Note

Norm equality for a basic elementary operator

Mohamed Barraa* and Mohamed Boumazgour

Department of Mathematics, Faculty of Sciences, Semlalia, Marrakesh B.P. 2390, Morocco
Received 28 July 2002
Submitted by R. Curto

Abstract

Let $\mathcal{L}(H)$ be the algebra of bounded linear operators on a Hilbert space $H$. For $A, B \in \mathcal{L}(H)$, define the elementary operator $M_{A,B}$ by $M_{A,B}(X) = AXB$ ($X \in \mathcal{L}(H)$). We give necessary and sufficient conditions for any pair of operators $A$ and $B$ to satisfy the equation $\|I + M_{A,B}\| = 1 + \|A\|\|B\|$, where $I$ is the identity operator on $H$.

Keywords: Norm; Numerical range; Elementary operators

Let $H$ be a complex Hilbert space and let $\mathcal{L}(H)$ be the Banach algebra of all bounded linear operators on $H$. For $A, B \in \mathcal{L}(H)$, let $L_A$ (respectively, $R_B$) denote the left (respectively, right) multiplication by $A$ (respectively, $B$). The basic elementary operator (two-sided multiplication) $M_{A,B}$ induced by the operators $A$ and $B$ is defined by $M_{A,B} = L_AR_B$. An elementary operator on $\mathcal{L}(H)$ is a finite sum $R = \sum_{i=1}^n M_{A_i,B_i}$ of basic ones. A familiar example of elementary operators is the generalized derivation $\delta_{A,B}$ defined by $\delta_{A,B} = L_A - R_B$.

Many facts about the relation between the spectrum of $R$ and spectrums of the coefficients $A_i$ and $B_i$ are known. This is not the case with the relation between the operator norm $R$ and norms of $A_i$ and $B_i$. Apparently, the only elementary operators on a Hilbert space for which the norm is computed are the basic ones and generalized derivations [10]. We refer to [2,4–11] for an intensive study of norms of elementary operators.

Let $A, B \in \mathcal{L}(H)$ and let $I$ denote the identity operator on $H$. It is well known and easy to prove that $\|M_{A,B}\| = \|A\|\|B\|$. Thus we always have $\|I + M_{A,B}\| \leq 1 + \|A\|\|B\|$.

* Corresponding author.

E-mail addresses: barraa@ucam.ac.ma (M. Barraa), boumazgour@ucam.ac.ma (M. Boumazgour).

0022-247X/ – see front matter © 2003 Published by Elsevier Inc.
doi:10.1016/S0022-247X(02)00657-1
In this note we shall give necessary and sufficient conditions for any pair of operators $A$ and $B$ to satisfy the equation $\|I + M_{A,B}\| = 1 + \|A\|\|B\|$.

In order to state our results in detail, we first recall some notation and results from the literature. Let $T \in \mathcal{L}(H)$. Following [10], the maximal numerical range of $T$ is defined by

$$ W_0(T) = \left\{ \lambda \in \mathbb{C} : \text{there exists } \{x_n\} \subseteq H, \|x_n\| = 1 \text{ such that } \lim_n \langle Tx_n, x_n \rangle = \lambda \text{ and } \lim_n \|Tx_n\| = \|T\| \right\}, $$

and its normalized maximal numerical range is given by

$$ W_N(T) = \begin{cases} W_0(T/\|T\|) & \text{if } T \neq 0, \\ 0 & \text{if } T = 0. \end{cases} $$

The set $W_0(T)$ is nonempty, closed, convex, and contained in the closure of the numerical range, see [10].

For $A \in \mathcal{L}(H)$, let $\sigma(A)$ and $\sigma_{ap}(A)$ denote, respectively, the spectrum and approximate point spectrum of $A$.

The next theorem is our main result.

**Theorem 1.** For $A, B \in \mathcal{L}(H)$ the following are equivalent:

1. $\|I + M_{A,B}\| = 1 + \|A\|\|B\|$.
2. $W_N(A^*) \cap W_N(B) \neq \emptyset$.

**Proof.** (1) $\Rightarrow$ (2) Suppose that $\|I + M_{A,B}\| = 1 + \|A\|\|B\|$. Then we can find two sequences $\{X_n\} \subseteq \mathcal{L}(H)$ and $\{x_n\} \subseteq H$ with $\|X_n\| = \|x_n\| = 1$ for each $n$ such that

$$ \lim_n \|X_nx_n + AX_nBx_n\| = 1 + \|A\|\|B\|. $$

Since

$$ \|X_nx_n + AX_nBx_n\| \leq \|X_nx_n\| + \|AX_nBx_n\| \leq 1 + \|A\|\|B\|, $$

it follows that

$$ \lim_n \|AX_nBx_n\| = \|A\|\|B\|. $$

On the other hand, we have for each $n$,

$$ \|X_nx_n + AX_nBx_n\|^2 = \|X_nx_n\|^2 + \|AX_nBx_n\|^2 + 2 \text{Re}\langle X_nx_n, AX_nBx_n \rangle. $$

Consequently, we derive that

$$ \lim_n \|X_nx_n, AX_nBx_n\| = \|A\|\|B\|. $$

Thus $\lim_n \|A^*X_nx_n\| = \|A\|$ and $\lim_n \|X_nBx_n\| = \|B\|$ because $|\langle X_nx_n, AX_nBx_n \rangle| \leq \|A^*X_nx_n\|\|X_nBx_n\|$. For each $n \geq 1$, we have

$$ \|\delta_{A^*, -B}\| \geq \|A^*X_n + X_nB\| \geq \|A^*X_n + X_nBx_n\|. $$

Since $\lim_n \|A^*X_n + X_nBx_n\| = \|A\| + \|B\|$ and $\|\delta_{A^*, -B}\| \leq \|A\| + \|B\|$, we conclude that $\|\delta_{A^*, -B}\| = \|A\| + \|B\|$. Thus, it follows from [10, Theorem 7] that $W_N(A^*) \cap W_N(B) \neq \emptyset$. 

(2) \(\Rightarrow\) (1) Let \(\mu \in W_N(A^*) \cap W_N(B)\). Then there exist two sequences \(\{x_n\}\) and \(\{y_n\}\) in \(H\) such that \(\|x_n\| = \|y_n\| = 1\), \(\lim_n \|A^*x_n\| = \|A\|\), \(\lim_n \|B y_n\| = \|B\|\), \(\lim_n \langle Ax_n, x_n \rangle = \mu \|A\|\), and \(\lim_n \langle By_n, y_n \rangle = \mu \|B\|\). Set also \(\Delta x_n = \alpha_n x_n + \beta_n u_n\), where \(\alpha_n, \beta_n \in \mathbb{C}\), \(u_n \in H\) with \(\|u_n\| = 1\) and \(\langle x_n, u_n \rangle = 0\). We may choose \(u_n\) so that \(\langle A^*x_n, u_n \rangle = \beta_n \geq 0\) for all \(n\).

Set also \(B y_n = \gamma_n y_n + \delta_n v_n\), where \(\gamma_n, \delta_n \in \mathbb{C}\), \(\|v_n\| = 1\), \(\langle y_n, v_n \rangle = 0\) and \(\langle B y_n, v_n \rangle = \delta_n \geq 0\).

Define a sequence \(\{X_n\} \subseteq \mathcal{L}(H)\) by

\[ X_n = \langle \cdot , y_n \rangle x_n + \langle \cdot , v_n \rangle u_n. \]

Then clearly \(\|X_n\| = 1\) for all \(n\), and we have

\[ \langle X_n y_n, AX_n B y_n \rangle = \langle A^* y_n, \gamma_n y_n + \delta_n u_n \rangle = \alpha_n \gamma_n + \beta_n \delta_n. \]

By the definitions of the sequences \(\{x_n\}\) and \(\{y_n\}\), we derive that \(\lim_n |\alpha_n|^2 + \beta_n^2 = \|A\|^2\) and \(\lim_n |\gamma_n| = |\mu| |A|\). Thus, \(\lim_n \beta_n = \sqrt{1 - |\mu|^2} \|A\|\). In a similar way we obtain \(\lim_n \delta_n = \sqrt{1 - |\mu|^2} \|B\|\). Hence,

\[ \lim_n \langle X_n y_n, AX_n B y_n \rangle = \lim_n \alpha_n \gamma_n + \beta_n \delta_n = |\mu|^2 \|A\| \|B\| + (1 - |\mu|^2) \|A\| \|B\| = \|A\| \|B\|. \]

From this we conclude that \(\lim_n \|AX_n B y_n\| = \|A\| \|B\|\). Now, we have for each \(n \geq 1\),

\[ 1 + \|A\| \|B\| \geq \|I + MA, B\| \geq \|X_n + AX_n B\| \geq \|X_n y_n + AX_n B y_n\|. \]

Therefore,

\[ \lim_n \|X_n y_n + AX_n B y_n\| = 1 + \|A\| \|B\| \leq \|I + MA, B\| \leq 1 + \|A\| \|B\|. \]

Consequently,

\[ \|I + MA, B\| = 1 + \|A\| \|B\|. \]

\[ \square \]

**Remark 2.** (i) Let \(A, B \in \mathcal{L}(H)\). It follows from Theorem 1, [10, Theorem 1], and [10, Theorem 8] that \(0 \in W_0(A)\) if and only if \(\|I - MA^* A\| = 1 + \|A\|^2\) if and only if \(\|\delta_{A, B}\| = 2 \|A\|\).

(ii) Also we conclude from Theorem 1 and [10] that the following are equivalent:

(1) \(\|I + MA, B\| = 1 + \|A\| \|B\|\).

(2) \(\|\delta_{A^*, -B}\| = \|A\| + \|B\|\).

(3) \(\|A\| \|B\| \leq \|A - \lambda\| \|B - \lambda\|\) for all \(\lambda \in \mathbb{C}\).

An immediate consequence of Theorem 1 is the following

**Corollary 3.** If \(A \in \mathcal{L}(H)\), then \(\|I + MA, A^*\| = 1 + \|A\|^2\).

Another consequence of Theorem 1 is the following result proved in [1,3].

**Corollary 4.** If \(A \in \mathcal{L}(H)\), then \(\|I + A\| = 1 + \|A\|\) if and only if \(A \in \sigma_{ap}(A)\).
Proof. If $B = I$ in Theorem 1, then we see that $\|I + A\| = 1 + \|A\|$ if and only if $1 \in WN(A^*)$. This is equivalent to the existence of a unit sequence $\{x_n\}_n$ in $H$ such that $\lim_n \langle Ax_n, x_n \rangle = \|A\|$ and $\lim_n \|Ax_n\| = \|A\|$. From this we conclude that $\lim_n \|Ax_n - \|A\|x_n\| = 0$, that is, $\|A\| \in \sigma_{ap}(A)$. 

References