Central kernels of subspaces of JB*-triples

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Abstract

An investigation of the norm central kernel \( k_n(L) \) of an arbitrary norm-closed subspace \( L \) of a JB*-triple and the central kernel \( k(L) \) of a weak*-closed subspace \( L \) of a JBW*-triple is carried out. It is shown that these geometrically defined objects have purely algebraic characterizations, the results providing new information about C*-algebras and W*-algebras.

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1. Introduction

This paper represents a further investigation into the central structure of Banach spaces. In the late sixties and early seventies, in ground-breaking work, Alfsen, Cunningham, Effros, and Roy [1,2,9,10] introduced the concepts of M-ideals, M-summands, and L-summands in real Banach spaces. In the following years their results were extended to complex Banach spaces, a full description being given in Behrends’ treatise [6].

For a complex Banach space \( A \) and any closed subspace \( L \) of \( A \), there exists a greatest M-ideal \( k_n(L) \) of \( A \) contained in \( L \), known as the norm central kernel of \( L \) in \( A \). In the case in which \( A \) is a dual space and \( L \) is weak*-closed, there exists a greatest M-summand \( k(L) \) of \( A \).
contained in $A$, known as the central kernel $k(L)$ of $L$ in $A$. It is the investigation of these two central kernels that is the subject of this paper.

A complex Banach space $A$ having the property that its open unit ball is a bounded symmetric domain possesses a canonical triple product $\langle \cdot \cdot \cdot \rangle : A \times A \times A \to A$ with respect to which $A$ forms a JB*-triple. In the case in which $A$ is a dual space, $A$ is said to be a JBW*-triple, and its predual $A_*$ is unique up to isometric isomorphism. The second dual of a JB*-triple is a JBW*-triple. The predual of a JBW*-triple has been proposed as a model for the state space of a physical system [26-29]. Such a space has the highly desirable property that its image under a contractive linear projection is of the same category [34,40]. In this case central properties of the JBW*-triple correspond to classical properties of the physical system. Examples of JB*-triples are C*-algebras, JB*-algebras, Hilbert C*-modules and spin triples. It is the interplay between the geometric, holomorphic, and algebraic structure of JB*-triples that has fascinated many authors over recent years.

Whilst much is known about the central structure of JB*-triples [13,23-25], no attention has yet been given to an investigation into the properties of the norm central kernel $k_n(L)$ of an arbitrary norm-closed subspace $L$ of a JB*-triple or the central kernel $k(L)$ of an arbitrary weak*-closed subspace $L$ of a JBW*-triple. The main results of the paper show that these objects, which are defined purely in geometrical terms can be described purely algebraically. The support space of a subset of the predual of a JBW*-triple plays an important part in the construction of contractive projections [15,16,20,31]. In the course of the investigations into the central structure of a weak*-closed subspace $L$ of a JBW*-triple $A$, a new algebraic characterization of the algebraic annihilator $s(L_\circ)\perp$ of the support space $s(L_\circ)$ of the topological annihilator $L_\circ$ of a weak*-closed subspace $L$ is discovered.

The paper is organised as follows. In Section 2, definitions are given, notation is established, and certain preliminary results are described. In Section 3, the norm central kernel of a norm-closed subspace of a JB*-triple is investigated, and, in Section 4, the results of Section 3 are applied to study the central kernel of a weak*-closed subspace of a JBW*-triple. The final section considers the applications of the main results to C*-algebras and W*-algebras.

2. Preliminaries

Let $A$ be a complex Banach space. A linear projection $S$ on $A$ is said to be an $M$-projection if, for each element $a$ in $A$,

$$\|a\| = \max\{\|Sa\|, \|a - Sa\|\}.$$ 

A closed subspace which is the range of an M-projection is said to be an $M$-summand of $A$, and $A$ is said to be the $M$-sum

$$A = SA \oplus_\infty (\text{id}_A - S)A$$

of the M-summands $SA$ and $(\text{id}_A - S)A$. A linear projection $T$ on a complex Banach space $E$ is said to be an $L$-projection if, for each element $x$ of $E$,

$$\|x\| = \|Tx\| + \|x - Tx\|.$$ 

A closed subspace which is the range of an L-projection is said to be an $L$-summand of $E$, and $E$ is said to be the $L$-sum

$$E = TE \oplus_1 (\text{id}_E - T)E$$

of the L-summands $TE$ and $(\text{id}_E - T)E$. 
For a subset $M$ of the complex Banach space $E$, having dual space $E^*$, let

$$M^0 = \{ x \in E^*: x(a) = 0, \forall a \in M \},$$

and, for a subset $L$ of $E^*$, let

$$L_0 = \{ a \in E: x(a) = 0, \forall x \in L \},$$

be the topological annihilators of $M$ and $L$, respectively. The mapping $M \mapsto M^0$ is a bijection from the family of $L$-summands of $E$ onto the family of weak*-closed $M$-summands of $E^*$. When ordered by set inclusion, the family of $L$-summands of $E$ forms a complete Boolean lattice, the lattice operations being defined for a family $\{M_j: j \in \Lambda\}$ of $L$-summands in $E$, by

$$\bigwedge_{j \in \Lambda} M_j = \bigcap_{j \in \Lambda} M_j, \quad \bigvee_{j \in \Lambda} M_j = \overline{\text{lin}} \left( \bigcup_{j \in \Lambda} M_j \right),$$

the closure being in the norm topology. It follows that for any family $\{L_j: j \in \Lambda\}$ of weak*-closed $M$-summands of the dual space $E^*$ of $E$, the weak*-closure of their linear span is also an $M$-summand. A norm-closed subspace $L$ of the complex Banach space $A$ is said to be an $M$-ideal if its topological annihilator $L^0$ is an $L$-summand of its dual space. It follows from the remarks above that for any family $\{L_j: j \in \Lambda\}$ of $M$-ideals in $A$, the norm-closure of their linear span is also an $M$-ideal in $A$. For details, the reader is referred to [1,2,9,10].

It can now be seen that, for each closed subspace $L$ of a complex Banach space $A$, there exists a greatest $M$-ideal $k_n(L)$ of $A$ contained in $L$. The $M$-ideal $k_n(L)$ is said to be the norm central kernel of $L$ in $A$. Similarly, for each complex Banach space $E^*$ which is a dual space and each weak*-closed subspace $L$ of $E^*$, there exists a greatest weak*-closed $M$-summand $k(L)$ of $E^*$ contained in $L$. The $M$-summand $k(L)$ is said to be the central kernel of $L$ in $E^*$.

A complex vector space $A$ equipped with a triple product $(a, b, c) \mapsto \{a b c\}$ from $A \times A \times A$ to $A$ which is symmetric and linear in the first and third variables, conjugate linear in the second variable and, for elements $a$, $b$, $c$ and $d$ in $A$, satisfies the identity

$$[D(a, b), D(c, d)] = D([a b c], d) - D(c, [d a b]), \quad (2.1)$$

where $[ , ]$ denotes the commutator, and $D$ is the mapping from $A \times A$ to the algebra of linear operators on $A$ defined by

$$D(a, b)c = \{a b c\},$$

is said to be a Jordan*-triple. For an element $a$ in the Jordan*-triple $A$ and for $n$ equal to $1, 2, \ldots$, define

$$a^{1} = a, \quad a^{2n+1} = \{a a^{2n-1} a\}.$$

Observe that for non-negative integers $l$, $m$, and $n$,

$$\{a^{2l+1} a^{2m+1} a^{2n+1}\} = a^{2(l+m+n)+3}. \quad (2.2)$$

A Jordan*-triple for which the vanishing of $a^3$ implies that $a$ itself vanishes is said to be anisotropic. For elements $a$ and $b$ in $A$, the conjugate linear mapping $Q(a, b)$ from $A$ to itself is defined, for each element $c$ in $A$, by

$$Q(a, b)c = \{a c b\}.$$

For details about the properties of Jordan*-triples the reader is referred to [35].
A Jordan*-triple $A$ which is also a Banach space such that $D$ is continuous from $A \times A$ to the Banach algebra $B(A)$ of bounded linear operators on $A$, and, for each element $a$ in $A$, $D(a, a)$ is hermitian in the sense of [7, Definition 5.1], with non-negative spectrum, and satisfies

$$\|D(a, a)\| = \|a\|^2,$$

is said to be a JB*-triple. A subspace $B$ of a JB*-triple $A$ is said to be a subtriple if $\{B, B, B\}$ is contained in $B$. A subspace $B$ is clearly a subtriple if and only if, for each element $a$ in $B$, the element $a^3$ lies in $B$. Observe that every subtriple of a JB*-triple is an anisotropic Jordan*-triple. A subspace $J$ of a JB*-triple $A$ is said to be an inner ideal if $\{J, A, J\}$ is contained in $J$ and is said to be an ideal if $\{A, A, J\}$ and $\{A, J, A\}$ are contained in $J$. Every norm-closed subtriple of a JB*-triple $A$ is a JB*-triple [33], and a norm-closed subspace $J$ of $A$ is an ideal if and only if $\{J, J, A\}$ is contained in $J$ [8]. For each element $a$ in a JB*-triple $A$, the smallest norm-closed subtriple $A(a)$ of $A$ containing $a$ is isometrically triple isomorphic to the commutative $C^*$-algebra $C_0(\sigma_A(a))$ of complex-valued continuous functions on the bounded, locally compact subset $\sigma_A(a)$ of $\mathbb{R}^+$ which have limit zero at zero. Under the isomorphism the element $a^{2n+1}$ is mapped into the function $t^{2n+1}$ defined, for each element $t$ in $\sigma_A(a)$, by

$$t^{2n+1}(t) = t^{2n+1}.$$

The isometric triple isomorphism from $C_0(\sigma_A(a))$ onto $A(a)$ is said to be the functional calculus corresponding to $a$. A JB*-triple $A$ which is the dual of a Banach space $A_*$ is said to be a JBW*-triple. In this case the predual $A_*$ of $A$ is unique up to isometric isomorphism and, for elements $a$ and $b$ in $A$, the operators $D(a, b)$ and $Q(a, b)$ are weak*-continuous. It follows that a weak*-closed subtriple $B$ of a JBW*-triple $A$ is a JBW*-triple. Examples of JB*-triples are JB*-algebras and examples of JBW*-triples are JBW*-algebras. The second dual $A^{**}$ of a JB*-triple $A$ is a JBW*-triple. For details of these results the reader is referred to [4,5,11,12,30,32–34,41,42].

When $A$ is a JB*-triple the M-ideals of $A$ coincide with its norm-closed ideals, and, when $A$ is a JBW*-triple its M-summands coincide with its weak*-closed ideals [4,32]. Hence, the norm central kernel $k_n(L)$ of a norm-closed subspace $L$ of the JB*-triple $A$ is the greatest norm-closed ideal of $A$ contained in $L$, and the central kernel $k(L)$ of a weak*-closed subspace $L$ of a JBW*-triple $A$ is the greatest weak*-closed ideal of $A$ contained in $L$ [24,25].

### 3. Subspaces of JB*-triples

This section is devoted to an investigation of the norm central kernel of a norm-closed subspace of a JB*-triple. The results are proved using a series of mainly algebraic lemmas.

#### Lemma 3.1

Let $A$ be a JB*-triple, let $L$ be a norm-closed subspace of $A$, and let

$$J_L = \{a \in A: D(b, c)a \in L, \forall b, c \in A]\}.$$

Then, $J_L$ is a norm-closed inner ideal of $A$ contained in $L$.

**Proof.** Since $L$ is a norm-closed subspace, by the linearity and separate norm-continuity of the triple product, it is clear that $J_L$ is a norm-closed subspace of $A$. Furthermore, by polarization, it can be seen that an element $a$ of $A$ lies in $J_L$ if and only if, for all elements $b$ in $A$, the element $D(b, b)a$ lies in $L$. Let $a$ be an element of $J_L$, and let $b$ and $c$ be elements of $A$. Then, by (2.1),
\[ D(b, b)\{a c a\} = D(b, b)D(a, c)a \]
\[ = D(a, c)D(b, b)a + D(D(b, b)a, c)a - D(a, D(b, b)c)a \]
\[ = 2D(D(b, b)a, c)a - D(a, D(b, b)c)a \]

which lies in \( L \). It follows that the element \( \{a c a\} \) lies in \( J_L \), and, again by polarization, \( J_L \) is an inner ideal in \( A \).

For an element \( a \) in \( J_L \), using the functional calculus, there exists a sequence \((d_j)\) in the norm-closed subtriple \( A(a) \) generated by \( a \) such that the sequence \( D(d_j, d_j)a \) converges in norm to \( a \). However, for \( j \) equal to 1, 2, ... , the element \( D(d_j, d_j)a \) lies in \( L \), and, since \( L \) is closed, the element \( a \) therefore lies in \( L \). This completes the proof of the lemma. \( \Box \)

The following lemmas, which are of a technical algebraic nature, aim to give an alternative algebraic description of the norm-closed inner ideal \( J_L \).

**Lemma 3.2.** Let \( A \) be a Jordan\(^*\)-triple, let \( L \) be a subspace of \( A \), and let \( a \) be an element of \( A \) such that, for all elements \( b \) in \( A \), the element \( D(a, a)b \) lies in \( L \). Then, for all elements \( b \) in \( A \), the elements \( Q(a, a^3)b \) and \( D(a, a^5)b \) lie in \( L \).

**Proof.** Observe that, by using [35, JP1], twice, for each element \( b \) in \( A \),
\[
Q(a, a^3)b = \{a b \{a a a\} a\} = \{a b a a a\} a \\
= \{a a b a\} = \{a a b a\}
\]
\[
= D(a, a)\{a b a\}, \tag{3.1}
\]
which lies in \( L \) by hypothesis. Using (2.1) observe that
\[
D(a, a^5)b = 2D(a, a)\{a^3 a b\} - Q(a, a^3)\{a b a\}, \tag{3.2}
\]
which, by hypothesis and (3.1), lies in \( L \).

**Lemma 3.3.** Let \( A \) be a Jordan\(^*\)-triple, let \( L \) be a subspace of \( A \), and let \( a \) be an element of \( A \) such that, for all elements \( b \) in \( A \), the element \( D(a, a)b \) lies in \( L \). Then, for \( j \) equal to 1, 2, ... , and all elements \( b \) in \( A \), the elements \( Q(a, a^4j-1)b \) and \( D(a, a^4j+1)b \) lie in \( L \).

**Proof.** That the result holds when \( j \) is equal to 1 follows from Lemma 3.2. Suppose, inductively, that the result holds when \( j \) is equal to \( n \). Then, using [35, JP1], twice, for each element \( b \) in \( A \),
\[
Q(a, a^{4(n+1)-1})b = \{a b a^{4n+3}\} = \{a b \{a a^{4n+1}\} a\} \\
= \{a b a^{4n+1} a\} = \{a \{a^{4n+1} a b\} a\} \\
= D(a, a^{4n+1})\{a b a\} \tag{3.3}
\]
which, by hypothesis, lies in \( L \). Using [35, JP9],
\[
D(a, a^{4(n+1)+1})b = 2D(a, a)\{a^{4n+3} a b\} - Q(a, a^{4n+3})\{a b a\}, \tag{3.4}
\]
which, by hypothesis and (3.3), lies in \( L \). This completes the proof of the lemma. \( \Box \)

The next result requires the use of the functional calculus and is therefore not necessarily valid for a Jordan\(^*\)-triple.
Lemma 3.4. Let $A$ be a JB*-triple, let $L$ be a norm-closed subspace of $A$, and let $a$ be an element of $A$ such that, for all elements $b$ of $A$, the element $D(a, a)b$ lies in $L$. Then, the following results hold.

(i) For $j$ equal to $0, 1, 2, \ldots$ and all elements $b$ of $A$, the elements $Q(a, a^{2j+3})b$ and $D(a, a^{2j+1})b$ lie in $L$.

(ii) For all elements $b$ in $A$, the element $Q(a, a)b$ lies in $L$.

Proof. Let $C_0(\sigma_A(a))$ be the commutative C*-algebra of continuous functions on the bounded locally compact subset $\sigma_A(a)$ of $\mathbb{R}^+$ that have limit zero at zero. Then, the norm-closed *-subalgebra of $C_0(\sigma_A(a))$ generated by the set of functions $\{t^{4j}: j = 1, 2, \ldots\}$ satisfies the conditions of the Stone–Weierstrass theorem for locally compact Hausdorff spaces, and, hence, coincides with $C_0(\sigma_A(a))$. It follows that, given a positive real number $\epsilon$, there exist a positive integer $n$ and complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, such that

$$\left\| t^2 - \sum_{j=1}^n \alpha_j t^{4j} \right\| < \frac{\epsilon}{\|a\|}.$$ Using the functional calculus, it follows that,

$$\left\| a^3 - \sum_{j=1}^n \alpha_j a^{4j+1} \right\| \leq \|t\| \left\| t^2 - \sum_{j=1}^n \alpha_j t^{4j} \right\| < \epsilon.$$

By Lemma 3.3, for $j$ equal to $1, 2, \ldots, n$ and any element $b$ in $A$, the element $D(a, a^{4j+1})b$ lies in $L$, and, since

$$\left\| D(a, a^3)b - \sum_{j=1}^n \alpha_j D(a, a^{4j+1})b \right\| < \|a\|\|b\|\epsilon,$$

it can be seen that the element $D(a, a^3)b$ lies in $L$.

Observe that, as in the proof of Lemma 3.3, for each element $b$ in $A$,

$$Q(a, a^3)b = \{a b \{a a^3 a\}\} = D(a, a^3)[a b a],$$

which, from above, lies in $L$. An induction argument similar to that used in the proof of Lemma 3.3 now shows that, for $j$ equal to $1, 2, \ldots$, and all elements $b$ in $A$, the elements $Q(a, a^{4j+1})b$ and $D(a, a^{4j-1})b$ lie in $L$. Combining these facts with the results of Lemma 3.3 completes the proof of (i).

Observe that, using (2.2), for $j$ equal to $0, 1, 2, \ldots$,

$$(a^3)^{2j+1} = a^{6j+3} = a^{2(3j+3)}.$$ Therefore, using (i), for $j$ equal to $0, 1, 2, \ldots$, and all elements $b$ in $A$, the element $Q(a, (a^3)^{2j+1})b$ lies in $L$. Since the family of finite linear combinations of elements of the form $(a^3)^{2j+1}$ is dense in the JB*-triple $A(a^3)$ generated by the element $a^3$, it follows from the linearity and separate norm-continuity of the triple product that, for all elements $c$ in $A(a^3)$ and $b$ in $A$, the element $Q(a, c)b$ lies in $L$. However, the function $t^{1/3}$ is continuous on the bounded locally compact set $\sigma_A(a^3)$, and has limit zero at zero. Therefore, the functional calculus shows that the element $a$ lies in $A(a^3)$. Consequently, the element $Q(a, a)b$ lies in $L$, as required. □

The final lemma paves the way for the main result of this section.
Lemma 3.5. Let $A$ be a JB$^*$-triple, let $L$ be a norm-closed subspace of $A$, and let

$$J_L = \{a \in A: D(b, c)a \in L, \ \forall b, c \in A\}.$$

Then,

$$J_L = \{a \in A: D(a, a)b \in L, \ \forall b \in A\}.$$

Proof. Observe that if $a$ is an element of $J_L$ then, for all elements $b$ in $A$,

$$D(a, a)b = D(b, a)a,$$

which is contained in $L$. Conversely, suppose that $a$ is an element of $A$ such that, for all elements $b$ in $A$, the element $D(a, a)b$ lies in $L$. Then, by (2.1),

$$D(b, b)a^3 = D(b, b)D(a, a)a = 2D(a, a)[b, b]a - Q(a)[b, b]a$$

which, by hypothesis and Lemma 3.4(ii), lies in $L$. By polarization it follows that, for all elements $b$ and $c$ in $A$, the element $D(b, c)a^3$ lies in $L$, and, hence, that the element $a^3$ lies in $J_L$.

However, by Lemma 3.1, $J_L$ is a norm-closed inner ideal in $A$, and, therefore, the JB$^*$-triple $A(a^3)$ is contained in $J_L$. Using the functional calculus as in the proof of Lemma 3.4, the cube root $a$ of $a^3$ lies in $A(a^3)$ and, hence, in $J_L$. This completes the proof of the lemma.

It is now possible to present the main result concerning JB$^*$-triples which gives the required algebraic characterization of the norm central kernel of an arbitrary norm-closed subspace of a JB$^*$-triple.

Theorem 3.6. Let $A$ be a JB$^*$-triple, let $L$ be a norm-closed subspace of $A$, having norm central kernel $k_n(L)$, and let

$$J_L = \{a \in A: D(b, c)a \in L, \ \forall b, c \in A\},$$

and

$$I_L = \{a \in A: Q(b, c)a \in L, \ \forall b, c \in A\}.$$

Then, $J_L$ is a norm-closed inner ideal of $A$, such that

$$I_L = k_n(L) \subseteq J_L \subseteq L.$$

Proof. That $J_L$ is a norm-closed inner ideal of $A$ contained in $L$ was proved in Lemma 3.1.

By the linearity and separate norm-continuity of the triple product it is clear that $I_L$ is a norm-closed subspace of $A$. Let $a$ be an element of $I_L$. Then, for each element $b$ in $A$, it can be seen that the element

$$D(a, a)b = Q(b, a)a$$

lies in $L$. It follows from Lemma 3.5 that the element $a$ lies in $J_L$, and, hence, $I_L$ is contained in $J_L$.

Now let $a$ be an element of $I_L$ and let $b$ and $c$ be elements of $A$. Then, the element $a$ lies in $J_L$, and, using (2.1), the element

$$Q(b, b)[c, a, a] = 2Q(b, [a, c, b])a - D([b, a, b], c)a,$$
lies in $L$. By polarization it can be seen that the element $\{c\ a\ a\}$ lies in $I_L$. It follows from [8, Proposition 1.3], that $I_L$ is a norm-closed ideal in $A$. Therefore, $I_L$ is contained in the norm central kernel $k_n(L)$ of $L$.

Finally, let $J$ be a norm-closed ideal of $A$ contained in $L$. Then, for each element $a$ in $J$ and $b$ in $A$, the element $\{b\ a\ b\}$ lies in $J$ and, hence, in $L$. It follows by polarization that the element $a$ lies in $I_L$. Therefore, $J$ is contained in $I_L$, from which it follows that $k_n(L)$ is contained in $I_L$ as required. □

When the norm-closed subspace $L$ of the JB*-triple $A$ discussed above is a subtriple of $A$, rather more can be said about its norm central kernel.

**Theorem 3.7.** Let $A$ be a JB*-triple, let $L$ be a norm-closed subtriple of $A$, having norm central kernel $k_n(L)$, and let

$$J_L = \{a \in A : D(b, c)a \in L, \forall b, c \in A\},$$

and

$$I_L = \{a \in A : Q(b, c)a \in L, \forall b, c \in A\}.$$

Then,

$$I_L = k_n(L) = J_L \subseteq L.$$

**Proof.** By Theorem 3.6, the set $J_L$ is a norm-closed inner ideal of $A$ contained in $L$. It will first be shown that $J_L$ is an ideal in the JB*-triple $L$. Let $a$ be an element of $J_L$ and let $c$ be an element of $L$. Then, using (2.1), for all elements $b$ in $A$,

$$D(b, b)\{b\ a\ c\} = D(b, b)D(a, a)c$$

$$= D(a, a)D(b, b)c + D\{b\ b\ a\}, a\}c - D(a, \{a\ b\ b\})c$$

$$= D(a, a)D(b, b)c + \{D(b, b)a\ a\ c\} - \{a\ D(b, b)a\ c\}.$$

Since $a$ lies in $J_L$, it follows from Lemma 3.5 that the element $D(a, a)D(b, b)c$ lies in $L$, and, from the definition of $J_L$, the element $D(b, b)a$ lies in $L$. Since $a, c$ and $D(b, b)a$ are elements of the subtriple $L$, it can be concluded that the element $D(b, b)\{a\ a\ c\}$ lies in $L$. By polarization, it can be seen that the element $\{a\ a\ c\}$ lies in $J_L$. Again, by polarization, it follows from [8, Proposition 1.3], that $J_L$ is an ideal in $L$.

In order to show that $J_L$ is an ideal in $A$, let $a$ be an element of $J_L$ and let $b$ be an element of $A$. Since $J_L$ is an inner ideal the element $a^3$ lies in $J_L$. By Lemma 3.5, the element $D(a, a)b$, and, by Lemma 3.4(ii), the element $Q(a, a)b$ lie in $L$. Since $J_L$ is an ideal in $L$, the elements $\{a^3\ a\ D(a, a)b\}$ and $\{a^3\ Q(a, a)b\}$ lie in $J_L$. Therefore, using [35, JP9],

$$\{a^3\ a^3\ b\} = D(a^3, a^3)b = 2D(a^3, a)D(a, a)b - Q(a^3, a)Q(a, a)b$$

$$= 2\{a^3\ a\ D(a, a)b\} - \{a^3\ Q(a, a)b\},$$

which implies that the element $\{a^3\ a^3\ b\}$ lies in $J_L$. Using the functional calculus, an arbitrary element $c$ in $J_L$ possesses a cube root $a$ in $J_L$. Applying the argument above, it follows that for each element $c$ in $J_L$ and each element $b$ in $A$, the element $\{c\ c\ b\}$ lies in $J_L$. Therefore, by polarization and [8, Proposition 1.3], it can be seen that $J_L$ is a norm-closed ideal in $A$.

However, by Theorem 3.6, the greatest norm-closed ideal $k_n(L)$ of $A$ that is contained in $L$ is contained in $J_L$. Therefore, $k_n(L)$ and $J_L$ coincide and the proof is complete. □
4. Subspaces of JBW*-triples

Recall that a JBW*-triple $A$ is a JB*-triple that is the dual of a complex Banach space $A^*$ and that the predual $A^*$ is unique up to isometric isomorphism. The techniques used to study the norm central kernel of a norm-closed subspace of a JB*-triple can easily be adapted to the study of the central kernel of a weak*-closed subspace of a JBW*-triple, and yield rather more detailed information.

**Theorem 4.1.** Let $A$ be a JBW*-triple, with predual $A^*$, let $L$ be a weak*-closed subspace of $A$, having central kernel $k(L)$, let
\[ J_L = \{ a \in A : D(b, c)a \in L, \, \forall b, c \in A \}, \]
and
\[ I_L = \{ a \in A : Q(b, c)a \in L, \, \forall b, c \in A \}. \]
Then, $J_L$ is a weak*-closed inner ideal of $A$, such that
\[ J_L = \{ a \in A : D(a, a)b \in L, \, \forall b \in A \}, \]
and
\[ I_L = k(L) \subseteq J_L \subseteq L. \]

**Proof.** Since the triple product is separately weak*-continuous and since $L$ is weak*-closed, it is clear that $J_L$ and $I_L$ are weak*-closed subspaces of $L$. Moreover, since the central kernel $k(L)$ of $L$ is a norm-closed ideal in $A$, $k(L)$ is contained in $k_n(L)$. On the other hand, the weak*-closure of $k_n(L)$ is a weak*-closed ideal of $A$ contained in $L$, and, hence, $k_n(L)$ is contained in $k(L)$. Therefore, the central kernel $k(L)$ and the norm central kernel $k_n(L)$ of $L$ coincide, and the result follows from Theorem 3.6. \qed

Recall that an element $u$ in a JBW*-triple $A$ is said to be a **tripotent** if $u^3$ is equal to $u$. The set of tripotents in $A$ is denoted by $U(A)$. For each tripotent $u$ in $A$, the linear operators $P_0(u)$, $P_1(u)$, and $P_2(u)$, defined by
\begin{align*}
P_0(u) &= \text{id}_A - 2D(u, u) + Q(u)^2, \\
P_1(u) &= 2(D(u, u) - Q(u)^2), \\
P_2(u) &= Q(u)^2,
\end{align*}
are mutually orthogonal weak*-continuous projection operators on $A$ with sum $\text{id}_A$. For $j$ equal to 0, 1, or 2, the range of $P_j(u)$ is the eigenspace $A_j(u)$ of $D(u, u)$ corresponding to the eigenvalue $\frac{1}{2}j$ and
\[ A = A_0(u) \oplus A_1(u) \oplus A_2(u) \]
is the **Peirce decomposition** of $A$ relative to $u$. In particular, $A_2(u)$ and $A_0(u)$ are a weak*-closed inner ideals in $A$. Observe that there exist mutually orthogonal contractive linear projections $P_0(u)^*$, $P_1(u)^*$, and $P_2(u)^*$ on the predual $A^*$ of $A$ with sum $\text{id}_{A^*}$, the ranges of which are the preduals $A_0^*(u)$, $A_1^*(u)$, and $A_2^*(u)$ of $A_0(u)$, $A_1(u)$, and $A_2(u)$, respectively [30].

For two tripotents $u$ and $v$ in the JBW*-triple $A$, write $u \leq v$ if $\{u, v, u\}$ is equal to $u$. This relation is a partial ordering on the set $U(A)$ of tripotents in $A$, and the set $U(A)$, consisting of
\[ U(A) \] with a largest element adjoined, forms a complete lattice. Two tripotents \( u \) and \( v \) are said to be compatible if their Peirce projections commute, or, equivalently, if

\[
A = \bigoplus_{j,k=0}^2 \left( A_j(u) \cap A_k(v) \right).
\]

If \( u \) lies in a Peirce space of \( v \), then \( u \) and \( v \) are compatible [37]. Two tripotents \( u \) and \( v \) are said to be orthogonal if \( u \) lies in the Peirce-zero space \( A_0(v) \) of \( v \), and two such tripotents are, therefore, compatible. For each element \( x \) of the predual \( A^\ast \) of the JBW*-triple \( A \) there exists a smallest element \( e(x) \) of \( U(A)^\ast \) for which

\[
x(e(x)) = \|x\|,
\]
and \( e(x) \) is said to be support tripotent of \( x \) [30]. More generally, for each subset \( M \) of the predual \( A^\ast \) of \( A \), the weak*-closed linear span \( s(M) \) of the set \( \{e(x): x \in M\} \) is said to be the support space of \( M \) [16]. Observe that the annihilator \( s(M)^\perp \) of the support space \( s(M) \), is given by

\[
s(M)^\perp = \bigcap_{x \in M} A_0(e(x)), \tag{4.1}
\]
which, being the intersection of weak*-closed inner ideals in \( A \), is itself a weak*-closed inner ideal in \( A \) [20,31].

Recall that for any subset \( L \) of the JBW*-triple \( A \), the kernel \( \text{Ker}(L) \) is the weak*-closed subspace of \( A \) consisting of elements \( a \) in \( A \) for which \( \{L a L\} \) is equal to zero, and the (algebraic) annihilator \( L^\perp \) is the weak*-closed inner ideal of \( A \) contained in \( \text{Ker}(L) \) consisting of elements of \( A \) for which \( \{L a A\} \) is equal to zero. A weak*-closed subtriple \( L \) of \( A \) is said to be complemented if

\[
A = L \oplus \text{Ker}(L).
\]
Such a subtriple is an inner ideal in \( A \) and every weak*-closed inner ideal arises in this way. A linear projection \( R \) on the JBW*-triple \( A \) is said to be a structural projection [36] if, for each element \( a \) in \( A \),

\[
RQ(a, a)R = Q(Ra, Ra).
\]
The range of a structural projection is a weak*-closed inner ideal, and every weak*-closed inner ideal arises in this manner. For each weak*-closed inner ideal \( L \) of \( A \), the annihilator \( L^\perp \) is a weak*-closed inner ideal and \( A \) enjoys the generalized Peirce decomposition

\[
A = L_2 \oplus L_1 \oplus L_0,
\]
relative to \( L \), where

\[
L_2 = L, \quad L_0 = L^\perp, \quad L_1 = \text{Ker}(L) \cap \text{Ker}(L^\perp).
\]
The structural projections the ranges of which are \( L_2 \) and \( L_0 \) are denoted by \( P_2(L) \) and \( P_0(L) \), respectively, and the projection

\[
P_1(L) = \text{id}_A - P_2(L) - P_0(L)
\]
denotes the projection onto \( L_1 \). Then \( P_0(L), P_1(L), \) and \( P_2(L) \) are mutually orthogonal weak*-continuous linear projections on \( A \) with sum \( \text{id}_A \). Observe that the pre-adjoints \( P_0(L)^\ast, P_1(L)^\ast, \)
and $P_2(L)_*$ are mutually orthogonal projections onto the preduals $L_{0,*}$, $L_{1,*}$, and $L_{2,*}$ of $L_0$, $L_1$, and $L_2$, respectively. For more details the reader is referred to [17,19–21].

Before proving the main result connecting the theory of support spaces to the earlier results, one more lemma is required.

**Lemma 4.2.** Let $A$ be a JBW$^*$-triple, with predual $A^*$, let $M$ be a subset of $A^*$ having support space $s(M)$ and topological annihilator $M^\circ$, and let $k(s(M)^\perp)$, $k(Ker(s(M)))$, and $k(M^\circ)$ be the central kernels of the annihilator $s(M)^\perp$ of $s(M)$, the kernel $Ker(s(M))$ of $s(M)$, and $M^\circ$, respectively. Then,

$$s(M)^\perp \subseteq Ker(s(M)) \subseteq M^\circ,$$

and

$$k(s(M)^\perp) = k(Ker(s(M))) = k(M^\circ).$$

**Proof.** It is clear that $s(M)^\perp$ is contained in $Ker(s(M))$. If $a$ is an element of $Ker(s(M))$ then, since $e(x)$ lies in $s(M)$, for all elements $x$ in $M$, $Q(e(x), e(x))a = \{ e(x) a e(x) \} = 0$.

Therefore,

$$P_2(e(x))a = Q(e(x), e(x))^2 a = 0,$$

and

$$x(a) = P_2(e(x)) x(a) = x(P_2(e(x)a)) = 0,$$

and $a$ is contained in $M^\circ$. This completes the first part of the proof. A proof of the second part can be found in [14].

It is now possible to present the most interesting result of this section of the paper.

**Theorem 4.3.** Let $A$ be a JBW$^*$-triple, with predual $A^*$, let $M$ be a subset of $A^*$ having support space $s(M)$ and topological annihilator $M^\circ$, let $k(s(M)^\perp)$, $k(Ker(s(M)))$, and $k(M^\circ)$ be the central kernels of the annihilator $s(M)^\perp$ of $s(M)$, the kernel $Ker(s(M))$ of $s(M)$, and $M^\circ$, respectively, and let

$$J_{M^\circ} = \{ a \in A: D(b,c)a \in M^\circ, \forall b,c \in A \},$$

and

$$I_{M^\circ} = \{ a \in A: Q(b,c)a \in M^\circ, \forall b,c \in A \}.$$

Then, the following results hold.

(i) $J_{M^\circ} = s(M)^\perp \subseteq Ker(s(M)) \subseteq M^\circ$.

(ii) $I_{M^\circ} = k(s(M)^\perp) = k(Ker(s(M))) = k(M^\circ)$.

**Proof.** (i) To show that $J_{M^\circ}$ is contained in $s(M)^\perp$, let $a$ be an element of $J_{M^\circ}$ and let $x$ be an element of $M$. Since $M^\circ$ is a weak$^*$-closed subspace of $A$, it follows from Theorem 4.1 that the element $D(a,a)e(x)$ lies in $M^\circ$, and, hence,

$$x(\{ a a e(x) \}) = 0.$$
By [3, Proposition 1.2], it follows that, for all elements \( x \) of \( M \), the element \( a \) lies in \( A_0(e(x)) \), and, hence, \( a \) lies in the weak*-closed inner ideal \( s(M) \downarrow \).

Now suppose that \( u \) is a tripotent in \( s(M) \downarrow \) and let \( x \) be an element of \( M \). Then the tripotent \( u \) lies in \( A_0(e(x)) \). The tripotents \( u \) and \( e(x) \) are orthogonal, and, hence, compatible, and, therefore the pre-adjoint \( D(u, u)_* \) of the weak*-continuous linear operator \( D(u, u) \) satisfies

\[
D(u, u)_* x = P_2(u)_* P_2(e(x))_* x + \frac{1}{2} P_1(u)_* P_2(e(x))_* x = 0,
\]

since, by compatibility, \( P_2(u)_* P_2(e(x))_* \) and \( P_1(u)_* P_2(e(x))_* \) are projections onto the zero subspace. Hence, for each element \( b \) in \( A \) and each element \( x \) in \( M \),

\[
x (\{ u \ u \ b \}) = x (D(u, u)b) = D(u, u)_* x(b) = 0,
\]

and the element \( D(u, u)b \) is contained in \( M_0 \). Therefore, by Theorem 4.1, the element \( u \) lies in \( J_{M^\circ} \). Since \( s(M) \downarrow \) and \( J_{M^\circ} \) are both inner ideals in \( A \), it follows from [19, Lemma 2.3], that

\[
s(M) \downarrow = \bigcup_{u \in \mathcal{U}(s(M) \downarrow)} A_2(u) \subseteq \bigcup_{u \in \mathcal{U}(J_{M^\circ})} A_2(u) = J_{M^\circ},
\]

and the proof of (i) is complete.

(ii) This follows immediately from Theorem 4.1 and Lemma 4.2. \( \square \)

Similar to the situation that pertains in the case of JB*-triples, when the weak*-closed subspace \( M^\circ \) of the JBW*-triple \( A \) is a subtriple of \( A \), rather more can be said.

**Theorem 4.4.** Let \( A \) be a JBW*-triple, with predual \( A_\ast \), let \( M \) be a subset of \( A_\ast \), having support space \( s(M) \) and topological annihilator \( M^\circ \), let \( k(s(M) \downarrow) \), \( k(Ker(s(M))) \), and \( k(M^\circ) \) be the central kernels of the annihilator \( s(M) \downarrow \) of \( s(M) \), the kernel \( Ker(s(M)) \) of \( s(M) \), and \( M^\circ \), respectively, and let

\[
J_{M^\circ} = \{ a \in A: D(b, c)a \in M^\circ, \ \forall b, c \in A \},
\]

and

\[
I_{M^\circ} = \{ a \in A: Q(b, c)a \in M^\circ, \ \forall b, c \in A \}.
\]

If \( M^\circ \) is a subtriple of \( A \) then the weak*-closed inner ideal \( J_{M^\circ} \) is an ideal in \( A \), and

\[
s(M) \downarrow = J_{M^\circ} = I_{M^\circ} = k(s(M) \downarrow) = k(Ker(s(M))) = k(M^\circ).
\]

**Proof.** Since \( M^\circ \) is a weak*-closed subtriple of \( A \) such that \( k_\circ (M^\circ) \) and \( k(M^\circ) \) coincide, it follows from Theorem 3.7 that \( J_{M^\circ} \) is a weak*-closed ideal of \( A \) contained in \( M^\circ \) and containing the central kernel \( k(M^\circ) \) of \( M^\circ \). Hence, the weak*-closed ideal \( J_{M^\circ} \) coincides with \( k(M^\circ) \), and the result follows from Theorem 4.3. \( \square \)

Since, for any weak*-closed subspace \( L \) of the JBW*-triple \( A \), the double topological annihilator \((L_\circ)^\circ \) coincides with \( L \), Theorems 4.3 and 4.4 apply when \( M^\circ \) is replaced by any weak*-closed subspace \( L \), in which case \( M \) is replaced by the topological annihilator \( L_\circ \).

The central kernel \( k(L) \) of a weak*-closed inner ideal in the JBW*-triple \( A \) has been extensively studied [24,25]. Applying Theorem 4.4 yields a new algebraic characterization of \( k(L) \).
Corollary 4.5. Let \( A \) be a JBW\(^*\)-triple, with predual \( A_* \), let \( L \) be a weak\(^*\)-closed inner ideal in \( A \), let \( L_0, L_1, \) and \( L_2 \), and \( L_{0,*}, L_{1,*}, \) and \( L_{2,*} \), be the Peirce spaces corresponding to \( L \) in \( A \) and \( A_* \), respectively, let \( k(L) \) be the central kernel of \( L \), and let
\[
J_L = \{ a \in A: D(b, c)a \in L, \forall b, c \in A \},
\]
and
\[
I_L = \{ a \in A: Q(b, c)a \in L, \forall b, c \in A \}.
\]
Then,
\[
s(L_1,* \oplus L_{0,*}) = J_L = I_L = k(s(L_1,* \oplus L_{0,*})) = k(L).
\]

Proof. Observing that the topological annihilator \( L_0 \) of \( L \) coincides with \( L_{1,*} \oplus L_{0,*} \), the result is immediate from Theorem 4.4. \( \square \)

It is worth remarking that the case in which the weak\(^*\)-closed subspace \( L \) is a subtriple far from exhausts the interesting situations. For example, if the subset \( M \) of the predual \( A_* \) of the JBW\(^*\)-triple \( A \), consists of the single point \( \{ x \} \), then
\[
s(M) = C e(x), \quad s(M)^\perp = A_0(e(x)), \quad \text{Ker}(s(M)) = A_0(e(x)) \oplus A_1(e(x)), \quad M^\circ = \ker(x),
\]
and the following corollary holds.

Corollary 4.6. Let \( A \) be a JBW\(^*\)-triple with predual \( A_* \), let \( x \) be an element of \( A_* \) having support tripotent \( e(x) \) and kernel \( \ker(x) \), for \( j \) equal to 0, 1, and 2, let \( A_j(e(x)) \) be the Peirce \( j \)-space corresponding to \( e(x) \), and let
\[
I = \{ a \in A: x(Q(b, c)a) = 0, \forall b, c \in A \}.
\]
Then, the central kernels satisfy
\[
I = k(A_0(e(x))) = k(A_0(e(x)) \oplus A_1(e(x))) = k(\ker(x)).
\]

Since it is very rarely the case that the weak\(^*\)-closed inner ideal \( A_0(e(x)) \) is an ideal in the JBW\(^*\)-triple \( A \), this result confirms the strength of the condition in Theorem 4.4 that \( M^\circ \) is a subtriple.

5. C\(^*\)-algebras and W\(^*\)-algebras

The results above are now able to throw some new light upon the theory of C\(^*\)-algebras and W\(^*\)-algebras for the properties of which the reader is referred to [38,39]. Recall that a norm-closed subspace of a C\(^*\)-algebra \( A \) is an M-ideal if and only if it is an algebraic ideal, and a weak\(^*\)-closed subspace of a W\(^*\)-algebra is an M-summand if and only if it is an algebraic ideal. Furthermore, with respect to the multiplication defined, for elements \( a, b, \) and \( c \) of \( A \) by
\[
\{ a \ b \ c \} = \frac{1}{2}(ab^*c + cb^*a),
\]
the C\(^*\)-algebra \( A \) is a JB\(^*\)-triple. Similarly, a W\(^*\)-algebra is a JBW\(^*\)-triple [41].
Theorem 5.1. Let $A$ be a $C^*$-algebra, let $L$ be a norm-closed subspace of $A$, having norm central kernel $k_n(L)$, and let
\[ J_L = \{ a \in A : bc^*a + ac^*b \in L, \forall b, c \in A \}, \]
\[ I_L = \{ a \in A : ba^*c + ca^*b \in L, \forall b, c \in A \}, \]
and
\[ \tilde{I}_L = \{ a \in A : ba^*c \in L, \forall b, c \in A \}. \]
Then, $J_L$ is a norm-closed inner ideal of $A$, such that
\[ J_L = \{ a \in A : aa^*b + ba^*a \in L, \forall b \in A \}, \]
and
\[ I_L = \tilde{I}_L = k_n(L) \subseteq J_L \subseteq L. \]

Proof. Most of the proposition is immediate from Lemma 3.5 and Theorem 3.6. Observe that it is clear that $\tilde{I}_L$ is contained in $I_L$. On the other hand, it can be seen that $\tilde{I}_L$ is a norm-closed algebraic ideal in $A$. Suppose that $J$ is any norm-closed algebraic ideal in $A$ contained in $L$. Since norm-closed algebraic ideals are $^*$-subalgebras, for elements $b$ and $c$ in $A$ and $a$ in $J$, the element $ba^*c$ lies in $J$ and, hence, in $L$. It follows that the element $a$ lies in $\tilde{I}_L$. Hence, $J$ is contained in $\tilde{I}_L$, which is, therefore, the greatest norm-closed ideal in $A$ contained in $L$. Since the sets of norm-closed ideals and M-ideals coincide, it follows that $\tilde{I}_L$ is equal to the norm central kernel $k_n(L)$ of $L$ as required. \(\square\)

Observe that Theorem 3.7 leads to the following result, which applies, for example, when $L$ is a norm-closed $^*$-subalgebra of the $C^*$-algebra $A$.

Theorem 5.2. Let $A$ be a $C^*$-algebra, let $L$ be a norm-closed subtriple of $A$, having norm central kernel $k_n(L)$, and let
\[ J_L = \{ a \in A : bc^*a + ac^*b \in L, \forall b, c \in A \}, \]
\[ I_L = \{ a \in A : ba^*c + ca^*b \in L, \forall b, c \in A \}, \]
and
\[ \tilde{I}_L = \{ a \in A : ba^*c \in L, \forall b, c \in A \}. \]
Then,
\[ I_L = \tilde{I}_L = k_n(L) = J_L \subseteq L. \]

Let $A$ be a $W^*$-algebra with unit $1_A$, and let $\mathcal{P}(A)$ be the complete orthomodular lattice of self-adjoint idempotents in $A$, the ordering being given by $e \leq f$ if and only if $ef$ is equal to $e$, and the orthocomplementation being given by
\[ e \mapsto e' = 1_A - e. \]
Let $Z(A)$ be the commutative $W^*$-algebra that is the algebraic centre of $A$. Then $\mathcal{P}(Z(A))$ coincides with the complete Boolean lattice that is the orthomodular lattice centre $Z\mathcal{P}(A)$ of $\mathcal{P}(A)$. For each element $e$ in $\mathcal{P}(A)$, the central support $c(e)$ of $e$ is defined by
\[ c(e) = \bigwedge \{ z \in Z\mathcal{P}(A) : e \leq z \}. \]
A pair \((e, f)\) of elements of \(\mathcal{P}(A)\) is said to be centrally equivalent if \(c(e)\) and \(c(f)\) coincide. The common central support is denoted by \(c(e, f)\). When endowed with the product ordering, the set \(\mathcal{C}(A)\) of centrally equivalent pairs of elements of \(\mathcal{P}(A)\) forms a complete lattice in which the lattice supremum coincides with the supremum in the product lattice, but, in general, the lattice infimum does not. The results of [18] show that the mapping \((e, f) \mapsto eAf\) is an order isomorphism from \(\mathcal{C}(A)\) onto the complete lattice of weak\(^{*}\)-closed inner ideals in \(A\). The restriction of the mapping to the complete Boolean sublattice \(\mathcal{Z}(A)\) of pairs \((z, z)\), where \(z\) lies in \(\mathcal{Z}(A)\), is an order isomorphism onto the complete Boolean lattice of weak\(^{*}\)-closed ideals in \(A\).

Observe that an element \(u\) in the \(W^*\)-algebra \(A\) is a tripotent if and only if \(uu^*u = u\), or, equivalently, if and only if \(u\) is a partial isometry with initial projection \(h(u)\) equal to \(u^*u\) and final projection \(g(u)\) equal to \(uu^*\). Observe that the Peirce spaces corresponding to \(u\) are given by

\[
\begin{align*}
A_0(u) &= g(u)'Ah(u)', \\
A_1(u) &= g(u)Ah(u)' + g(u)'Ah(u), \\
A_2(u) &= g(u)Ah(u).
\end{align*}
\]

For a subset \(M\) of \(A_\ast\), the annihilator \(s(M)^\perp\) of the support space \(s(M)\) is a weak\(^{*}\)-closed inner ideal in \(A\), which, using [22, Lemma 2.4 and Corollary 2.5] is given by

\[
s(M)^\perp = \bigcap_{x \in M} A_0(e(x)) = \bigcap_{x \in M} g(e(x))'Af(e(x))' = g'Ah',
\]

where \(e(x)\) is the support partial isometry of \(x\) and

\[
g' = \bigwedge_{x \in M} g(e(x))' = \left( \bigvee_{x \in M} g(e(x)) \right)', \tag{5.1}
\]

\[
h' = \bigwedge_{x \in M} h(e(x))' = \left( \bigvee_{x \in M} h(e(x)) \right)'.
\]

Observe that the central supports \(c(g)\) and \(c(h)\) are equal and are given by

\[
c(g, h) = c\left( \left( \bigvee_{x \in M} g(e(x)) \right) \right) = \bigvee_{x \in M} c(e(x)e(x)^*) = \bigvee_{x \in M} c(e(x)^*e(x)).
\]

It is now possible to apply the results of Section 4 to \(W^*\)-algebras.

**Theorem 5.3.** Let \(A\) be a \(W^*\)-algebra, with predual \(A_\ast\), let \(M\) be a subset of \(A_\ast\), having support space \(s(M)\) and topological annihilator \(M^\circ\), let \(k(s(M)^\perp)\), \(k(Ker(s(M)))\), and \(k(M^\circ)\) be the central kernels of the annihilator \(s(M)^\perp\) of \(s(M)\), the kernel \(Ker(s(M))\) of \(s(M)\), and \(M^\circ\), respectively, let

\[
J_{M^\circ} = \{ a \in A : bc^*a + ac^*b \in M^\circ, \forall b, c \in A \},
\]

\[
I_{M^\circ} = \{ a \in A : ba^*c + ca^*b \in M^\circ, \forall b, c \in A \},
\]

and

\[
\tilde{I}_{M^\circ} = \{ a \in A : ba^*c \in M^\circ, \forall b, c \in A \},
\]
and let $g$ and $h$ be the projections in $A$ defined in (5.1)–(5.3), having common central support $c(g, h)$ given by (5.4). Then, the following results hold.

(i) $g'Ah' = J_{M^o} = s(M)^\perp \subseteq \ker(s(M)) \subseteq M^o$.

(ii) $c(g, h)'A = I_{M^o} = \tilde{I}_{M^o} = k(s(M)^\perp) = k(\ker(s(M))) = k(M^o)$.

**Proof.** Much of the proof follows immediately from Theorem 4.3. Observe that it is clear that $\tilde{I}_{M^o}$ is contained in $I_{M^o}$. On the other hand, it can be seen that $\tilde{I}_{M^o}$ is a weak*-*closed ideal of $A$. Suppose that $J$ is any weak*-*closed ideal in $A$ contained in $M^o$. Since weak*-*closed ideals of $A$ are *-subalgebras of $A$, for elements $b$ and $c$ in $A$ and $a$ in $J$, the element $ba^*c$ lies in $J$ and hence $M^o$. It follows that the element $a$ lies in $\tilde{I}_{M^o}$. Hence, $J$ is contained in $\tilde{I}_{M^o}$, which is, therefore, the greatest weak*-*closed ideal in $A$ contained in $M^o$. Since the sets of weak*-*closed ideals and $M$-summands coincide, it follows that $\tilde{I}_{M^o}$ is equal to the central kernel $k(M^o)$ of $M^o$ as required.

Observe that, from (5.2) and (5.3), it can be seen that the weak*-*closed inner ideals $g'Ah'$ and $s(M)^\perp$ coincide, thereby completing the proof of (i). However, it is not necessarily true that the projections $g'$ and $h'$ have a common central support. Therefore, the element of $CP(A)$ corresponding to the weak*-*closed inner ideal $s(M)^\perp$ is $(c(h')g', c(g')h')$. By [24, Theorem 4.1], it follows that the central kernel $k(s(M)^\perp)$ is given by

\[
k(s(M)^\perp) = c((c(h')g')'c((c(g')h')'A = c((c(h') \vee g))'c(c(g') \wedge h)'A
\]

\[
= (c(h') \vee c(g))'c(c(g') \wedge c(h)'A = (c(h') \wedge c(g, h))'c(c(h') \wedge c(g, h)'A
\]

since $c(g, h)'$ is majorized by both $c(g')$ and $c(h')$. This completes the proof of (ii). \qed

The restrictive situation in which the topological annihilator $M^o$ of the subset $M$ of the predual $A_*$ of the $W^*$-algebra $A$ is a subtriple can be considered. This, of course, occurs if, for example, $M^o$ is a *-subalgebra of $A$.

**Theorem 5.4.** Let $A$ be a $W^*$-algebra with predual $A_*$, let $M$ be a subset of $A_*$ having support space $s(M)$ and topological annihilator $M^o$, let $k(s(M)^\perp)$, $k(\ker(s(M)))$, and $k(M^o)$ be the central kernels of the annihilator $s(M)^\perp$ of $s(M)$, the kernel $\ker(s(M))$ of $s(M)$, and $M^o$, respectively, let

\[
J_{M^o} = \{a \in A: bc^*a + ac^*b \in M^o, \forall b, c \in A\},
\]

\[
I_{M^o} = \{a \in A: ba^*c + ca^*b \in M^o, \forall b, c \in A\},
\]

and

\[
\tilde{I}_{M^o} = \{a \in A: ba^*c \in L, \forall b, c \in A\},
\]

and let $g$ and $h$ be the projections in $A$ defined in (5.1)–(5.3). If $M^o$ is a subtriple of $A$ then, the projections $c(h')g$ and $c(g')h$ are central and satisfy

\[
c(h')g + c(h')' = c(g')h + c(g')' = c(g, h), \quad (5.5)
\]

and

\[
c(g, h)'A = s(M)^\perp = J_{M^o} = I_{M^o} = \tilde{I}_{M^o} = k(s(M)^\perp) = k(\ker(s(M))) = k(M^o).
\]
Proof. Since, by Theorem 4.4, the weak*‐closed inner ideal \( g'Ah' \) is an ideal in \( A \), the elements \((c(h')g', c(g'h'))\) and \((c(g, h'), c(g, h'))\) of \( CP(A) \) coincide, and a calculation shows that (5.5) holds. The rest of the result follows from Theorem 5.3. □

References