered by I. JOÓ [11] which uses neither KKM nor Hahn–Banach's theorems.

For the beginning, we give a minimax theorem for two functions which follows by Theorem 1.3 and which clearly implies Theorem 3.1. This contains, in particular SIMONS [25], Theorem 1.4, and H. NIKAIDÓ [24].

THEOREM 3.2. *Let $(X, h_1, \mathcal{F}_1)$ and $(Y, h_2, \mathcal{F}_2)$ be two pseudoconvex spaces with Y compact, $f, g : X \times Y \to \mathbb{R}$ with $f \leq g$ on $X \times Y$ such that*

(22) f *is* l.s.c. *in its second variable and quasiconcave in its first variable;*

(23) g *is* u.s.c. *in its first variable and quasiconvex in its second variable. Then*

$$\min_{y\in Y}\sup_{x\in X} f(x,y) \le \sup_{x\in X}\inf_{y\in Y} g(x,y).$$

PROOF. Suppose first that X is compact. If the result were false, we could choose $r\in\mathbb{R}$ such that

(24) $$\min_{y}\sup_{x} f(x,y) > r > \sup_{x}\inf_{y} g(x,y).$$

Let $Z=X\times Y$. Then $(Z,h,\mathcal{F})$ is a compact pseudoconvex space, where h and $\mathcal{F}$ are defined as follows: if $F=\{(x_0,y_0),\dots,(x_n,y_n)\}$, $F_1=\{x_0,\dots,x_n\}$, $F_2= =\{y_0,\dots,y_n\}$ then $\psi_F:=\psi_{1,F_1}\times\psi_{2,F_2}:\Delta^n\to Z$ (i.e. $\psi_F(t)=(\psi_{1,F_1},\psi_{2,F_2}(t))$ if $t\in\Delta^n$) and $\mathcal{F}=\{\psi_F : F\subset Z \text{ is finite}\}$. The convex hull $h(F)$ of a finite set $F\subset Z$ is defined by $\psi_F(\Delta^n)(n+1=\operatorname{card}F)$ and the convex hull of any set $A\subset Z$ is determined by $h(A)=\cap\{h(F), F\subset A \text{ is a finite set}\}$.

Consider the function $\Phi: Z\times Z\to\mathbb{R}$ defined by

$$\Phi((x,y),(\hat{x},\hat{y}))=\min\{f(x,\hat{y})-r, r-g(\hat{x},y)\}.$$

It is easy to see that Φ verifies the conditions of Theorem 3.1 (with $f=g:= =\Phi$). We have $\Phi((x,y),(x,y))\le 0$ for each $(x,y)\in Z$. Thus, there exists $(\hat{x},\hat{y})\in Z$ such that $\Phi((x,y),(\hat{x},\hat{y}))\le 0$ for each $(x,y)\in Z$, or, in other words $f(x,\hat{y})\le r$ or $g(\hat{x},y)\ge r$ for each $(x,y)\in Z$. This contradicts to (24). Now, considering the general case, it can be seen that for each $x_1,\dots,x_n\in X$ $\bigcap_{i=1}^{n}\{y\in \in Y : f(x_i,y)\le\alpha\}\ne\emptyset$ where $\alpha:=\sup_x\inf_y g(x,y)$. Using the finite intersection property for compact sets, we have $\bigcap_{x\in X}\{y\in Y : f(x,y)\le\alpha\}\ne\emptyset$, as required.

In paper [19], G. KASSAY gave a simple proof for König's minimax theorem [22] based on geometrical properties on $\mathbb{R}^2$. König's theorem has extended by S. SIMONS [25] for two functions. We prove Simon's theorem using the method in [19].

THEOREM 3.3 (S. SIMONS, [25]). *Let X be a nonempty set, Y a nonempty compact topological space, $f,g : X\times Y\to\mathbb{R}$ with $f\le g$ on $X\times Y$ such that*

f is l.s.c. *and 1/2 convex in its second variable, i.e. for all y_1, $y_2\in Y$ there exists $y_3\in Y$ such that*

(25) $$f(x,y_3)\le\frac{f(x,y_1)+f(x,y_2)}{2}\quad \textit{for each}\quad x\in X;$$

g is 1/2 concave in its first variable, i.e. for all x_1, $x_2 \in X$ there exists $x_3 \in X$ such that

$$g(x_3,y) \geq \frac{g(x_1,y)+g(x_2,y)}{2} \quad \text{for each} \quad y \in Y. \tag{26}$$

Then

$$\min_{y\in Y}\sup_{x\in X} f(x,y) \leq \sup_{x\in X}\inf_{y\in Y} g(x,y). \tag{27}$$

Let $c_* := \sup_x \inf_y g(x,y)$. We have the following statement which has proved for one function by I. Joó [13].:

LEMMA 3.4. *(27) holds iff* $\bigcap_{x\in X} \{y \in Y : f(x,y) \leq c\} \neq \emptyset, \quad c > c_*.$

The proof is analogous to that of Theorem 2 [13]; we omit the details.

PROOF OF THEOREM 3.3. Since Y is compact and f is l.s.c. in its second variable, hence the sets $H_x^c := \{y : f(x,y) \leq c\}$ $(c > c_*)$ are compact. Thus, it is enough to prove that the family of sets $\{H_x^c : x \in X\}$ $(c > c_*)$ has the finite intersection property. We prove that any two sets of this family have nonempty intersection. Suppose the contrary, i.e. that there exists $c > c_*$ and x_1, $x_2 \in X$ such that $H_{x_1}^c \cap H_{x_2}^c = \emptyset$ and define $p : Y \to \mathbb{R}^2$ by $p(y) = (c - f(x_1,y), c - f(x_2,y))$. If $K = \{(s,t) \in \mathbb{R}^2 : s \geq 0,\ t \geq 0\}$ then $p(Y) \cap K = \emptyset$.We show that $\operatorname{co} p(Y) \cap \operatorname{int} K = 0$ ($\operatorname{co} A$ denotes the standard convex hull of $A \subset \mathbb{R}^2$). For this, suppose that there exist $\lambda_1,\dots,\lambda_k \in [0,1]$ with $\sum_{i=1}^k \lambda_i = 1$ and $y_0, y_2, \dots, y_k \in Y$ such that $\sum_{i=1}^k \lambda_i p(y_i) \in int K$. Using (25) we can choose $\overline{\lambda_1}, \overline{\lambda_2}, \dots, \overline{\lambda_k}$ with $\overline{\lambda_i} \geq 0$ $(1 \leq i \leq k)$ and $\sum_{i=1}^k \overline{\lambda_i} = 1$ such that $\sum_{i=1}^k \overline{\lambda_i} p(y_i) \in K$ and for which there exists $\overline{y} \in Y$ such that $p(\overline{y}) - \sum_{i=1}^k \overline{\lambda_i} p(y_i) \in K$. This means that $p(\overline{y}) \in K$, which contradicts the hypothesis. By the well-known separation theorem of Hahn–Banach in $\mathbb{R}^2$, there exists a line which separates the sets $\operatorname{co} p(Y)$ and K. That is, there exists $b = (b_1, b_2) \in K$ with $b_1 + b_2 = 1$ such that $\langle u, b \rangle \leq 0$ for all $u \in p(Y)$ or, in other words $b_1 f(x_1,y) + b_2 f(x_2,y) \geq c$ for every $y \in Y$. Let $c_1 \in \mathbb{R}$ such $c_* < c_1 < c$ and $d := c_1 - c$. Then we have $b_1[c_1 - f(x_1,y)] + b_2[c_1 - f(x_2,y)] \leq d$ for every $y \in Y$, hence the set $p_1(Y)$ is separated from K by the line $b_1 s + b_2 t = d$, where $p_1(y) = (c_1 - f(x_1,y), c_1 - f(x_2,y))$ $(d < 0)$. Since $f(x_1,\cdot)$ and $f(x_2,\cdot)$ are l.s.c. on Y, hence there exist $\alpha, \beta > 0$ such that $p_1(Y) \subset (-\infty,\alpha] \times (-\infty,\beta]$. The line $b_1 s + b_2 t = d$ intersects at least one of the lines $s = \alpha$ and $t = \beta$; suppose that it intersects the second one. It is clear then, that the line $b_1 \beta s + (-d + b_2 \beta) t = 0$ separates $p_1(Y)$ and K. Let $g_1(y) = (c_1 - g(x_1, y), c_1 - g(x_2, y))$. Since $p_1(y) - g_1(y) \in K$ for every $y \in Y$, each line which separates $p_1(Y)$ and K, separates $g_1(Y)$ and K too. Using (26), choose $\mu \in [0,1]$ and $x_\mu \in X$ such that $g(x_\mu, y) \geq \mu g(x_1, y) + (1-\mu) g(x_2, y)$ for every $y \in Y$ and such that $\mu s + (1-\mu) t = 0$ separates $g_1(Y)$ and K or, in other words $\mu [c_1 - g(x_1, y)] + (1-\mu)[c_1 - g(x_2, y)] \leq 0$ for every $y \in Y$. Therefore, $g(x_\mu, y) \geq c_1$ for every $y \in Y$ which leads to $\sup_x \inf_y g(x,y) \geq c_1 > c_*$. This is a contradiction. Hence any two sets of the family $\{H_x^c : x \in X\}$ have nonempty intersection. (In order to prove that for any $c > c_*$ and $x_1, \ldots, x_n \in X$ we have $\bigcap_{i=1}^{n} H_{c_i}^c \neq \emptyset$ we use induction. For the details, see [13]). This completes the proof.

Observe that Theorem 3.3 implies Theorem 3.1 (note that 1/2 convexity implies convexity in a pseudoconvex space):

Finally, Theorem 3.1 follows by a result of I. Joó ([14]. Theorem 3).

THEOREM 3.4 (I. Joó, [14]). *Let X be an interval space, Y a compact interval space and $f : X \times Y \to \mathbb{R}$ such that*

(28) *f is* u.s.c. *and quasiconvex in its first variable;*

(29) *f is* l.s.c. *and quasiconcave in its second variable.*

Then

$$\min_{y \in Y} \sup_{x \in X} f(x,y) = \sup_{x \in X} \min_{y \in Y} f(x,y).$$

4. Application to variational inequalities

In this section we apply the results from section 1 to obtain an existence theorem for variational inequalities. Our statement will be an extension of F. E. BROUWER [2].

Let X be a nonempty convex subset of a topological vector space E and let $T : X \to E^*$, where E^* denotes dual space of E. The problem is to find $y \in X$ such that

$$\langle Ty, y - x \rangle \leq 0 \quad \text{for every} \quad x \in X. \tag{30}$$

This problem has been studied by many mathematicians who have solved (30) under some hypothesis on X and T. (see for instance [2], [23], [25]). Now we will study a generalized form of (30). Let X be a convex subset of a topological vector space E, $(Y,h,\mathcal{F})$ a pseudoconvex space and E^* the dual space of E. Let $T: Y \to E^*$, $g: Y \to E$.

We study the following problem: find an element $y \in Y$ such that

(31) $$\langle Ty, g(y) - x) \rangle \leq 0 \quad \text{for every} \quad x \in X.$$

THEOREM 4.1. *Let X be a convex subset of a topological vector space E, $(Y,h,\mathcal{F})$ a pseudoconvex space, $T: Y \to E^*$, $g: Y \to E$ and $\varphi: X \to Y$.*

Suppose that

(32) *for each $x_1, x_2, \ldots, x_n \in X$, $h\{\varphi(x_1), \ldots, \varphi(x_n)\} \subset \varphi(\text{co}\{x_1, \ldots, x_n\})$.*

(33) *$\langle T(\varphi(x)), g(\varphi(x)) - x \rangle \leq 0$ for every $x \in X$;*

(34) *$T: Y \to E^*$ and $g: Y \to E$ are continuous;*

(35) *there is a compact subset D of Y and $x_0 \in X$ such that $\langle Ty, g(y) - x_0 \rangle >$ > 0 for every $y \in Y \setminus D$.*

Then there exists $y_0 \in D$ solution of (31).

PROOF. Let $f: X \times X \to \mathbb{R}$ defined by $f(x,y) = \langle Ty, g(y) - x \rangle$ ($x \in X$, $y \in Y$). It is easy to see that f satisfies the conditions of Theorem 1.3 (considering $f = g$). Since $\sup_{x \in X} f(x, \varphi(x)) \leq 0$ by (33), there exists $y_0 \in D$ such that $f(x, y_0) \leq 0$, as desired.

COROLLARY 4.1. *Suppose that* (32), (33), (34) *hold and Y is compact. Then there exists $y_0 \in Y$, solution of* (31).

COROLLARY 4.2 (F. E. BROUWER, [2]). *Let X be a compact convex subset of a topological vector space E and $T: X \to E^*$ a continuous map. Then there exists $y_0 \in X$ such that*

$$\langle Ty_0, y_0 - x \rangle \leq 0 \quad \textit{for every} \quad x \in X.$$

PROOF. Apply Theorem 4.1 for $X = Y$, $g(x) = \varphi(x) = x$ for every $x \in X$.

5. Upward-downward functions
A disprove for two conjectures of S. Simons

In [27] S. SIMONS has given the following definitions. Let X and Y be nonempty sets. A function $f: X \times Y \to \mathbb{R}$ is said to be *upward* on Y if $\forall \varepsilon > 0 \;\; \exists \;\; \delta > 0$ such that $\forall y_1, y_2 \in Y \;\; \exists y_3 \in Y$ such that:

(36) $$\begin{cases} \forall x \in X: f(x, y_3) \leq \max\{f(x, y_1), f(x, y_2)\} \quad \text{and} \\ f(x, y_3) > \max\{f(x, y_1), f(x, y_2)\} - \delta \Rightarrow |f(x, y_1) - f(x, y_2)| < \varepsilon. \end{cases}$$

One says that f is *downward* on X if $\forall\varepsilon > 0 \; \exists \; \delta > 0$ such that $\forall x_1, x_2 \in X$, $\exists x_3 \in X$ such that:

$$(36') \quad \begin{cases} \forall y \in Y : f(x_3,y) \geq \min f(x_1,y), f(x_2,y) \quad \text{and} \\ f(x_3,y) < \min\{f(x_1,y), f(x_2,y)\} + \delta \Rightarrow |f(x_1,y) - f(x_2,y)| < \varepsilon. \end{cases}$$

SIMONS has proved in [27] that if f is upward on Y, downward on X, Y is a compact topological space and f is l.s.c. in its second variable then $\min_y \sup_x f(x,y) = \sup_x \min_y f(x,y)$. The question whether this statement remains true in case of two functions remained open. Namely, [27] concludes with the following conjectures:

(37) Suppose that $f,g : X \times Y \to \mathbb{R}$, f is upward on Y, g is downward on X, $f \leq g$ on $X \times Y$, X_0 is a nonempty finite subset of X. Then $\inf_{y \in Y} \max_{x \in X_0} f(x,y) \leq \sup_{x \in X} \inf_{y \in Y} g(x,y)$.

(38) Suppose that $f,g : X \times Y \to \mathbb{R}$, f is upward on Y, g is downward on X, $f \leq g$ on $X \times Y$, Y is a compact topological space and f is l.s.c. in its second variable. Then

$$\min_{y \in Y} \sup_{x \in X} f(x,y) \leq \sup_{x \in X} \inf_{y \in Y} g(x,y).$$

We shall give negative answer for both conjectures of S. SIMONS [27].

Our counterexample shows that these conjectures fail even in a special case: X and Y are both compact, $f,g : X \times Y \to \mathbb{R}$ are continuous, f is upward on Y, g is downward on X and $f \leq g$ on $X \times Y$. However, $\min_{y \in Y} \sup_{x \in X} f(x,y) > \sup_{x \in X} \inf_{y \in Y} g(x,y)$.

To this end, we need the following Lemma:

LEMMA 5.1. *Let X and Y be compact subsets of $\mathbb{R}$ and $f : X \times Y \to \mathbb{R}$, $f(x,\cdot)$ is either strictly monotone, or a constant function. If Y is convex, then f is upward on Y.*

PROOF. It is enough to show that for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $x \in X$ and $y_1, y_2 \in Y$, $f\left(x, \frac{y_1+y_2}{2}\right) \leq \max\{f(x,y_1), f(x,y_2)\}$ and $|f(x,y_1) - f(x,y_2)| \geq \varepsilon \Rightarrow f\left(x, \frac{y_1+y_2}{2}\right) \leq \max\{f(x,y_1), f(x,y_2)\} - \delta$. The first relation is trivial. For the second on let's suppose that there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ there exist $x_n \in X$, $y_1^n, y_2^n \in Y$ with $|f(x_n,y_1^n) - f(x_n,y_2^n)| \geq \varepsilon$ and $f\left(x, \frac{y_1^n+y_2^n}{2}\right) > \max\{f(x_n,y_1^n), f(x_n,y_2^n)\} - \frac{1}{n}$ for all $n \in \mathbb{N}$. Using compactness, one can suppose (x_n), (y_1^n), (y_2^n) convergent (if it is necessary, we choose subsequences). Since f is uniformly continuous on $X \times Y$, there exists $\delta_0 > 0$ such that $| y_1^n - y_2^n | \geq \delta_0$ for all $n \in \mathbb{N}$. Let $x_n \to x$, $y_1^n \to y_1$, $y_2^n \to y_2$. Then we obtain the following contradiction:

$$f\left(x, \frac{y_1 + y_2}{2}\right) \geq \max\{f(x,y_1), f(x,y_2)\}.$$

A similar property assures that g is downward on x.

Now let $X = Y = [-1,1]$, $f, g : X \times Y \to \mathbb{R}$ given by

$$f(-1,-1) = g(-1,-1) = f(1,1) = g(1,1) = 2,$$
$$f(-1,1) = g(-1,1) = f(1,-1) = g(1,-1) = -2,$$
$$f(-1,0) = g(-1,0) = f(1,0) = g(1,0) = 1,$$
$$f(0,-1) = g(0,-1) = f(0,1) = g(0,1) = -1;$$

on the segments

$$[(-1,-1),(-1,0)], [(-1,0),(-1,1)], [(-1,1),(0,1)], \ldots, [(0,-1),(-1,-1)]$$

let $f = g$ be affine; $f(0,y) = -1$ for each $y \in [-1,1]$ and $g(x,0) = 1$ for each $x \in [-1,1]$ (see figure 1). At the interior points of the square $[-1,1] \times [-1,1]$ we construct f such that for each $y \in (-1,1)$, $f(x,y)$ is affine for $x \in (-1,0)$ and $f(x,y)$ is affine for $x \in (0,1)$ (see figure 2); g such that for each $x \in (-1,1)$, $g(x,y)$ is affine for $y \in (-1,0)$ and $g(x,y)$ is affine for $y \in (0,1)$ (see figure 3).

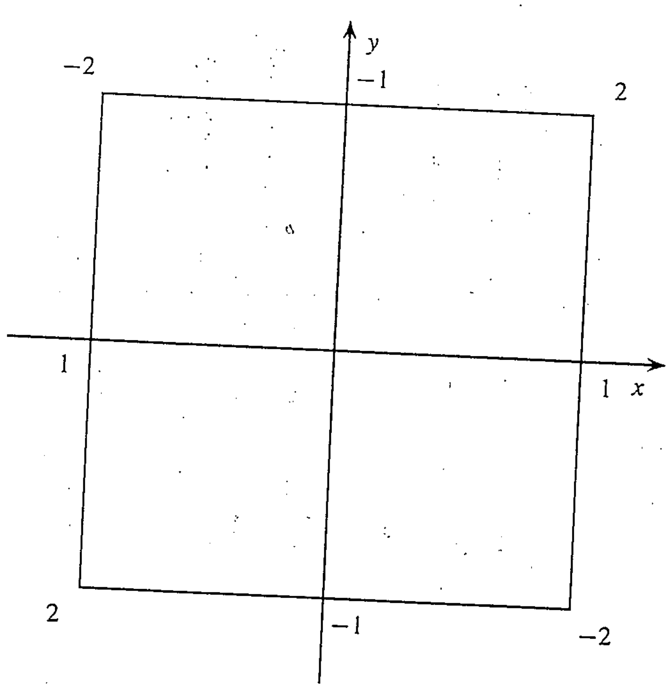

Fig. 1

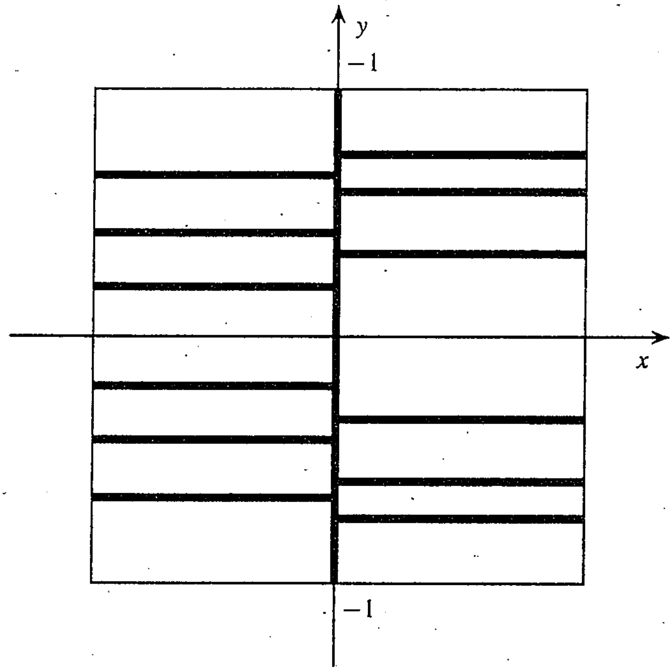

Fig. 2

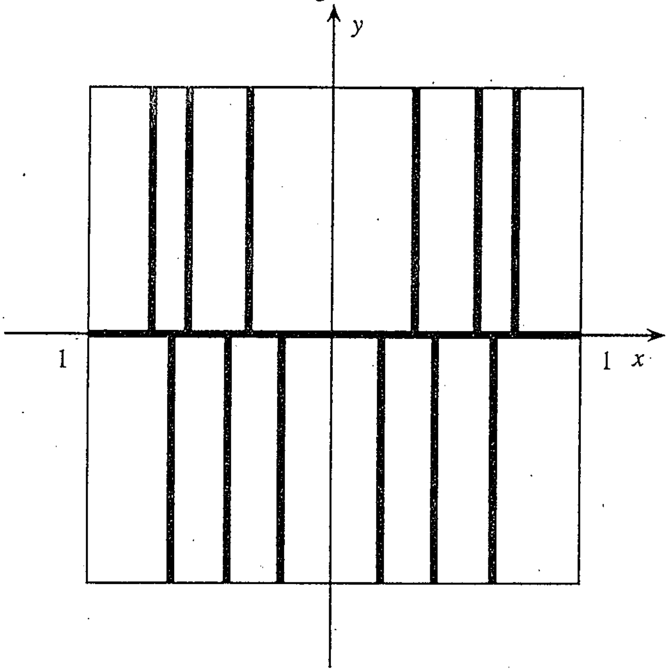

Fig. 3

Geometrically, both f and g are formed by two conoidal surfaces.

Using Lemma 5.1, f is upward on Y and g is downward on X. It is easy to verify that $f \leq g$ on $[-1,1] \times [-1,1]$. On the other hand, we have $\inf_y \sup_x f(x,y) = 1$ and $\sup_x \inf_y g(x,y) = -1$.

Therefore, the second conjecture of S. SIMONS fails (i.e. (38)). The first one (i.e. (37)) fails too: let $X_0 = \{-1,1\}$.

It remains an open question: under which additional hypothesis (37) and (38) can be proved?

We answer to this question in case when one of the sets X and Y is finite. To this end we prove the following lemmas.

LEMMA 5.2. *Suppose that $f : X \times Y \to \mathbb{R}$ is downward on X and X is a finite set. Then there exists $x^* \in X$ such that $f(x^*,y) \geq f(x,y)$ for each $x \in X$ and $y \in Y$.*

PROOF. Let $X = \{x_1, x_2, \ldots, x_n\}$. It is easy to see that (36) implies

$$\forall x_1, x_2 \in X : \exists x_3 \in X \quad \text{such that}$$

(39) $$\begin{cases} \forall y \in Y : f(x_3,y) \geq \min\{f(x_1,y), f(x_2,y)\} & \text{and} \\ f(x_1,y) \neq f(x_2,y) \Rightarrow f(x_3,y) > \min\{f(x_1,y), f(x_2,y)\}. \end{cases}$$

We use induction. If $n = 1$ the conclusion is trivial. Suppose that for $n = k$ the property holds and prove it for $n = k+1$. First we shall prove

(40) There exist

$i, j \in \{1,2,\ldots,k+1\}$ with $i \neq j$ such that $f(x_i,y) \geq f(x_j,y)$ for each $y \in Y$.

Suppose the contrary. Then the sets $A_1 = \{y \in Y : f(x_1,y) < f(x_2,y)\}$ and $A_2 = \{y \in Y : f(x_1,y) > f(x_2,y)\}$ are both nonempty. Choose x_3 as in (39) and let $A_3 = \{y \in Y : f(x_3,y) < f(x_1,y)\}$. If $A_3 = \emptyset$, then (40) holds. Suppose $A_3 \neq \emptyset$. It is easy to see by (39) that

$$f(x_3,y) > f(x_1,y) \quad \text{for every} \quad y \in A_1 \qquad \text{and}$$
$$f(x_3,y) > f(x_2,y) \quad \text{for every} \quad y \in A_2.$$

We also have $A_3 \subset A_2$. Let x_4 be the element which corresponds to x_1 and x_3 by (39). Then $f(x_4,y) > f(x_3,y)$ for every $x \in A_3$. Let $A_4 := = \{y \in Y : f(x_4,y) < f(x_1,y)\}$. Then $A_4 \subset A_3$. If $A_4 = \emptyset$, we have (40). Else, we continue this procedure. It is also clear, that the elements $x_1, x_2, x_3, \ldots$ are distinct (supposing that the corresponding sets $A_1, a_2, A_3, \ldots$ are nonempty). Hence, it is impossible to continue this procedure indefinitely since X is finite. Then, for an $i \in \{1,2,\ldots,k+1\}$ we must have $A_i = \emptyset$, which implies

(40). Choose $i,j \in \{1,2,\ldots,k+1\}$, $i \neq j$ such that $f(x_i,y) \geq f(x_j,y)$ for every $y \in Y$. Then (39) remains true for $\{x_1,x_2,\ldots,x_{k+1}\}\setminus\{x_j\}$, or in other words the functions $f(x_p,\cdot)$, $p \in \{1,2,\ldots,k+1\}\setminus\{j\}$ satisfy (39). Using induction, the conclusion of Lemma 5.2 follows.

LEMMA 5.3. *Suppose $f : X \times Y \to \mathbb{R}$ is upward on Y and Y is a finite set. Then there exists $y^* \in Y$ such that $f(x,y^*) \leq f(x,y)$ for each $x \in X$ and $y \in Y$.*

The proof is analogous to the proof of Lemma 5.2. We omit the details.

PROPOSITION 5.1. *Suppose that f is downward on X and X is finite. Then*

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) = \sup_{x\in X}\inf_{y\in Y} f(x,y).$$

PROOF. By Lemma 5.2 there exists $x^* \in X$ such that $f(x^*,y) \geq f(x,y)$ for every $x \in X$ and $y \in Y$. Then $\sup_{x\in X} f(x,y) = f(x^*,y)$ for every $y \in Y$, from which we have

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) = \inf_{y\in Y} (x^*,y) \leq \sup_{x\in X}\inf_{y\in Y} f(x,y).$$

PROPOSITION 5.2. *Suppose that f is upward on Y and Y is finite. Then*

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) = \sup_{x\in X}\inf_{y\in Y} f(x,y).$$

The proof is similar to the proof of Proposition 5.1 and uses Lemma 5.3.

COROLLARY 5.1. *Suppose that $f,g : X \times Y \to \mathbb{R}$ are such that $f \leq g$ on $X \times Y$.*

(41) If f is upward on Y and Y is finite, then

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \leq \sup_{x\in X}\inf_{y\in Y} g(x,y);$$

(42) if g is downward on X and X is finite, then

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \leq \sup_{x\in X}\inf_{y\in Y} g(x,y).$$

6. Saddle points and generalized Kuhn–Tucker theorems

In this section, using König's. minimax theorem [22], we extend the Kuhn–Tucker principle (see for instance [30]) for noconvex optimization problems with side conditions. Our results contain, in particular, those of [30]. For other extensions of Kuhn–Tucker type theorems which uses KÖNIG [22] see S. SIMONS [28].

First we discuss the case when the constrains are given in operator form.

Let X be a compact Hausdorff topological space, Y be reflexive Banach space, Y^* the dual space of Y and K a closed convex cone of Y such that its interior is nonempty. Let K^* be the dual cone of K, i.e. the set $\{y^* \in Y^* : y^*(y) \geq 0 \text{ for all } y \in K\}$. Consider the ordering relation $\ll$ on Y defined by $y_1 \ll y_2$ iff $y_2 - y_1 \in K$.

Let $f : X \to \mathbb{R}$, $G : X \to Y$ and $A := \{x \in X : G(x) \ll 0\}$. Consider the problem

$$\text{(P)} \qquad \begin{cases} f(x) \to \min \\ x \in A \end{cases}$$

Then we prove

THEOREM 6.1. *Suppose that the following conditions hold:*

(43) f *is* l.s.c. *on* X, G *is* l.s.c. *in the sense that the functionals* $x \to y^*(G(x))$ *are* l.s.c. *on* X *for all* $y^* \in K^*$;

(44) *for each* $x_1, x_2 \in X$, *there exists* $x_3 \in X$ *such that* $2f(x_3) \leq f(x_1) + f(x_2)$ *and* $2G(x_3) \ll G(x_1) + G(x_2)$;

(45) *there exists* $\overline{x} \in X$ *such that* $-G(\overline{x} \in \operatorname{int} K$ *(Slater condition).*

Then the following two assertions are equivalent:

(46) (P) *has a solution* x_0;

There exists $y_0^* \in K^*$ *such that*

$$(47) \qquad f(x_0) + y_0^*(G(x_0)) = \min_{x \in X}\{f(x) + y_0^*(G(x))\} \quad \textit{and} \quad y_0^*(G(x_0)) = 0$$

PROOF. Define the lagrange function $L : X \times K^* \to \mathbb{R}$ by $L(x, y^*) = f(x) + y^*(G(x))$. Let $K_n^* = \{y^* \in K^* : \|y^*\| \leq n\}$. Then we have:

For each x_1, $x_2 \in X$, there exists $x_3 \in X$ such that

$$(48) \qquad 2L(x_3, y^*) \leq L(x_1, y^*) + L(x_2, y^*) \quad \text{for each} \quad y^* \in K_n^*;$$

(49) the functions $L(x, \cdot)$ are affine and continuous on K_n^* for each $x \in X$; therefore they are weakly continuous:

The functions $L(\cdot,y^*)$ are l.s.c. on X for each $y^* \in K^*$. Since K_n^* is weakly compact, by König's theorem [22], there exists a saddle point (x_n,y_n^*) of $L(x,y^*)$ on $X \times K_n^*$, i.e. $L(x_n,y^*) \le L(x_n,y_n^*) \le L(x,y_n^*)$ for each $x \in X$ and $y^* \in K_n^*$. In particular

$$L(x_n,0) \le L(x_n,y_n^*) \le L(\overline{x},y_n^*). \tag{50}$$

It is easy to see that the sequence (y_n^*) is bounded. Otherwise, by (45) one can choose a neighborhood of $-G(\overline{x})$ which is contained in $int K$, that is, there exists $r>0$ such that

$$y^*(-G(\overline{x})-rh) \ge 0 \quad \text{for each} \quad y^* \in K^*$$

and $h \in Y$ with $\|h\| \le 1$. Therefore

$$r\|y^*\| = \sup_{\|h\|=1} y^*(rh) \le y^*(-G(\overline{x})) \quad \text{for each} \quad y^* \in K^*. \tag{51}$$

Now if $\|y_n^*\| \to \infty$, by (10), $-y_n^*(G(\overline{x})) \to \infty$. Since $L(\cdot,0)$ is l.s.c. on X, this contradicts (50). Without loss of generality, we may suppose $x_n \to z \in X$, $y_n^* \to y_0^* \in K^*$ (converges weakly) and $L(x_n,y_n^*) \to \alpha$ as $n \to \infty$. Therefore

$$L(z,y^*) \le \varliminf_{n\to\infty} L(x_n,y^*) \le \alpha \le \varliminf_{n\to\infty} L(x,y_n^*) = L(x,y_0^*)$$

for each $x \in X$ and $y^* \in K^*$. In particular, $L(x,y_0^*)=\alpha$, hence (x,y_0^*) is a saddle point of L on $X \times K^*$, i.e.

$$f(z)+y^*(G(z)) \le f(z)+y_0^*(G(z)) \le f(x)+y_0^*(G(x))$$

for each $x \in X$ and $y^* \in K^*$.

(52) In particular $z \in A$ and $y_0^*(G(z))=0$.

Now let x_0 be a solution of (P). Then $y_0^*(G(\overline{x})) \le 0$ (since $x_0 \in A$ and $f(x_0)=\inf_{x\in A}\sup_{y^*\in K^*} L(x,y^*)$.

Indeed, $L(x,y^*) \le f(x)$ for each $x \in A$ and $y^* \in K^*$, hence

$$\sup_{y^*\in K^*} L(x,y^*) \le f(x)$$

for each $x \in A$. Therefore

$$\inf_{x\in A}\sup_{y^*\in K^*} L(x,y^*) \le \sup_{y^*\in K^*} L(x_0,y^*) \le f(x_0).$$

It is easy to see that we also have $f(x_0) \ge \inf_{x\in X}\sup_{y^*\in K^*} L(x,y^*)$ using the fact that x_0 is a solution of (P). Thus $f(x_0)=\alpha$, hence $f(x_0)+y_0^*(G(x_0)) \le f(x_0) \le$ $\le f(x)+y_0^*(G(x))$ for each $x \in X$. On the other hand, if we put x_0 instead of

x and 0 instead of y^* in (52), we obtain $y_0^*(G(x_0)) = 0$. Hence (46) $\Rightarrow$ (47). (47) $\Rightarrow$ (46) is trivial. This completes the proof.

Now consider the case when the constraints are given in the form of inequalities. Let X be a compact Hausdorff topological space, $f_0, f_1, \ldots, f_m$ be real functions on X. Consider the problem:

$$(\mathrm{P}') \qquad \begin{cases} f_0(x) \to \min, & x \in X \\ f_1(x) \le 0, \ldots, f_m(x) \le 0 \end{cases}$$

COROLLARY 6.1. *Suppose*

(53) f_i *is* l.s.c. *on* X *for each* $i \in \{0, \ldots, m\}$*;*

(54) *for each* $x_1, x_2 \in X$*, there exists* $x_3 \in X$ *such that*

$$2f_i(x_3) \le f_i(x_1) + f_i(x_2), \quad \text{for each} \quad i \in \{0, \ldots, m\};$$

(55) *there exists* $\overline{x} \in X$ *such that* $f_i(\overline{x}) < 0$ *for each* $i \in \{1, 2, \ldots, m\}$.

Then the following two assertions are equivalent:

$$(56) \qquad x_0 \quad \text{is a solution of} \quad (P');$$

There exist (Lagrange multipliers) $\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*$ *such that*

$$(57) \qquad f_0(x_0) + \sum_{i=1}^{m} \lambda_i^* f_i(x_0) = \min_{x \in X} \{f_0(x) + \sum_{i=1}^{m} \lambda_i^* f_i(x)\} \quad \text{and}$$

$$\lambda_i^* f_0(x_0) = 0 \quad \text{for each} \quad i \in \{1, \ldots, m\}.$$

PROOF. Take $Y = \mathbb{R}^m$, $K = \{h = (h_1, \ldots, h_m) \in \mathbb{R}^m;\ h_i \ge 0,\ i \in \{1, \ldots, m\}\}$, $f = f_0$, $G = (f_1, \ldots, f_m) : X \to \mathbb{R}^m$ and apply Theorem 6.1.

Finally we give an account on the the connection of the results considered, in Figure 4.

References

[1] H. BRÉZIS, L. NIRENBERG and G. STAMPACCHIA, A remark on Ky Fan's minimax principle. *Boll. Un. Mat. Ital.*, **6** (1972), 293–300.

[2] F. E. BROUWER, The fixed point theory of multi-valued mappings in topological vector spaces, *Math. Ann.*, **177** (1968), 283–301.

[3] K. FAN, Fixed-point and minimax theorems in locally convex topological linear spaces, *Proc. Natl. Acad. Sci. ASA*, **38** (1952), 121–126.

[4] K. FAN, A generalization of Tychonoff's fixed point theorem, *Math. Ann.*, **142** (1961), 305–310.

[5] K. FAN, A minimax inequality and its applications, in "Inequalities" (O. Shisha ed.) Vol 3, 103–113. *Academic Press, New York,* 1972.

[6] K. FAN, Some properties of convex sets related to fixed point theorems, *Math. Ann.*, **266** (1984), 519–537.

[7] K. FAN, A survey of some results closely related to the Knaster–Kuratowski–Mazurkiewicz theorem, *in Game Theory and Applications*, 358–370, *Academic Press*, 1990.

[8] C. HORVÁTH, Quelques theoremes en theorie des Mini-Max, *C. R. Acad. Sci. Paris,* **310** (1990), 269–272.

[9] C. HORVÁTH, A connectivity approach to minimax inequalities, (to appear).

[10] C. HORVÁTH, and A. SÖVEGJÁRTÓ, On convex functions, *Ann. Univ. Sci. Budapest, Sect Math.*, **29** (1986), 193–198.

[11] I. JOÓ, A simple proof for van Neumann's minimax theorem, *Acta Sci. Math. Szeged,* **42** (1980), 91–94.

[12] I. JOÓ, and L. L. STACHÓ, A note on Ky Fan's minimax theorem, *Acta Math. Acad. Sci. Hung.*, **39** (1982), 401–407.

[13] I. JOÓ, Note on my paper "A simple proof for von Neumann minimax theorem", *Acta Math. Hung.*, **44** (1984), 363–365.

[14] I. JOÓ, On some convexities, *Acta Math. Hung.*, **54** (1989), 163–172.

[15] I. JOÓ, and G. KASSAY, Convexity, minimax theorems and their applications, Preprint, 1991.

[16] I. JOÓ, A general minimax theorem, *Publ. Math. Debrecen*, **41** (1992) (to appear).

[17] G. KASSAY, and I. KOLUMBÁN, On the Knaster–Kuratowski–Mazurkiewicz and Ky Fan's theorems, *Babes-Bolyai Univ. Cluj*, Preprint, **7** (1990), 87–100.

[18] G. KASSAY, On Brézis–Nirenberg–Stampacchia's minimax principle *Babes-Bolyai Univ. Cluj*, Preprint, **7** (1991).

[19] G. KASSAY, A simple proof for Konig's minimax theorem, *Acta Math. Hung.*, (1995) (to appear).

[20] H. KOMIYA, Convexity in a topological space, *Fund. Math.*, **111** (1981), 107–113.

[21] V. KOMORNIK, Minimax theorems for upper semicontinuous functions *Acta Math. Acad. Sci. Hung.*, **40** (1982), 159–163.

[22] H. KÖNIG, Uber das von Neumannsche Minimax-Theorem, *Arch. Math.,* **19** (1968), 482–487.

[23] J. L. LIONS, and G. STAMPACCHIA, Variational inequalities, *Comm. Pure and Appl. Math.*, **20** (1967), 493–519.

[24] H. NIKAIDÓ, On von Neumann's minimax theorem, *Pacific J. Math.*, **4** (1954), 65–72.

[25] S. SIMONS, Two-functions minimax theorems and variational inequalities for functions on compact and noncompact sets with some comments on fixed-point theorems, *in Proc. Sump. Pure Math.* (F. E. Brouwer ed.) Vol. 45, 377–392. Amer. Math. Soc,. Providence, Rhide Island, 1986.

[26] S. SIMONS, Abstract Kuhn–Tucker theorems, *J. Opt. Theory and Appl.*, **58** (1988), 148–152.

[27] S. SIMONS, An upward-downward minimax theorem, *Arch. Math.*, **55** 91990), 275–279.

[28] L. L. STACHÓ, Minimax theorems beyond topological vector spaces, *Acta Sci. Math. Szeged,* **42** (1980), 157–164.

[29] F. TERKELSEN, Some minimax theorems, *Math. Scand.*, **31** (1972), 405–413.

[30] E. ZEIDLER, *Nonlinear Functional Analysis and its Applications* Part III, Springer-Verlag, Berlin, 1985.

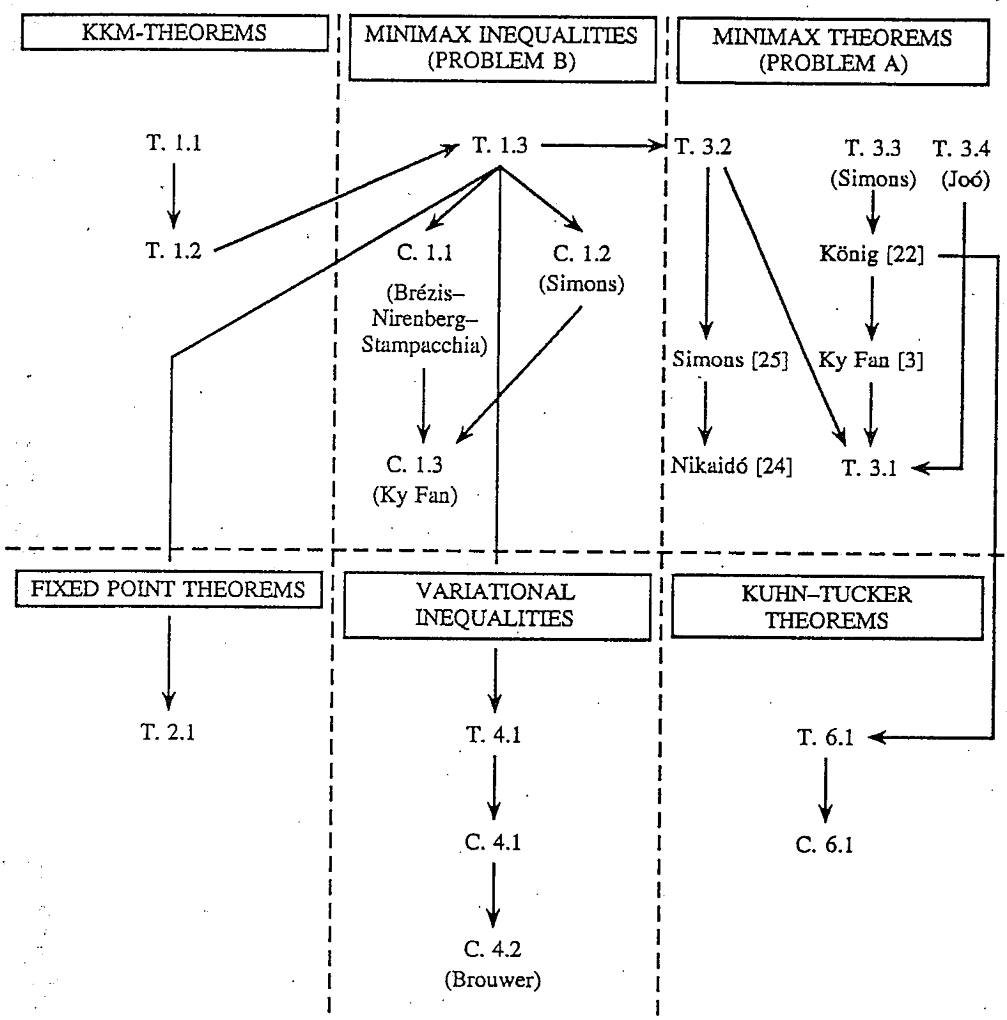

Fig. 4