For every $i$ and $j$ we have $M_j \supset M_i^{i+1}$. Indeed, if $p \in M_i^{i+1}$, then $(p \cdot m_j) \otimes \chi \in V[\text{for all } \chi \in V_{i \leq j}]$. By (27), this implies $(p \cdot m_j) \otimes \chi \in V[\text{for all } \chi \in V_{i \leq j}]$, that is, $p \in M_i^{i+1}$. Now it follows from Lemma 8, that for every $j$ there is an index $i(j)$ such that $M_j = M_i^{i(j)}$ for every $i \geq i(j)$. Let $i_j = \max\{i_0, i(1), \ldots, i(k)\}$. Then we have $M_j = M_i^{i_j}$ for every $i \geq i_j$.

By the definition of $\Sigma$ we can find polynomials $p_j \in M_j^j$ ($j = 1, \ldots, k$) such that (26) holds for every $x \in F$. If $i > i_j$ then $M_j = M_j^{j}$ for every $j$, and hence $(p_j \cdot m_j) \otimes \chi \in V[\text{for all } \chi \in V_{i \leq j}]$ for every $i > i_j$ and $j = 1, \ldots, k$.

It follows from (27) that for every $j = 1, \ldots, k$ there is a function $m_j : Q^+ \to C$ such that $m_j \otimes \chi = m_{i_j}$ for every $i \geq i_j$. It is clear that $m_j$ is an exponential on $Q^+$. By $(p_j \cdot m_j) \otimes \chi \in V[\text{for all } \chi \in V_{i \leq j}]$ we can find a function $g_j \in V$ such that $g_j \otimes \chi = (p_j \cdot m_j) \otimes \chi$. It is easy to see that the sequence $(g_j)_{i \geq i_j}$ converges pointwise to the function $(p \cdot m_j) \otimes \chi$, and thus $(p \cdot m_j) \otimes \chi \in V$.

Now $\psi = \sum (p \cdot m_j) \otimes \chi$ is an exponential polynomial belonging to $V$. Taking into consideration that $m_j$ is an extension of $m_{i_j}$ and that $F \subset G_i \times T \subset G_i \times T$, it follows from (26) that $(p \cdot m_j) \otimes \chi \in V[\text{for all } \chi \in V_{i \leq j}]$ for every $x \in F$. This completes the proof.

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Orthogonal pairs of weak*-closed inner ideals in a JBW*-triple

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Abstract

Pre-symmetric complex Banach spaces have been proposed as models for state spaces of physical systems. A neutral GL-projection on a pre-symmetric space represents an operation on the corresponding system, and has as its range a further pre-symmetric space which represents the state space of the resulting system. Two neutral GL-projections $S$ and $T$ on the pre-symmetric space $A_{\star}$ are said to be $L$-orthogonal if for all elements $x \in S_{A_{\star}}$ and $y \in T_{A_{\star}}$,

$$\|x \pm y\| \leq \|x\| + \|y\|.$$ 

By studying the algebraic properties of the dual space $A_{\star}$, which is a JBW*-triple, it is shown that, provided that the orthogonal neutral GL-projections $S$ and $T$ satisfy a certain geometrical condition, there exists a smallest neutral GL-projection $S \vee T$ majorizing both $S$ and $T$, and that $S$ and $T$ form a compatible family.

1. Introduction

This paper presents a further investigation into the structure of JBW*-triples, examples of which include JBW*-algebras, Hilbert spaces, spin triples and W*-algebras. A complex Banach space $A_{\star}$ is said to be pre-symmetric if the open unit ball in its dual space $A$ is a bounded symmetric domain. In this case the holomorphic structure of the open unit ball leads to the existence of a triple product on $A$ with respect to which it is a JBW*-triple. Since the predual of a JBW*-triple is unique, there is a bijection $A_{\star} \cong A$ from the set of pre-symmetric Banach spaces onto the set of JBW*-triples $[3, 4, 10, 11, 33, 35]$. The motivation for using the apparently redundant notion of pre-symmetry is the fact that many purely geometric properties of the pre-symmetric space $A_{\star}$ are equivalent to purely algebraic properties of the JBW*-triple $A$. Furthermore, in one approach to the theory of classical physical systems the state space of the system is represented by a pre-symmetric space $A_{\star}$, the geometric properties of which represent physical properties of the system in question [27–30]. Although some of the results presented in this paper are mainly algebraic and analytic in nature, and present new information about the structure of JBW*-triples, it is their geometric and physical analogues that might be thought to be of greatest interest.

Operations on the physical system the state space of which is represented by the pre-symmetric space $A_{\star}$ are represented by contractive linear projections on $A_{\star}$, or, equivalently, by weak*-continuous contractive linear projections on $A$. Groundbreaking work of Kaup [36] and Stachó [41] may be applied to show that the range of a contractive linear projection on the pre-symmetric complex Banach space $A_{\star}$ is itself pre-symmetric, a highly desirable
property for a model of a physical system. Properties of physical operations give rise to geometric properties of contractive projections. For example, a contractive linear projection \( S \) on \( A \), is said to be neutral if an element \( x \in A \), for which \( \|xS\| = \|x\| \) coincide necessarily lies in the range of \( S \), and is said to be a GL-projection if the L-orthogonal complement

\[
(SA_x)^* = \{x \in A : \|x \pm y\| = \|x\| + \|y\|, \forall y \in SA_x\}
\]

de the range \( SA_x \) of \( S \) is contained in the kernel of \( S \). Both of these geometrical properties may be interpreted physically. A linear projection \( R \) on the JBW*-triple \( A \) is said to be structural if, for all elements \( a, b, c \) in \( A \),

\[
R(aRbRc) = (RaRbRc).
\]

It was shown in [17, 19, 20] that structural projections are automatically contractive and weak*-continuous, and that the mapping \( R \rightarrow RA \) is a bijection from the complete lattice \( S(A) \) of structural projections on \( A \) onto the complete lattice \( \mathcal{Z}(A) \) of weak*-closed inner ideals in \( A \). More recently, in [15], it was shown that the mapping \( S \rightarrow S^* \) is a bijection between the set of neutral GL-projections on \( A \) and the complete lattice \( S(A) \), thereby linking the purely physical and geometric properties of the pre-symmetric space \( A \) with the purely algebraic properties of \( A \).

For each element \( J \) of \( \mathcal{Z}(A) \), the kernel \( \ker(J) \) of \( J \) is defined to be the set of elements \( a \in A \) for which the triple product \( \langle J a J, a \rangle \) is equal to zero, and the annihilator \( J^\perp \) of \( J \) is defined to be the set of elements \( a \in A \) for which \( \langle J a, a \rangle \) is equal to zero. For each element \( J \) in \( \mathcal{Z}(A) \), the annihilator \( J^\perp \) also lies in \( \mathcal{Z}(A) \), and \( A \) enjoys the generalized Peirce decomposition

\[
A = J_0 \oplus J_1 \oplus J_2,
\]

where,

\[
\begin{align*}
J_0 &= J^\perp, \\
J_1 &= \ker(J) \cap \ker(J^\perp), \\
J_2 &= J - J_1.
\end{align*}
\]

The structural projections onto \( J \) and \( J^\perp \) are denoted by \( P_J(J) \) and \( P_{J^\perp}(J) \), respectively, and the projection \( \text{id}_A = P_J(J) + P_{J^\perp}(J) \) onto \( J \) is denoted by \( P_J(J) \). Furthermore,

\[
\{a J_0 J_2 \} = \{0\}, \quad \{a J_2 J_0 \} = \{0\},
\]

and, for \( j, k \) and \( l \) equal to 0, 1 or 2, the Peirce arithmetic relations,

\[
\{J_l J_k J_l \} \subseteq J_{l+k},
\]

when \( j + l - k \) is equal to 0, 1 or 2 and

\[
\{J_l J_0 J_l \} = \{0\},
\]

otherwise, hold, except in the cases when \( (j, k, l) \) is equal to \((0, 1, 1), (1, 1, 0), (1, 0, 1), (2, 1, 1), (1, 1, 1) \) or \((0, 1, 1) \). For \( j \) equal to 0, 1 or 2, writing \( P(J) \) for the pre-adjoint of \( P_J(J) \) and \( J_J \) for its range, it is clear that \( A_a \) also enjoys a Peirce decomposition

\[
A_a = J_0 \oplus J_2 \oplus J_2,
\]

and that \( P_J(J) \) and \( P_{J^\perp}(J) \) are neutral GL-projections. In general, however, \( J_1 \) is not a JBW*-triple and \( P_J(J) \) and, hence, \( P_J(J) \), is not contractive. A remarkable result, proved in [21], shows that the Peirce-one projections \( P_J(J) \) and \( P_{J^\perp}(J) \), are contractive if and only if the Peirce arithmetic relations (1-4) and (1-5) hold in all cases. In this case \( J \) is said to be a Peirce inner ideal. In general, the physical operation corresponding to \( P_J(J) \), has a natural complementary operation corresponding to \( P_{J^\perp}(J) \), and, in some sense, the space \( J_J \) represents the information lost in performing the operations. In the case in which \( J \) is Peirce, \( P_J(J) \), not only is contractive but also is a GL-projection [15]. This could be interpreted as indicating that the information, apparently lost in the measurement process, can possibly be retrieved using the operation corresponding to \( P_J(J) \).

Two weak*-closed inner ideals \( J \) and \( K \) in the JBW*-triple \( A \) are said to be compatible when, for \( j \) and \( k \) equal to 0, 1 or 2, the Peirce projections \( P_j(J) \) and \( P_k(J) \) commute [16]. The corresponding physical operations may be thought to be simultaneously performable. A weak*-closed inner ideal \( J \) is compatible with all weak*-closed inner ideals in \( A \) if and only if it is an ideal in \( A \), or, equivalently, if and only if \( P_J(J) \) is an L-projection on \( A_a \), or, equivalently, if and only if the Peirce-one space \( J \) is zero [16]. The sets \( \mathcal{Z}(A) \) of weak*-closed ideals in \( A \) and \( S(A) \) of central elements of \( S(A) \), or \( M \)-projections, form order isomorphic Boolean sub-complete lattices of \( \mathcal{Z}(A) \) and \( S(A) \), respectively, and both are order isomorphic to the complete Boolean lattice of L-projections on \( A \) [1, 2, 5, 8, 9, 16]. It is clear that physical operations represented by L-projections on \( A_a \) may be considered to be classical.

It is now possible to describe the material that appears in this paper. Two weak*-closed inner ideals \( J \) and \( K \) are said to be orthogonal if \( J \) is contained in the annihilator \( K^\perp \) of \( K \). The relationship is clearly symmetric, and it is shown in [14] that orthogonality of \( J \) and \( K \) is equivalent to L-orthogonality of the of the pre-symmetric spaces \( J_a \) and \( K_a \). It is the purpose of this paper to investigate the compatibility of two orthogonal weak*-closed inner ideals \( J \) and \( K \) with each other, and with various weak*-closed inner ideals which contain them. Since it is not known if, in general, two weak*-closed inner ideals, one of which is contained in the other, are compatible, it is hardly surprising that some additional condition is needed in order to make any progress. What can be shown is that, provided that the larger of the two inner ideals is Peirce, then the inner ideals are compatible. Using this fact, in the first main result it is proved that, if one of the orthogonal pair \( J \) and \( K \) is Peirce then \( J \) and \( K \) are compatible. In order to make further progress it appears to be necessary to consider the situation in which both \( J \) and \( K \) are Peirce. In this case, it is easily shown that

\[
B = J \oplus K \oplus J_1 \cap K_1
\]

is a weak*-closed inner ideal in \( A \). However, it may not be Peirce, and a deep analysis of \( B \) is required in order to show that \( B \) is compatible with \( J \) and \( K \). Since \( B \) is a weak*-closed inner ideal containing \( J \) and \( K \) it is clear that the smallest weak*-closed inner ideal \( J \vee K \) containing \( J \) and \( K \) is also contained in \( B \). The final results of the paper describe \( J \vee K \) and show that \( J \) and \( K \) are compatible with \( J \vee K \).

The paper is organized as follows. In Section 2 definitions are given and notation is established. In Section 3 an analysis of the weak*-closed inner ideal \( B \) described above, is carried out, and in Section 4, the properties of the smallest weak*-closed inner ideal containing an orthogonal pair of weak*-closed inner ideals is investigated. The final section is devoted to a consideration of examples.

2. Preliminaries

A complex vector space \( A \) equipped with a triple product \( (a, b, c) \mapsto [a b c] \) from \( A \times A \) to \( A \), which is symmetric and linear in the first and third variables, conjugate linear in
the second variable and, for elements $a, b, c$ and $d$ in $A$, satisfies the identity
\[ D(a, b), D(c, d) = D((a \ b c), d) - D(c, (d \ a b)), \] (2.1)
where \([a, b]\) denotes the commutator, and $D$ is the mapping from $A \times A$ to the algebra of linear operators on $A$ defined by
\[ D(a, b) c = (a b c), \]
is said to be a Jordan-*triple*. A Jordan-*triple* $A$ for which the vanishing of \([a, a]\) implies that $a$ itself vanishes is said to be anisotropic. For each element $a$ in $A$, the conjugate linear mapping $\overline{Q}(a)$ from $A$ to itself is defined, for each element $b$ in $A$, by
\[ Q(a)b = (a b a). \]
For details about the properties of Jordan-*triples* the reader is referred to [37].

A Jordan-*triple* $A$ which is also a Banach space such that $D$ is continuous from $A \times A$ to the Banach algebra $B(A)$ of bounded linear operators on $A$, and, for each element $a$ in $A$, $D(a, a)$ is hermitian in the sense of [6, definition 5.1], with non-negative spectrum, and satisfies
\[ \|D(a, a)\| = \|a\|^2, \]
is said to be a JB-*triple*. A subspace $B$ of a JB-*triple* $A$ is said to be a subtriple if $B \subseteq B \subseteq B$ is contained in $B$. A subspace $B$ is clearly a subtriple if and only if, for each element $a$ in $B$, the element $[a, a]$ lies in $B$. Observe that every subtriple of a JB-*triple* is an anisotropic Jordan-*triple*. For each element $a$ in a JB-*triple* the smallest closed subtriple $A(a)$ containing $a$ is isometric to a commutative $C^*$-algebra, the Gelfand representation of $A(a)$ giving rise to a functional calculus. A subspace $J$ of a JB-*triple* $A$ is said to be an inner ideal if $[J, A]$ is contained in $J$ and is said to be an ideal if $[J, A]$ is contained in $J$. Every norm-closed subtriple of a JB-*triple* $A$ is a JB-*triple* [35], and a norm-closed subspace $J$ of $A$ is an ideal if and only if $[J, A]$ is contained in $J$. A JB-*triple* $A$ which is the dual of a Banach space $A_*$ is said to be a JBW-*triple*. In this case the predual $A_*$ of $A$ is unique and, for elements $a$ and $b$ in $A$, the operators $D(a, b)$ and $Q(a)$ are weak*-continuous. It follows that a weak*-closed subtriple $B$ of a JBW-*triple* $A$ is a JBW-*triple*. The second dual $A^{**}$ of a JBW-*triple* is a JBW-*triple*. For details of these results the reader is referred to [3, 4, 10, 11, 31, 34–36, 42, 43]. Examples of JB-*triples* are JB*-algebras and examples of JBW*-triples are JBW*-algebras, for the properties of which the reader is referred to [12, 32, 44, 45].

An element $u$ in a JBW-*triple* $A$ is said to be a tripotent if $u u u = u$. The set of tripotents in $A$ is denoted by $U(A)$. For each tripotent $u$ in $A$, the weak*-continuous linear operators $P_0(u), P_1(u)$ and $P_2(u)$, defined by
\[ P_0(u) = id_A - 2D(u, u) + Q(u)^2, P_1(u) = 2(D(u, u) - Q(u)^2), P_2(u) = Q(u)^2, \]
(2.2)
are mutually orthogonal projection operators on $A$ with unit $I_A$. For $j$ equal to 0, 1 or 2, the range of $P_j(u)$ is the weak*-closed eigenspace $A_j(u)$ of $D(u, u)$ corresponding to the eigenvalue $(1/2)^j$ and
\[ A = A_0(u) \oplus A_1(u) \oplus A_2(u) \] (2.3)
is the Peirce decomposition of $A$ relative to $u$. Moreover, $A_0(u)$ and $A_2(u)$ are inner ideals of $A$.

Orthogonal pairs of weak*-closed inner ideals
in $A$, $A_j(u)$ is a subtriple of $A$ and $A_j(u)$ is said to be the Peirce $j$-space corresponding to the tripotent $u$. Furthermore,
\[ \{ A_0(u), A_0(u) \} = \{ A, A_0(u) \} \] (2.4)
and, for $j, k$ and $l$ equal to 0, 1 or 2,
\[ \{ A_j(u), A_k(u), A_l(u) \} \subseteq A_{j+k+l}(u) \] (2.5)
when $j + l - k$ is equal to 0, 1 or 2, and
\[ \{ A_j(u), A_k(u), A_l(u) \} = \{ 0 \} \] (2.6)
otherwise.

A pair $a$ and $b$ of elements in a JBW-*triple* $A$ is said to be orthogonal when $D(a, b)$ is equal to zero. For a subset $L$ of $A$, denote by $L^R$ the subset of $A$ which consists of all elements in $A$ which are orthogonal to all elements in $L$. The subset $L^R$ is said to be the annihilator of $L$. Then $L^R$ is a weak*-closed inner ideal in $A$. Moreover, for subsets $L, M$ of $A$, $L^R \cap L \subseteq \{ 0 \}$, $L \subseteq L^{**}$, $L \subseteq M$ implies that $M^R \subseteq L^R$ and $L^R$ and $L^{**}$ coincide.

For each non-empty subset $B$ of the JBW-*triple* $A$, the kernel $\ker(B)$ of $B$ is the weak*-closed subspace of elements $a$ in $A$ for which $\{ B a \} = \{ 0 \}$. It follows that the annihilator $B^R$ of $B$ is contained in $\ker(B)$ and that $B \cap \ker(B)$ is contained in $\{ 0 \}$. A subtriple $B$ of $A$ is said to be complemented [20] if $B$ coincides with $B \oplus \ker(B)$. It can easily be seen that every complemented subtriple is a weak*-closed inner ideal. A linear projection $R$ on the JBW-*triple* $A$ is said to be a structural projection [38] if, for each element $a$ in $A$,
\[ R(a) R = Q(Ra). \]
(2.7)

The main results of [17], [19] and [20] show that the range $R$ of a structural projection $R$ is a complemented subtriple, that the kernel $\ker(R)$ of the map $R$ coincides with $\ker(RA)$, that every structural projection is contractive and weak*-continuous, and, most significantly, that every weak*-closed inner ideal is complemented.

Let $\tau(A)$ denote the complete lattice of weak*-closed inner ideals in the JBW-*triple* $A$ and let $\mathcal{S}(A)$ denote the set of structural projections on $A$. The results of [17] can be used to show that the set $\mathcal{S}(A)$ of structural projections on $A$ is a complete lattice with respect to the ordering defined, for elements $Q$ and $R$, by $Q \leq R$ if $QR = R$ is equal to $R$, and the mapping $R \mapsto RA$ is an order isomorphism from $\mathcal{S}(A)$ onto the complete lattice $\tau(A)$ of weak*-closed inner ideals in $A$.

For each element $J$ of $\tau(A)$, the annihilator $J^R$ also lies in $\tau(A)$ and $A$ enjoys the generalized Peirce decomposition described in (1.1)-(1.3). The Peirce relations given in (1.4) and (1.5) hold, except in the cases when $j, k, l$ is equal to $(0, 1, 1), (1, 0, 0), (0, 1, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)$ or $(1, 0, 1)$. When the relations hold in all cases then $J$ is said to be a Peirce inner ideal. When $J$ is the Peirce-two-space $A_0(u)$ corresponding to a tripotent $u$ the generalized Peirce decomposition reduces to that described in (2.2)-(2.6). A pair $J$ and $K$ of elements of $\tau(A)$ is said to be compatible if, for $j$ and $k$ equal to 0, 1 or 2,
\[ [P_j(J), P_j(K)] = 0. \]
(2.8)

Let $A$ be a complex Banach space. A linear projection $R$ on $A$ is said to be an $M$-projection if, for each element $a$ in $A$,
\[ [a] = \max\{ \| Ra \|, \| a - Ra \| \}. \]
A closed subspace which is the range of an M-projection is said to be an M-summand of A, and A is said to be equal to the M-sum

$$A = RA \oplus_M (id_A - R)A$$

of the M-summands RA and (id_A - R)A. For details the reader is referred to [1, 2, 8, 9]. The results of [3, 34] show that the set of M-summands of a JBW*-triple A coincides with its weak*-closed ideals.

A structural projection $P_J(I)$ on the JBW*-triple A which commutes with every structural projection on A is said to be central. It is shown in [16] that a weak*-closed inner ideal $I$ in A is an ideal if and only if one of the following equivalent conditions holds: $P_J(I)$ is central; $I$ is compatible with every weak*-closed inner ideal in A; the Peirce one-space $I_1$ is equal to $[0]$. The set $Z(A)$ of all weak*-closed ideals in A is a Boolean sub-complete lattice of $Z(A)$, such that the annihilator $J_I$ of each element $I$ in $Z(A)$ also lies in $Z(A)$. The central hull $c(L)$ of a subspace $L$ of A is defined by

$$c(L) = \{I \in Z(A) : L \subseteq I\},$$

the smallest weak*-closed ideal in A that contains L.

3. Orthogonal pairs of weak*-closed inner ideals

In this section some properties of orthogonal pairs of weak*-closed inner ideals in a JBW*-triple are investigated. Before attempting this, several preliminary results are required. The proof of the following result can be found in [21, theorem 4-8].

**Lemma 3.1.** Let A be a JBW*-triple and let K be a weak*-closed inner ideal in A, with corresponding Peirce projections $P_K(K)$, $P_K(K)$, and $P_K(K)$. Then, K is a Peirce inner ideal if and only if the linear mapping $\phi_K$ defined by

$$\phi_K = 2P_K + 2P_K - id_A = id_A - 2P_K$$

(3-1)

is an isometry from A onto itself.

Observe that by [35, proposition 5-5], the linear isometry $\phi_K$ appearing in Lemma 3.1 is a triple automorphism of A.

**Lemma 3.2.** Let A be a JBW*-triple, let J and K be weak*-closed inner ideals in A with K Peirce, let $P_J(J)$, $P_J(J)$ and $P_J(K)$, $P_K(K)$ and $P_K(K)$ be the Peirce projections corresponding to J and K respectively, and let $\phi_K$ be the triple automorphism of A given by

$$\phi_K = 2P_K + 2P_K - id_A.$$

Then, the following conditions are equivalent:

(i) $\phi_K(\leq J$;
(ii) $\phi_K = J$;
(iii) $J \leq \phi_K$;
(iv) $\phi_K = P_K(\phi_K)$.

**Proof.** Observe that $\phi_K$ coincides with $id_A$. If (i) holds then

$$J = \phi_K(\leq \phi_K(J),$$

and (ii) and (iii) hold. Similarly, if (iii) holds so also do (ii) and (i). Hence, (i), (ii) and (iii) are equivalent.

If (ii) holds it follows that

$$P_K(\phi_K = P_K(J),$$

(3-2)

Let a be an element of the kernel $\ker(P_K)$ of the structural projection $P_K$, which, by [20, lemma 4-4], coincides with the kernel $\ker(J)$ of J. Then, since $\phi_K$ is a triple homomorphism,

$$J = \phi_K(J) = \phi_K(J) = \phi_K(J) = \phi_K(J) = [0].$$

Therefore, $\phi_K(\ker(P_K))$ is contained in $\ker(P_K)$. Arguing as before,

$$\ker(P_K) = \ker(P_K) \subseteq \ker(P_K),$$

and it follows that $\phi_K(\ker(P_K))$ and $\ker(P_K)$ coincide. Hence,

$$id_A = P_K = P_K = P_K = P_K = P_K = P_K = P_K.$$

(3-3)

Combining (3-2) and (3-3), it can be seen that (iv) holds. Conversely, if (iv) holds it is clear that (i) holds. This completes the proof of the lemma.

**Lemma 3.3.** Let A be a JBW*-triple and let J and K be weak*-closed inner ideals in A, with K Peirce and J contained in K. Then, J and K form a compatible pair.

**Proof.** Since J is contained in K it is clear that

$$P_K(J) = P_K(J).$$

(3-4)

Moreover, the annihilator $K^\perp$ of K is contained in the annihilator $J^\perp$ of J, and therefore

$$P_K(P_K) = P_K(K).$$

(3-5)

Furthermore, by [17, theorem 5-3],

$$J \subseteq K, \quad (K^\perp) \subseteq (J^\perp),$$

and it follows that

$$P_K(J) = P_K(J), \quad P_K(P_K) = P_K(K).$$

Taking adjoints,

$$P_K(K) = P_K(K), \quad P_K(P_K) = P_K(K).$$

(3-6)

From (3-4)–(3-6),

$$[P_K(J), P_K(K)] = [P_K(J), P_K(K)] = 0.$$ 

(3-7)

Moreover, by (3-6),


(3-8)

from which it follows that

$$[P_K(J), P_K(K)] = 0.$$ 

(3-9)
Observe that, by (3-6) and (3-8),
\[
\phi_K(J) = (2P_0(K) + 2P_2(K) - \text{id}_A)J = (2P_0(K) + 2P_2(K) - \text{id}_A)P_2(J)A \subseteq J.
\] (3-10)
Let \( a \) be an element of \( J^- \). Then, using (3-10) and the fact that \( \phi_K \) is a triple homomorphism,
\[
[\phi_K(a)J] \subseteq [\phi_K(a) \phi_K(J) \phi_K(A)] = [\phi_K(\langle a J A \rangle)] = [0],
\]
and it follows that \( \phi_K(J^-) \) is contained in \( J^- \). Using Lemma 3-2, it can be seen that
\[
[P_0(J), \phi_K] = 0.
\] (3-11)
By (3-7) and (3-11),
\[
[P_0(J), P_2(K)] = 0.
\] (3-12)
Since
\[
P_0(J) + P_1(J) + P_2(J) = P_0(K) + P_1(K) + P_2(K) = \text{id}_A,
\]
equations (3-7), (3-9), and (3-12) are sufficient to show that, for \( j \) and \( k \) equal to 0, 1 and 2,
\[
[P_j(J), P_k(K)] = 0,
\]
as required.

It is now possible to prove the first important result.

**Theorem 3-4.** Let \( J \) and \( K \) be orthogonal weak*-closed inner ideals in a JBW*-triple \( A \) one of which is a Peirce inner ideal. Then, \( J \) and \( K \) form a compatible pair.

**Proof.** Without loss of generality suppose that \( K \) is a Peirce inner ideal. Then, by [13, theorem 4-2], \( K^- \) and \( K^{\bot \bot} \) are also Peirce inner ideals. Since \( J \) is contained in \( K^- \), by Lemma 3-3, \( J \) and \( K^- \) form a compatible pair, and, hence, by [13, theorem 4-4], \( J \) and \( K^{\bot \bot} \) form a compatible pair. Using [13, lemma 4-1(ii)], for \( j \) equal to 0, 1 and 2,
\[
[P_j(J), P_0(K)] = [P_j(J), P_0(K^{\bot \bot})] = 0.
\] (3-13)
Moreover, by [25, lemma 3-12],
\[
P_2(J)P_2(K) = P_2(K)P_2(J) = 0.
\] (3-14)
Since \( J \) is contained in \( K^- \),
\[
P_0(J)P_0(K) = P_0(J)
\] (3-15)
and, using [17, theorem 5-3],
\[
J_\infty \subseteq K_\infty^-.
\]
Therefore,
\[
P_2(J)P_0(K) = P_0(J).
\] (3-16)
and, taking adjoints,
\[
P_0(K)P_0(J) = P_0(J).
\]
It follows from (3-15) and (3-16) that
\[
[P_0(J), P_0(K)] = 0.
\] (3-17)

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Since
\[
P_0(J) + P_1(J) + P_2(J) = P_0(K) + P_1(K) + P_2(K) = \text{id}_A,
\]
equations (3-13), (3-14) and (3-17) are sufficient to show that, for \( j \) and \( k \) equal to 0, 1 and 2,
\[
[P_j(J), P_k(K)] = 0,
\]
as required.

This theorem has the following corollary, that is an immediate consequence of [13, corollary 4-5].

**Corollary 3-5.** Let \( J \) and \( K \) be orthogonal Peirce weak*-closed inner ideals in a JBW*-triple \( A \). Then,
\[
\{J, K, J^\perp, K^\perp, J^{\perp \perp}, K^{\perp \perp}, J^{\perp \perp} \cap J_1, K^{\perp \perp} \cap K_1\}
\]
forms a family of pairwise compatible Peirce weak*-closed inner ideals in \( A \).

The next result displays the existence of a weak*-closed inner ideal containing a pair of orthogonal Peirce weak*-closed inner ideals that is, in general, equal to \( A \).

**Theorem 3-6.** Let \( J \) and \( K \) be orthogonal Peirce weak*-closed inner ideals in the JBW*-triple \( A \), with corresponding Peirce spaces \( J_0, J_1, J_2, K_0, K_1, K_2 \). Then, the subspace \( B \) of \( A \) defined by
\[
B = J_2 \oplus K_1 \oplus J_1 \cap K_1,
\]
is a weak*-closed inner ideal in \( A \) with kernel \( \text{Ker}(B) \) given by
\[
\text{Ker}(B) = J_0 \cap K_0 \oplus J_1 \oplus J_2 \cap K_1.
\]

**Proof.** Observe that, since \( J \) and \( K \) are inner ideals in \( A \)
\[
\{J_2 A, J_2 \} \subseteq J_2, \quad \{K_2 A, K_2 \} \subseteq K_2.
\] (3-18)
By Theorem 3-4, \( J \) and \( K \) form a compatible pair, and, therefore, by [16, theorem 3-4], the intersection table of \( A \) relative to the pair \( J \) and \( K \) is given by:

<table>
<thead>
<tr>
<th>( \cap )</th>
<th>( J_2 )</th>
<th>( J_1 )</th>
<th>( J_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_2 )</td>
<td>(0)</td>
<td>K_2</td>
<td></td>
</tr>
<tr>
<td>( K_1 )</td>
<td>(0)</td>
<td>J_1 \cap K_1</td>
<td>J_0 \cap K_1</td>
</tr>
<tr>
<td>( K_0 )</td>
<td>J_2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( J_1 \cap K_0 )</td>
<td></td>
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</tr>
<tr>
<td>( J_0 \cap K_0 )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Since \( J_2 \) is contained in \( K_1 \) and \( K_2 \) is contained in \( J_0 \), using (1-3)-(1-5) and the intersection table above,
\[
\{J_2 A, K_2 \} = \{J_2, J_2 \} \oplus J_1 \oplus J_2 \cap K_2
\]
\[
= \{J_0, J_2 \} + \{J_1, J_2 \} + \{J_2, J_2 \}
\]
\[
\subseteq \{0\} + \{J_1 \} J_0 \cap K_1 \cap J_0 \cap K_2 + \{J_2, J_2 \}
\]
\[
\subseteq \{0\} + J_1 \cap (K_1 \cap \{0\}) + \{0\} = J_1 \cap K_1.
\] (3-19)
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which implies that \( B \) is a complemented inner ideal in \( A \). Therefore, by [17, lemma 3-2], \( B \) is weak*-closed and, by (3-33),

\[
\ker(B) = J_0 \cap K_1 \oplus J_1 \cap K_0 \oplus J_0 \cap K_0,
\]
as required.

The theorem above has two important corollaries.

**Corollary 3-7.** Under the conditions of Theorem 3-6,

\[
\{J_1^+, K_1^-, B^+, J_2^+, K_2^-, K_3^+\}
\]
forms a family of pairwise compatible weak *-closed inner ideals in \( A \).

**Proof.** Recall that, by [13, theorem 4-2], \( J_1^+, K_1^-, J_2^+, K_2^-, K_3^+ \) are Peirce weak*-closed inner ideals. Since \( J \) is contained in \( B \), it follows that \( B^\perp \) is contained in the Peirce weak*-closed inner ideal \( J^\perp \). Therefore, by Lemma 3-3, \( B^\perp \), \( J^\perp \) is a compatible pair. Using [13, theorem 4-4], it can be seen that \( (B^\perp, J^\perp) \) forms a compatible pair. The same applies when \( J \) is replaced by \( K \), and the result follows.

**Corollary 3-8.** Under the conditions of Theorem 3-6, the structural projection with range \( B \) is given by

\[
P_2(B) = P_2(J) + P_2(K) + P_2(J)P_2(K).
\]

**Proof.** Again using the compatibility of the pair \( J \) and \( K \) and the intersection diagram given above, it can be seen that the mapping \( R \) is given by

\[
R = P_2(J) + P_2(K) + P_2(J)P_2(K).
\]

is a projection onto \( B \) with kernel equal to the kernel \( \ker(B) \) of \( B \). The result follows from [17, theorem 3-4].

In order to study the compatibility of the weak*-closed inner ideals \( J, K \), and \( B \), a detailed analysis of the structure of \( B \) is required. As a first step, the Peirce decomposition of \( A \) relative to the weak*-closed inner ideal \( J_0 \cap K_0 \) is considered.

**Lemma 3-9.** Let \( J \) and \( K \) be orthogonal Peirce weak*-closed inner ideals in the JBW* tritope \( A \), with corresponding Peirce spaces \( J_0, J_1, J_2, \) and \( K_1, K_2, K_3 \). Then, the kernel \( \ker(J_0 \cap K_0) \) of the weak*-closed inner ideal \( J_0 \cap K_0 \) in \( A \) is given by

\[
\ker(J_0 \cap K_0) = J_2 \oplus K_3 \oplus J_1 \oplus K_1 \oplus J_0 \cap K_0 \oplus J_0 \cap K_1.
\]

**Proof.** Observe that, by (1-3)-(1-5), since both \( J \) and \( K \) are Peirce inner ideals,

\[
J_0 \cap K_0 = J_2 \oplus K_3 \oplus J_1 \cap K_1 \oplus J_1 \cap K_0 \oplus J_0 \cap K_0 \oplus J_0 \cap K_1 \cap K_1
\]

and, using the intersection table of \( A \) corresponding to the compatible pair \( J \) and \( K \) given above, it follows that

\[
A = (J_0 \cap K_0) \oplus (J_2 \oplus K_3 \oplus J_1 \cap K_1 \oplus J_1 \cap K_0 \oplus J_0 \cap K_0 \oplus J_0 \cap K_1)
\]

\[
\subseteq (J_0 \cap K_0) \oplus \ker(J_0 \cap K_0) = A,
\]

from which it follows that

\[
\ker(J_0 \cap K_0) = J_2 \oplus K_3 \oplus J_1 \cap K_1 \oplus J_1 \cap K_0 \oplus J_0 \cap K_0 \oplus J_0 \cap K_1.
\]

as required.
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from which the result follows.

In order to study the Peirce decomposition of $A$ corresponding to $B$ it is necessary to prove the following fairly general lemma.

**Lemma 3.12.** Let $A$ be a JBW$^*$-triple, let $M$ be a weak$^*$-closed inner ideal in $A$ and let $I$ be a weak$^*$-closed ideal in $A$. Then, the annihilator $M^*$ and kernel $\text{Ker}(M)$ of $M$ have the following properties:

(i) $M \cap I = (M \cap I)^{\perp} \cap I$,

(ii) $\text{Ker}(M) \cap I = \text{Ker}(M \cap I) \cap I$.

**Proof.** (i) Since $M$ and $I$ are compatible,

$$M = (M \cap I) \oplus (M \cap I^\perp)$$

and, hence,

$$M^* = (M \cap I)^{\perp} \cap (M \cap I^\perp)^{\perp} \cap I.$$

Therefore,

$$M^* \cap I = (M \cap I)^{\perp} \cap (M \cap I^\perp)^{\perp} \cap I.$$

However, since $M \cap I^\perp$ is contained in $I^\perp$,

$$I = I^\perp \subseteq (M \cap I^\perp)^{\perp}$$

and it follows from (3.35) that

$$M^* \cap I = (M \cap I)^{\perp} \cap I,$$

as required.

(ii) Since $M \cap I$ is contained in $M$, it follows that $\text{Ker}(M)$ is contained in $\text{Ker}(M \cap I)$ and, hence, that

$$\text{Ker}(M) \cap I \subseteq \text{Ker}(M \cap I) \cap I.$$  

(3.36)

Suppose that $a$ lies in $\text{Ker}(M \cap I) \cap I$. Then, since $M$ is complemented, there exist elements $b$ in $M$ and $c$ in $\text{Ker}(M)$ such that

$$a = b + c.$$

Then,

$$\{b \ a \ b\} = \{b \ b \ b\} + \{c \ b \ b\} = \{b \ b \ b\},$$

from which it follows that $\{b \ b \ b\}$ lies in $I$. Using the functional calculus it follows that $b$ lies in $I$ and, by linearity, that $c$ also lies in $I$. It follows that $c$ lies in $\text{Ker}(M \cap I)$ which, by (3.36), is contained in $\text{Ker}(M \cap I) \cap I$. Again using linearity, $b$ is contained in $\text{Ker}(M \cap I) \cap (M \cap I)$ and is, therefore, equal to zero. Hence, $a$ is contained in $\text{Ker}(M \cap I)$ and the proof is complete.

It is now possible to prove the main result of this section.

**Theorem 3.13.** Let $J$ and $K$ be orthogonal Peirce weak$^*$-closed inner ideals in the JBW$^*$-triple $A$, with corresponding Peirce spaces $J_0$, $J_1$, and $J_2$, and $K_0$, $K_1$, and $K_2$, and let $B$ be the weak$^*$-closed inner ideal in $A$ given by

$$B = J_2 \oplus K_2 \oplus J_1 \cap K_1.$$
Then,
\[ (J, K, B, J^\perp, K^\perp, J^\perp \cap J_1, K^\perp \cap K_1) \]
is a family of pairwise compatible weak*-closed inner ideals in \(A\).

**Proof.** Let \(I\) be the weak*-closed ideal in \(A\) defined in Lemma 3.11. Then, since \(I\) is compatible with every weak*-closed inner ideal, using Theorem 3.6 and Lemmas 3.9, 3.11 and 3.12,
\[ (B \cap I)^2 \cap I = B \cap I = (J_1 \cap I) \oplus (K_1 \cap I) \oplus ((J \cap K) \cap I), \quad (3.37) \]
\[ (B \cap I)^2 \cap I = B \cap I = (J_1 \cap K_1) \cap I, \quad (3.38) \]
\[ (B \cap I)^2 \cap I = \ker(B \cap I) \oplus \ker(B^* \cap I) \cap I \]
\[ = \ker(B) \cap \ker(J_1 \cap K_1) \cap I \]
\[ = (J_1 \cap K_1 \oplus J_2 \cap K_2 \oplus J_3 \cap K_3 \oplus J_4 \cap K_4) \]
\[ \cap (J_2 \cap K_2 \oplus J_3 \cap K_3 \oplus J_4 \cap K_4 \cap I) \]
\[ = (J_1 \cap K_1 \oplus J_2 \cap K_2 \cap I) \cap I \]
\[ = ((J_1 \cap K_1) \cap I) \oplus ((J_0 \cap K_0) \cap I). \quad (3.39) \]

Since there is a unique structural projection onto a weak*-closed inner ideal, it follows from Corollary 3.8 that
\[ P_2(B \cap I)P_2(I) = (P_2(J) + P_2(K) + P_2(J)P_2(K))P_2(I), \quad (3.40) \]
\[ P_0(B \cap I)P_0(I) = P_0(J)P_0(K)P_0(I) \quad (3.41) \]
and, hence, that
\[ P_1(B \cap I)P_1(I) = (P_1(J)P_2(K) + P_2(J)P_1(K))P_2(I). \quad (3.42) \]

Applying similar arguments with \(I^\perp\) replacing \(I\), it can be seen that
\[ (B \cap I^\perp)^2 \cap I^\perp = (J_1 \cap I^\perp) \oplus (K_1 \cap I^\perp) \oplus ((J \cap K^\perp) \cap I^\perp), \]
\[ (B \cap I^\perp)^2 \cap I^\perp = B \cap I^\perp = [0], \]
\[ (B \cap I^\perp)^2 \cap I^\perp = \ker(B \cap I^\perp) \cap I^\perp \]
\[ = (J_1 \cap K_1 \oplus J_2 \cap K_2 \oplus J_3 \cap K_3 \cap I^\perp \]
\[ = ((J_1 \cap K_1) \cap I^\perp) \oplus ((J_0 \cap K_0) \cap I^\perp). \]

It follows from Corollary 3.8 that
\[ P_2(B \cap I^\perp)P_2(I) = (P_2(J) + P_2(K) + P_1(J)P_2(K))P_2(I), \]
\[ P_0(B \cap I^\perp)P_0(I) = 0, \quad (3.43) \]
\[ P_1(B \cap I^\perp)P_1(I) = (P_1(J)P_0(K) + P_0(J)P_1(K) + P_0(J)P_0(K))P_0(I). \]

Using (3.40)–(3.43), it follows that
\[ P_2(B) = P_2(J) + P_2(K) + P_1(J)P_1(K), \]
\[ P_0(B) = P_0(J)P_0(K)P_1(I), \]
\[ P_1(B) = P_1(J)P_0(K) + P_0(J)P_1(K) + P_0(J)P_0(K)P_0(I). \]

Hence, for \(j\) and \(k\) equal 0, 1 or 2, \(P_j(B)\) commutes with \(P_j(K)\) and \(P_j(K)\), and \(B\) is compatible with both \(J\) and \(K\). That \(B\) is compatible with \(J^\perp, K^\perp, J^\perp \cap J_1, K^\perp \cap K_1\) follows from [13, theorem 4.4].

**4. The supremum of an orthogonal pair of weak*-closed inner ideals**

In the previous section the properties of a weak*-closed inner ideal \(B\) containing two orthogonal Peirce weak*-closed inner ideals \(J\) and \(K\) were investigated. Before going on to discuss the smallest weak*-closed inner ideal containing \(J\) and \(K\) one further property of \(B\) is required.

**Lemma 4.1.** Let \(J\) and \(K\) be orthogonal Peirce weak*-closed inner ideals in the JBW*-triple \(A\), with corresponding Peirce spaces \(J_0\), \(J_1\) and \(J_2\), and \(K_0\), \(K_1\) and \(K_2\), and let \(B\) be the weak*-closed inner ideal in \(A\) given by
\[ B = J_2 \oplus K_2 \oplus J_1 \cap K_1. \]

Then, the Peirce decomposition of \(B\) associated with \(I\) is given by
\[ B = (J)_{2,2} \oplus (J)_{2,1} \oplus (J)_{2,0} = J_2 \oplus J_1 \cap K_1 \oplus K_2 \]
and \(J\) and \(K\) form a compatible pair of Peirce weak*-closed inner ideals in \(B\) such that
\[ J^\perp \cap B = K, \quad K^\perp \cap B = J. \]

**Proof.** By Theorems 3.6 and 3.13, using the compatibility of \(J\), \(K\) and \(B\),
\[ J^\perp \cap B = K, \quad K^\perp \cap B = J \]
and
\[ \ker_B(J) = \ker(J) \cap B = K_2 \oplus J_1 \cap K_1. \]
\[ \ker_B(J) \cap B = \ker_B(K) = J_2 \oplus J_1 \cap K_1. \]

Hence, again using Theorem 3.6,
\[ (J)_{2,1} = \ker_B(J) \cap \ker_B(J^\perp \cap B) = J_1 \cap K_1 \]
and the relative Peirce decomposition of \(B\) associated with \(J\) is as stated above. Furthermore, using Corollary 3.8, since \(P_j(B)\) is a projection on \(B\) with rank equal to \(J\) and \(K\) and \(K\), it follows from [17, theorem 3.4], that the Peirce projection \(P_{B,j}(J)\) on \(B\) corresponding to \(J\) is given by \(P_j(B)(J)\). By symmetry the same applies to \(K\) and it follows that
\[ P_{B,2}(J) = P_2(J), \quad P_{B,1}(J) = P_1(J)P_2(K), \quad P_{B,2}(J) = P_2(K). \]

In particular \(P_{B,1}(J)\) is contractive and, by [21, theorem 4.8], \(J\) is a Peirce weak*-closed inner ideal in \(B\). By symmetry the same applies to \(K\), and by the compatibility of \(J\) and \(K\) in \(A\) it can be seen that, for \(j\) and \(k\) equal to 0, 1 or 2, the relative Peirce projections \(P_{B,j}(J)\) and \(P_{B,k}(K)\) commute and \(J\) and \(K\) form a compatible pair in \(B\).

The key result allowing the supremum of the orthogonal Peirce weak*-closed inner ideals \(J\) and \(K\) to be defined is the following lemma.
Lemma 4.2. Let \( J \) be a Peirce weak\(^*\)-closed inner ideal in a JBW\(^*\)-triple \( A \) and let \( J_0 \), \( J_1 \) and \( J_2 \) be the Peirce spaces corresponding to \( J \). Then, the smallest weak\(^*\)-closed inner ideal \( J_2 \vee J_0 \) containing \( J_1 \) and \( J_0 \) is given by

\[
J_2 \vee J_0 = J_2 \oplus J_0 \oplus \text{lin}(J_0, J_1, J_2)^{\omega*},
\]

where \( \text{lin}(J_0, J_1, J_2)^{\omega*} \) is the weak\(^*\)-closure of the linear span of the set

\[
\{ J_0, J_1, J_2 \} = \{ (a_0, a_1, a_2) : a_j \in J_j, j = 0, 1, 2 \}.
\]

Proof. Observe that, by (1-4), \( J_0, J_1, J_2 \) is contained in \( J_1 \) and, therefore, \( \text{lin}(J_0, J_1, J_2)^{\omega*} \) is also contained in the weak\(^*\)-closed subtriple \( J_1 \) of \( A \). Let \( M \) be a weak\(^*\)-closed inner ideal in \( A \) containing \( J_1 \) and \( J_0 \). Then,

\[
( J_0, J_1, J_2 ) \subseteq \{ M, M, M \} \subseteq M
\]

and it can be seen that

\[
J_2 \oplus J_0 \oplus \text{lin}(J_0, J_1, J_2)^{\omega*} \subseteq M.
\]

Since the weak\(^*\)-closure of an inner ideal is an inner ideal, it remains to show that

\[
N = J_2 \oplus J_0 \oplus \text{lin}(J_0, J_1, J_2)
\]

is an inner ideal in \( A \). Since \( J_2 \) and \( J_0 \) are inner ideals in \( A \), to complete the proof it must be shown that

\[
( J_0, A, J_2 ) \subseteq N, \tag{4.1}
\]

\[
( J_0, A, J_0, J_2 ) \subseteq N, \tag{4.2}
\]

\[
( J_0, A, J_1, J_2 ) \subseteq N, \tag{4.3}
\]

\[
( J_0, J_1, J_2 ) \subseteq N. \tag{4.4}
\]

Observe that

\[
( J_0, J_0, J_2 ) = ( J_0, J_0 \oplus J_0 \oplus J_0 ) = ( J_0, J_1, J_2 ) \leq N
\]

and (4-1) holds. Using (1-3)-(1-5),

\[
( J_0, J_0, J_2 ) = ( J_0, J_0 \oplus J_0 ) \oplus ( J_0, J_1 ) \leq ( J_0, J_0 \oplus J_1 ) \oplus ( J_0, J_1, J_2 ) \leq ( J_0, J_0 ) \oplus J_0 \oplus J_2 = J_0 \oplus J_0 \oplus J_2 = J_0 \oplus J_2 \oplus J_0 = J_2 \oplus J_0 \oplus J_0 \oplus J_2 \leq ( J_0, J_0, J_2 ) \leq N.
\]

Moreover, by (2-1), using the fact that \( D(J_2, J_0) \) is equal to zero,

\[
D((J_0, J_1, J_2), J_0, J_0) = D(J_0, J_0, J_2) + (D(J_0, J_1), J_0, J_0)
\]

\[
= ( J_2, J_0, J_0 ) = ( J_0, J_0, J_2 ) \subseteq ( J_0, J_0, J_2 ). \tag{4.5}
\]

It follows from (4-5)-(4-6) that (4-2) holds, and (4-3) is proved in a similar manner. Observe that, by (1-3)-(1-5),

\[
( J_0, J_1, J_2 ) \subseteq ( J_0, J_0, J_2 ) \subseteq J_0 \subseteq N,
\]

\[
( J_0, J_1, J_2 ) \subseteq ( J_0, J_0, J_2 ) \subseteq J_0 \subseteq N
\]

and in order to complete the proof of (4-4) and, hence, that of the lemma, it remains to prove that

\[
( J_0, J_1, J_2 ) \subseteq J_1 \subseteq N. \tag{4.7}
\]

For \( j \) equal to 0, 1 and 2, let \( a_j \) be elements of \( J_j \). Using (2-1),

\[
D((a_0, a_1, a_2) b_1, (c_0, c_1, c_2)) = D((a_0, a_1, a_2), b_1) D((c_0, c_1, c_2))
\]

\[
= D((a_0, a_1, a_2), b_1) c_2 + D((c_0, c_1, c_2)) c_1 c_2
\]

\[
D((a_0, a_1, a_2), b_1) c_2 - D((c_0, c_1, c_2)) c_2. \tag{4.8}
\]

Since

\[
D((a_0, a_1, a_2), b_1) c_2 \leq (a_0, a_1, a_2) b_1 c_2 \leq J_1 \subseteq J_2,
\]

it follows that

\[
D((a_0, a_1, a_2), b_1) c_2 \leq (a_0, a_1, a_2) b_1 c_2 \leq J_1 \subseteq J_2 \subseteq J_0. \tag{4.9}
\]

Moreover, since

\[
\{ (a_0, a_1, a_2), b_1, c_0 \} \subseteq \{ J_1, J_0 \} \subseteq J_0,
\]

it can be seen that

\[
D((a_0, a_1, a_2), b_1, c_0, c_2) \leq (a_0, a_1, a_2) b_1, c_0, c_2 \leq (J_0, J_1, J_2) \subseteq J_0 \subseteq N. \tag{4.10}
\]

and, since

\[
( a_0, a_1, a_2, b_1, c_0 ) \in \{ c_1, (a_0, a_1, a_2), b_1 \} \subseteq J_1,
\]

it can be seen that

\[
D((a_0, a_1, a_2), b_1, c_0, c_2) \leq (a_0, a_1, a_2) b_1, c_0, c_2 \leq (J_0, J_1, J_2) \subseteq J_0 \subseteq N. \tag{4.11}
\]

Therefore, (4-7) follows from (4-8)-(4-11) and the proof is complete.

It is now possible to prove the first important result of this section.

Theorem 4.3. Let \( J \) and \( K \) be orthogonal Peirce weak\(^*\)-closed inner ideals in the JBW\(^*\)-triple \( A \) with corresponding Peirce spaces \( J_0, J_1, J_2, J_0, K_1, K_2 \) and \( K_3 \). Then, the smallest weak\(^*\)-closed inner ideal \( J \oplus K \) containing \( J \) and \( K \) is given by

\[
J \oplus K = J_2 \oplus K_2 \oplus \text{lin}(J_0, J_1, K_0)^{\omega*},
\]

where \( \text{lin}(J_0, J_1, K_0)^{\omega*} \) is the weak\(^*\)-closure of the linear span of the set

\[
\{ J_0, J_1, K_0 \} = \{ (a_0, a_1, a_2) : a_k \in J_0 \cap K_0, j, k = 0, 1, 2 \}.
\]

Proof. By Lemma 4.1, \( J \oplus K \) is a weak\(^*\)-closed inner ideal in the weak\(^*\)-closed inner ideal \( B = J_1 \oplus K_1 \oplus J_1 \cap K_1 \) in \( A \) given by

\[
B = J_2 \oplus K_2 \oplus J_1 \cap K_1,
\]

in which \( J \) is Peirce with relative Peirce decomposition given by

\[
( J )_{B,2} = J_2, \quad ( J )_{B,1} = J_1 \cap K_1, \quad ( J )_{B,0} = K_2.
\]

The result follows immediately from Lemma 4.2.
LEMMA 4.4. Let $J$ be a Peirce weak*-closed inner ideal in a JBW*-triple $A$, let $J_0$, $J_1$, and $J_2$ be the Peirce spaces corresponding to $J$ and let $J_2 \vee J_0$ be the smallest weak*-closed inner ideal in $A$ containing $J_2$ and $J_0$. Then, the weak*-closed subspace $\text{lin}(J_0 J_1 J_2)^{\perp \perp}$ of the weak*-closed subtriple $J_0 J_1 J_2$ of $A$ is an ideal in $J_1$, and the Peirce decomposition of $A$ associated with $J_2 \vee J_0$ is given by

$$A = (J_2 \vee J_0) \oplus (J_2 \vee J_0) \oplus (J_2 \vee J_0)^0 = (J_2 \vee J_0) \oplus (J_0 J_1 J_2)^{\perp} \cap J_1 \oplus [0].$$

Proof. Observe that, by (1.3)–(1.5) and (2.1),

$$\{J_1 J_2, \{J_0 J_1 J_2\}\} = D(J_1 J_2)D(J_0 J_1 J_2) = D(J_0 J_1 J_2 J_1 J_2) = D(J_0 J_1 J_2) = \{J_0 J_1 J_2\} \cap J_1 \cap J_0 \cap J_2$$

It follows that

$$\{J_0 J_1 J_2\} \subseteq \text{lin}(J_0 J_1 J_2)^{\perp \perp} \cap J_1.$$

and, by the weak*-continuity of the triple product,

$$\{J_1 J_2, \text{lin}(J_0 J_1 J_2)\} \subseteq \text{lin}(J_0 J_1 J_2)^{\perp \perp}.$$

Therefore, by (7, proposition 1.3), $\text{lin}(J_0 J_1 J_2)^{\perp \perp}$ is a weak*-closed ideal in the JBW*-triple $J_1$.

Notice that

$$(J_2 \vee J_0)^{\perp} = (J_2 \oplus J_0 \oplus \text{lin}(J_0 J_1 J_2)^{\perp \perp})^{\perp \perp} = J_1 \cap J_1 \cap \text{lin}(J_0 J_1 J_2)^{\perp \perp} = [0].$$

(4.12)

Moreover, since $J_2$, $J_0$, and $\text{lin}(J_0 J_1 J_2)^{\perp \perp}$ are contained in $J_2 \vee J_0$, it is clear that $\text{Ker}(J_2 \vee J_0)$ is contained in $\text{Ker}(J_2) \cap \text{Ker}(J_0)$, which coincides with $\text{Ker}(\text{lin}(J_0 J_1 J_2)^{\perp \perp}) \cap J_1$. However, since $\text{lin}(J_0 J_1 J_2)^{\perp \perp}$ is a weak*-closed ideal in $J_1$, it follows that

$$\text{Ker}(J_2 \vee J_0) \subseteq \text{lin}(J_0 J_1 J_2)^{\perp \perp} \cap J_1 = \text{lin}(J_0 J_1 J_2)^{\perp \perp} \cap J_1.$$

(4.13)

Furthermore, since $\{J_0 J_1 J_2\}$ lies in $\text{lin}(J_0 J_1 J_2)^{\perp \perp}$, it follows that

$$\text{lin}(J_0 J_1 J_2)^{\perp \perp} \subseteq \{J_0 J_1 J_2\} \subseteq \{J_0 J_1 J_2\} \cap J_1.$$

and, by the weak*-continuity and linearity of the triple product, it follows that the reverse inclusion holds. Therefore, using (4.13),

$$\text{Ker}(J_2 \vee J_0) \subseteq \{J_0 J_1 J_2\} \cap J_1.$$

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and, since $J_2 \vee J_0$ is a weak*-closed inner ideal in $A$ and, therefore, complemented,

$$A = (J_2 \vee J_0) \oplus \text{Ker}(J_2 \vee J_0) \subseteq J_2 \oplus \text{lin}(J_0 J_1 J_2) \cap \{J_0 J_1 J_2\} \cap J_1 \subseteq A.$$

Hence,

$$\text{Ker}(J_2 \vee J_0) = \{J_0 J_1 J_2\} \cap J_1.$$

and, using (4.12), the proof is complete.

Using Lemma 4.4, an immediate corollary of Theorem 4.3 can now be stated.

COROLLARY 4.5. Under the conditions of Theorem 4.3 the following are equivalent:

(i) $J \vee K = J_2 \oplus K_2 \oplus J_1 \cap K_1$;
(ii) $\text{lin}(J_0 J_1 J_2 K_2) = J_1 \cap K_1$;
(iii) $(J_2 \cap K_1 K_2) \cap J_1 \cap K_1 = [0]$.

In the special case in which $J$ and $K$ coincide with the Peirce two-spaces of two orthogonal tripotents in the JBW*-triple $A$ a little more can be said.

COROLLARY 4.6. Let $u$ and $v$ be orthogonal tripotents in the JBW*-triple $A$ having Peirce spaces $A_j(u)$ and $A_j(v)$ for $j$ equal to 0, 1 and 2. Then

$$\text{lin}(A_j(u) \cap A_j(v) A_2(v))^{\perp \perp} = A_1(u) \cap A_1(v).$$

Proof. Since $v$ is contained in the weak*-closed ideal $A_2(v)$ in $A$, it follows that $A_2(v)$ is contained in $A_2(u)$ and the Peirce weak*-closed inner ideals $A_2(u)$ and $A_2(v)$ are orthogonal. Since $u$ and $v$ lie in $A_2(u) \vee A_2(v)$ it follows that the tripotent $u + v$ lies in $A_2(u) \vee A_2(v)$, and, hence, that $A_2(u + v)$ is contained in $A_2(u) \oplus A_2(v)$. By (26, lemma 5.3),

$$A_2(u) \oplus A_2(u) \oplus A_2(u) \cap A_2(v) = A_2(u + v) \subseteq A_2(u) \cap A_2(v)$$

and the result follows from Corollary 4.5.

The second main result of the section is now proved.

THEOREM 4.7. Let $J$ and $K$ be orthogonal Peirce weak*-closed inner ideals in the JBW*-triple $A$ and let $J \vee K$ be the smallest weak*-closed inner ideal in $A$ containing $J$ and $K$. Then,

$$(J, K, J \vee K, J^\perp, J_1 \perp, K^\perp, K_1 \perp, J_1 \cap K_1, J_1 \cap K_1)$$

is a family of pairwise compatible weak*-closed inner ideals in $A$.

Proof. Let $\phi_j$ be the triple automorphism of $A$ defined as in (3.1), by

$$\phi_j = 2P_j(J) + 2P_j(J) - i\text{id}_A = i\text{id}_A - 2P_j(J)$$

(4.14)

and observe that, for an element $a$ of the subtriple $J_2 \oplus J_0$,

$$\phi_j(a) = a.$$

Since $K_2$ is contained in $J_0$, the subspace $J_2 + K$ is contained in $J_2 \oplus J_0$ and it follows that $\phi_j$ is the identity on $J + K$. Since $J + K$ is contained in $J \vee K$ it can be seen that $J \vee K$ is the
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lattice centre $\mathcal{Z}(C)$ of $\mathcal{P}(C)$. Moreover, with respect to the Jordan triple product defined, for elements $a, b$ and $c$ in $C$, by

$$\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a).$$

$C$ is a JBW*-triple. For details, the reader is referred to [39, 40, 42].

For each element $e$ in $\mathcal{P}(C)$, the central support $c(e)$ of $e$ is defined by

$$c(e) = \bigwedge \{z \in \mathcal{Z}(C) : : e \leq z\}.$$ A pair $(e, f)$ of elements of $\mathcal{P}(C)$ is said to be centrally equivalent if $c(e)$ and $c(f)$ coincide. The common central support is denoted by $c(e, f)$. When endowed with the product ordering, the set $\mathcal{P}(C)$ of centrally equivalent pairs of elements of $\mathcal{P}(C)$ forms a complete lattice in which the lattice supremum coincides with the supremum in the product lattice, but, in general, the lattice infimum does not. The results of [18] show that the mapping $(e, f) \mapsto c(e, f)$ is an order isomorphism from $\mathcal{P}(C)$ onto $\mathcal{Z}(C)$.

A JBW*-triple $A$ is said to be rectangular if there exists a $W^*$-algebra $C$ and an element $(p, q)$ of $\mathcal{P}(C)$ such that $A$ is isomorphic to the JBW*-triple $pC_q$. In what follows the rectangular JBW*-triples $A$ and $pC_q$ will be identified. Let $\mathcal{P}(C)_{(p,q)}$ denote the principal order ideal in $\mathcal{P}(C)$ consisting of elements $(e, f)$ such that

$$(e, f) \leq (p, q).$$ Then, the mapping $(e, f) \mapsto eAf$ is an order isomorphism from $\mathcal{P}(C)_{(p,q)}$ onto the complete lattice $\mathcal{Z}(A)$ of weak*-closed inner ideals in $A$. Therefore, there exists a corresponding order isomorphism from $\mathcal{P}(C)_{(p,q)}$ onto $\mathcal{P}(A)$.

The mapping $z \mapsto pz$ is a *-isomorphism from the commutative $W^*$-algebra $C(p, q)z(C)$ onto the centre $Z(pCp)$ of the hereditary sub-$W^*$-algebra $pCp$ of $C$. It follows that the same mapping determines an order isomorphism from the complete Boolean lattice $\mathcal{Z}(pCp)$ onto $\mathcal{Z}(pCp)$ or, equivalently, $\mathcal{Z}(\mathcal{P}(C))$. In order to simplify notation, for $e$ in the principal order ideal $\mathcal{P}(C)_{(p,q)}$ of $\mathcal{P}(C)$, let

$$c^e(e) = \bigwedge \{zp : z \in \mathcal{Z}(pCp), e \leq z\}.$$ It is clear that $c^e(e)$ coincides with $c(e)p$. The results of [23] show that the mapping $\mu$, defined, for each element $z$ of the complete Boolean lattice $\mathcal{Z}(\mathcal{P}(C)_{(p,q)})$ and each element $a$ in $A$, by

$$\mu(z)(a) = za,$$ is an order isomorphism onto the complete Boolean lattice of $M$-projections on $A$. It follows that the mapping $z \mapsto zA$ is an order isomorphism from $\mathcal{Z}(pCp)$ onto the complete Boolean lattice $\mathcal{Z}(A)$ of weak*-closed ideals in $A$.

For each element $(e, f)$ in $\mathcal{P}(C)_{(p,q)}$, and each element $z$ in $\mathcal{Z}(pCp)$, write

$$e^r = p - e, \quad f^s = q - f, \quad z^{\omega_e} = e(p, q) - z.$$ For an element $(e, f)$ in $\mathcal{P}(C)_{(p,q)}$, let

$$(e, f)^{\omega_e} = (c^e(e)^r e^r, c^f(f)^s f^s).$$ Then, the mapping $(e, f) \mapsto (e, f)^{\omega_e}$ is order reversing, and if $J$ is the weak*-closed inner ideal $eAf$ in $A$, then the annihilator $J^\perp$ coincides with $c^e(e)^r e^r A c^f(f)^s f^s$. It follows that
the generalized Peirce decomposition of $A$ corresponding to the weak*-closed inner ideal $J$ is given by

$$J = J_0 \oplus J_1 \oplus J_2,$$

where

$$J_2 = eAf, \quad J_0 = c^0_1(e^r)Ac^0_1(e^r)f,$$

and

$$J_1 = ec^0_1(e^r)Ac(e, f)f^r + c(e, f)e^rAc^0_1(e^r)f.$$

Furthermore, every weak*-closed inner ideal $J$ is a Peirce inner ideal.

The results of [22, 23] show that for two elements $(e, f)$ and $(g, h)$ of $CP(C)_{(p,q)}$ the corresponding weak*-closed inner ideals

$$J = eAf, \quad K = gAh,$$

are orthogonal if and only if $(e, f) \leq (g, h)^{\text{adj}}$, or, equivalently, if and only if, in $T(C)$,

$$e + g \leq p, \quad f + h \leq q.$$ 

In this case the general results take on a fairly straightforward form.

**THEOREM 5.1.** Let $C$ be a $W^*$-algebra, let $(p, q)$ be an element of the complete lattice $CP(C)$ of pairs of centrially equivalent projections in $C$ and let $A$ be the rectangular $JBW^*$-triple $pCq$. Let $(e, f)$ and $(g, h)$ be orthogonal elements of the complete lattice $CP(C)_{(p,q)}$. Let $J$ and $K$ be the weak*-closed inner ideals $eAf$ and $gAh$ and let $J_0, J_1, J_2$, and $K_0, K_1, K_2$ be the corresponding generalized Peirce spaces defined above. Let $B$ be the weak*-closed inner ideal in $A$ given by

$$B = J_2 \oplus K_2 \oplus J_1 \cap K_1,$$

and let $J \vee K$ be the smallest weak*-closed inner ideal in $A$ containing $J$ and $K$. Then, the weak*-closed inner ideals $B$ and $J \vee K$ are equal and both coincide with the weak*-closed inner ideal $(e + g)A(f + h)$ in $A$.

Proof. Observe that the projections $e$ and $g$ commute, as do $f$ and $h$. Notice that $eAf$ and $e^rAh$ are ideals in the $JBW^*$-triple $J_0$, as are $gAh$ and $g^rAh$ in $K_1$. Therefore, using the orthogonality of the pairs $e$ and $g$, and $f$ and $h$, it can be seen that

$$J_1 \cap K_1 = eAf \cap gAh = eAf \cap e^rAh \cap gAh = eAh \oplus gAf.$$

It follows that

$$B = eAf \oplus gAh \oplus eAh \oplus gAf = (e + g)A(f + h).$$

Let $(r, s)$ be the element of $CP(C)_{(p,q)}$ corresponding to the weak*-closed inner ideal $J \vee K$. Then,

$$(e, f) \leq (r, s), \quad (g, h) \leq (r, s)$$

and both $e$ and $g$ are majorised by $r$, and $f$ and $h$ are majorised by $s$. Hence, $e + g = e \vee g \leq r$, $f + h = f \vee h \leq s$ and it follows that

$$J \vee K \subseteq B = (e + g)A(f + h) \subseteq rAs = J \vee K,$$

as required.

When $A$ is a $W^*$-algebra the situation is described by choosing both $p$ and $q$ equal to the unit in the theorem above.

In conclusion, it is worth repeating that the main results of the paper could equally well be stated about $L$-orthogonal subspaces of the pre-symmetric space $A$, that is the predual of the JBW*-triple $A$. The existence of a smallest subspace that is the range of a neutral GL-projection containing two $L$-orthogonal such spaces having the Peirce property, and the fact that all these subspaces are pairwise compatible clearly has deep physical significance in any theory that uses pre-symmetric spaces as models for state spaces.

**REFERENCES**


The Hausdorff dimension of pulse-sum graphs

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Abstract

We consider random functions formed as sums of pulses

\[ F(t) = \sum_{n=1}^{\infty} \eta^{\alpha/D} G(n^{1/D} (t - X_n)) \quad (t \in \mathbb{R}^D) \]

where \( X_n \) are independent random vectors, \( 0 < \alpha < 1 \), and \( G \) is an elementary "pulse" or "bump". Typically such functions have fractal graphs and we find the Hausdorff dimension of these graphs using a novel variant on the potential theoretic method.

1. Introduction

Many types of random fractal function have been proposed to model a wide range of phenomena from internet traffic to stock prices. One class of construction, studied in \([1]\) and \([5]\), depends on the superposition of randomly located "pulses" or "bumps" with width and amplitude decreasing in a self-similar manner. Here we investigate the Hausdorff dimension of the graph of such pulse-sum functions, which provides a measure of the irregularity or volatility of the process.

Let \( g: \mathbb{R} \to \mathbb{R} \) be an even continuous function, decreasing on \([0, 1]\), equal to 0 on \([1, \infty)\) and such that \( g(0) = 1 \). We define the elementary pulse or elementary bump \( G: \mathbb{R}^D \to \mathbb{R} \) to be the symmetrical function

\[ G(t) = g(|t|) \]

where \( |t| = \max(t_i) \) for \( t = (t_1, \ldots, t_D) \in \mathbb{R}^D \). (The simplest instance to bear in mind is the "triangular bump" on \( \mathbb{R} \), where \( G(t) = g(t) = \max(1 - |t|, 0) \).) Given a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), we study the random function \( F: \mathbb{R}^D \to \mathbb{R} \) given by a sum of randomly centred pulses

\[ F(t) = \sum_{n=1}^{\infty} \eta^{\alpha/D} G(n^{1/D} (t - X_n)) \]

where \( 0 < \alpha < 1 \) and \( (X_n)_{n \geq 1} \) is a sequence of independent random variables uniformly distributed.