ON THE HOLOMORPHIC RIGIDITY OF LINEAR OPERATORS ON COMPLEX BANACH SPACES

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1. Introduction

In 1931 H. Cartan [4] proved the following uniqueness theorem: Let \( D \subset \mathbb{C}^n \) be a bounded domain and let \( f : D \to D \) be a holomorphic mapping. Then \( f \) is the identity on \( D \) if it has a fixed point \( a \in D \) at which the Jacobian \( f'(a) \) is the identity matrix. Another way of expressing this is as follows: In the space of all holomorphic mappings \( D \to D \) a biholomorphic mapping \( f \) is uniquely determined by the first two terms \( f(a), f'(a) \) of its power series expansion about \( a \). Cartan’s proof uses an iteration argument that can immediately be extended to bounded domains in complex Banach spaces (clearly the Jacobian \( f'(a) \) has to be interpreted as linear operator—the Fréchet derivative \( df(a) \) of \( f \) at \( a \)). In finite as well as in infinite dimensions Cartan’s uniqueness theorem has been the key for many important results.

In the present paper we study holomorphic mappings \( f : B \to D \) between domains in complex Banach spaces that are rigid at \( a \in B \) in the following sense: \( f = g \) for every holomorphic mapping \( g : B \to D \) with \( f(a) = g(a) \) and \( df(a) = dg(a) \). Our main interest is concentrated to the special case where \( B, D \) are the open unit balls of the complex Banach spaces \( E, F \) and where \( a = 0 \) is the origin. Then, if \( f : B \to D \) with \( f(0) = 0 \) is rigid at the origin it necessarily must be of the form \( f = L|B \) for a linear operator \( L : E \to F \) with \( \|L\| = 1 \). Such linear operators \( L \) we also call rigid.

The paper is organized as follows:

In Section 2 we present the basic background for holomorphic mappings between domains \( U, V \) in complex Banach spaces as needed later for the linear case. In particular, for every \( m \in \mathbb{N} \) and every holomorphic mapping \( f : U \to V \) we define \( f \) to be \( m \)-rigid at the point \( a \in U \) if \( f \) is uniquely determined within the space of all holomorphic mappings \( U \to V \) by all derivatives of order \( < m \) at \( a \). This, in case \( m = 2 \), is just what occurs for biholomorphic \( f \) in Cartan’s uniqueness theorem. We also introduce the more general notion of infinitesimal \( m \)-rigidity at \( a \) in case \( f \) is uniquely determined within families of holomorphic mappings \( g_t : U \to V \) depending holomorphically on a one-dimensional parameter \( t \). Using a theorem of Kakutani we show that every biholomorphic automorphism of a bounded domain is infinitesimally 1-rigid, and is even infinitesimally 0-rigid if \( U \) is the open unit ball of a complex Banach space.
In Section 3 we study linear operators $L : E \to F$ with $\|L\| = 1$ and call $L$ rigid if the induced map between the open unit balls is 2-rigid at the origin. In the case of Hilbert spaces $E$, $F$ for instance, $L$ is rigid if and only if $L$ is a (not necessarily surjective) isometry.

In Section 4 we introduce for every given linear operator $L : E \to F$ of norm 1 and every $m \in \mathbb{N}$ numerical invariants $\alpha_m \in [0, 1]$ that measure the non-rigidity of $L$ in connection with homogeneous polynomials of degree $m$. It turns out that $L$ is rigid if and only if $\alpha_m$ vanishes. Furthermore, $(\alpha_m)_{m \in \mathbb{N}}$ is an increasing sequence, and $L$ is rigid if and only if $\alpha_m = 0$ for all $m$ (i.e. the limit $\alpha_\infty$ vanishes). Besides the invariants $\alpha_m$ we also introduce invariants $\pi_m$ that measure certain eccentricities.

In Section 5 we determine the invariants $\alpha_m$ and estimate $\pi_m$ for some special examples. Also, in the case of contractive projections $L$ we relate rigidity properties of $L$ with smoothness properties of the unit spheres.

In Section 6 we introduce various types of tangent spaces and correlate them to the rigidity problem.

In Section 7 we apply the methods to JB*-triples. These are generalizations of operator algebras where the algebra product is replaced by a certain ternary product, the Jordan triple product. Our main result—Theorem 7.14—solves completely the rigidity problem for $w^*$-closed inner ideals in JBW*-triples, the triple generalizations of $W^*$-algebras.

**NOTATION .** Throughout, $E$ and $F$ are complex Banach spaces with open unit balls $B \subset E$ and $D \subset F$. The notation $E \subset F$ means that $E$ carries the induced norm from $F$, i.e. $B = D \cap F$. By $\mathcal{L}(E, F)$ we denote the Banach space of all bounded linear operators $E \to F$. Furthermore $\mathcal{L}(E) := \mathcal{L}(E, E)$ is the Banach algebra of all continuous endomorphisms and $E^* := \mathcal{L}(E, \mathbb{C})$ is the dual of $E$. The group of all invertible operators in $\mathcal{L}(E)$ is denoted by $\text{GL}(E)$.

The $\ell^p$-sum of $E$ and $F$ will be denoted by $E \oplus_p F$, that is $E \oplus F$ with norm satisfying $\|(z, w)\| = \max(\|z\|, \|w\|)$ if $p = \infty$ and $\|(z, w)\|^p = \|z\|^p + \|w\|^p$ if $1 \leq p < \infty$. For complex Hilbert spaces we denote the inner product always by $(z|w)$ where the conjugate linear variable is $w$.

The boundary of $B$ (the unit sphere in $E$) is denoted by $\partial B$. The subset of all extreme boundary points of $B$ is denoted by $\partial_e B$ and $\partial_e \mathbb{C}$ is the set of all complex extreme boundary points of $B$. Always $\Delta := \{t \in \mathbb{C} : |t| < 1\}$ is the open unit disc and $\mathbb{T} := \partial \Delta$ is the circle group. The set $\mathbb{N}$ of natural numbers always includes 0.

2. **Rigid holomorphic mappings**

For an open subset $U$ of the Banach space $E$ a mapping $f : U \to F$ is called **holomorphic** if for every $a \in U$ the Fréchet-derivative $df(a) \in \mathcal{L}(E, F)$ exists.
Then it is known that the derivative \( df : U \to \mathcal{L}(E, F) \) again is holomorphic and the second derivative \( d^2 f = df \) takes values in \( \mathcal{L}(E, \mathcal{L}(E, F)) \), which as Banach space can be identified in a natural way with the space of all bounded bilinear mappings \( E^2 \to F \). More generally, the \( n \)-th derivative \( d^n f(a) \) exists as a bounded symmetric \( n \)-linear mapping \( E^n \to F \) for every \( n \in \mathbb{N} \) (by definition \( d^n f = f \)).

For arbitrary subsets \( S \subseteq E \) and \( T \subseteq F \) a mapping \( f : S \to T \) is called holomorphic if there exists an open subset \( U \subset E \) and a holomorphic mapping \( h : U \to F \) with \( S \subset U \) and \( f = h|S \). The space of all holomorphic mappings \( S \to T \) will be denoted by \( \text{Hol}(S, T) \). With \( \text{Aut}(S) \subset \text{Hol}(S, S) \) we denote the group of all biholomorphic automorphisms of \( S \). For every point \( a \in S \) denote by
\[
\circ_a(f) = \sup\{n \in \mathbb{N} : \|f(z)\| = O(\|z - a\|^n) \quad \text{as} \quad z \to a\}
\]
the vanishing order of the holomorphic mapping \( f : S \to T \) at \( a \). For any pair \( f, g \in \text{Hol}(S, T) \) then \( \circ_a(f, g) := \circ_a(f - g) \) is the order of contact at \( a \). In case \( a \) is an inner point of \( S \) the condition \( \circ_a(f, g) \geq m \) is equivalent to \( d^n f(a) = d^n g(a) \) for all \( n < m \).

**Definition 2.1.** A subset \( A \subseteq E \) is called a set of determinacy in \( E \) if for every open connected neighbourhood \( U \subseteq E \) of \( A \) the restriction operator \( \text{Hol}(U, \mathbb{C}) \to \text{Hol}(A, \mathbb{C}) \) is injective.

The following statement is an easy consequence of the Hahn–Banach theorem.

**Lemma 2.2.** Let \( f : U \to F \) be a holomorphic mapping for a domain \( U \subseteq E \). Let furthermore \( R \subseteq F \) be a closed linear subspace and \( A \subseteq U \) a set of determinacy in \( E \). Then \( f(U) \subseteq R \) if \( f(A) \subseteq R \).

**Proof.** Fix \( \lambda \in F^* \) with \( \lambda|_R = 0 \). Then the holomorphic function \( \lambda \circ f \) vanishes on \( A \).

**Lemma 2.3.** Let \( A \subseteq E \) be a balanced set of determinacy in \( E \). Then also \( A \cap B \) is a set of determinacy in \( E \).

**Proof.** Let \( U \) be an open connected neighbourhood of \( A \cap B \) and fix a holomorphic function \( f : U \to \mathbb{C} \) vanishing on \( A \cap B \). We have to show that \( f = 0 \). Since \( A \cap B \) contains the origin we may assume without loss of generality that \( U = B \). Expand \( f \) into a series \( \sum f_n \), where every \( f_n : E \to \mathbb{C} \) is homogeneous of degree \( n \) (compare Section 4). Every \( f_n \) vanishes on \( A \cap B \) and hence also on \( A \), i.e. \( f_n = 0 \) for all \( n \).

**Definition 2.4.** Let \( S \subseteq E \) and \( T \subseteq F \) be connected subsets, let \( m \geq 0 \) be an integer and let \( \Delta \subseteq \mathbb{C} \) be the open unit disc. Then \( f \in \text{Hol}(S, T) \) is called \( m \)-rigid at \( a \in S \) if the equality \( g = f \) holds for every \( g \in \text{Hol}(S, T) \) with \( \circ_a(f, g) \geq m \). If \( f \) is \( m \)-rigid at every point of \( S \) we call \( f \) \( m \)-rigid everywhere. The mapping \( f \) is called infinitesimally \( m \)-rigid at \( a \in S \) if \( g_i \equiv f \) holds for
every family \((g_t)_{t \in \Delta}\) in \(\text{Hol}(S, T)\) satisfying the following properties: (i) \(g_0 = f\), (ii) \(\sup_t \|g_t\| > m\) for all \(t \in \Delta\) and (iii) \(g_t\) depends holomorphically on \(t\), i.e. the mapping \(\Delta \times S \to T\) defined by \((t, s) \mapsto g_t(s)\) is holomorphic, where \(\Delta \times S\) is considered as a subset of \(\mathbb{C} \times E\).

Clearly, stronger versions of infinitesimal rigidity could be introduced by requiring for instance that \((g_t)\) depends real analytically or only \(C^r\) on the parameter \(t\). Our main interest however is in the case of linear operators where all these notions are equivalent to simple rigidity (at least if \(m \geq 2\), compare 4.3).

All degrees of rigidity may occur: \(f \in \text{Hol}(\Delta, \overline{\Delta})\) defined by \(f(z) = z^m\), \(m \in \mathbb{N}\), is \(1\)-rigid but not \(m\)-rigid at 0 \(\in \Delta\).

We start with some trivial statements

**Remark 2.5.** Suppose that \(f : S \to T\) is a holomorphic mapping and \(g : T = 1, h : S \to \tilde{S}\) are biholomorphic mappings. Then for every \(a \in S\), every integer \(m \geq 0\) and \(f := g \circ f \circ h^{-1} : \tilde{S} \to T\) the following holds

(i) \(f\) is (infinitesimally, resp.) \(m + 1\)-rigid at \(a\) if \(f\) is (infinitesimally, resp.) \(m\)-rigid at \(a\).

(ii) \(f\) is (infinitesimally) \(m\)-rigid at \(a\) if \(f\) is \(m\)-rigid at \(a\).

(iii) \(f\) is (infinitesimally, resp.) \(m\)-rigid at \(h(a)\) if \(f\) is (infinitesimally, resp.) \(m\)-rigid at \(a\).

**Remark 2.6.** Suppose that the holomorphic mappings \(f_i : S \to T_i\) are (infinitesimally, resp.) \(m\)-rigid at \(a \in S\) for \(i = 1, 2\). Then also \(f = (f_1, f_2) : S \to T_1 \times T_2\) is (infinitesimally, resp.) \(m\)-rigid at \(a\).

As a consequence of Liouville’s theorem every holomorphic mapping \(E \to T\) is \(1\)-rigid everywhere if \(T \subset F\) is bounded. Also, every biholomorphic mapping \(f : U \to V\) is \(2\)-rigid everywhere as a consequence of Cartan’s uniqueness theorem if \(U \subset E\) is a bounded domain. We even have

**Proposition 2.7.** Suppose that \(f : U \to V\) is a biholomorphic mapping where \(U \subset E\) is a bounded domain. Then \(f\) is infinitesimally \(1\)-rigid everywhere.

**Proof.** We may assume that \(U = V\) and that \(f\) is the identity on \(U\). Fix \(a \in U\) and consider on \(E\) the Carathéodory norm \(v\) defined by

\[ v(v) = \sup\{|df(a)v| : f \in \text{Hol}(U, \Delta), f(a) = 0\} \]

for all \(v \in E\). Then \(V := (E, v)\) also is a complex Banach space. Now suppose that \((g_t)\) is a family in \(\text{Hol}(U, U)\) depending holomorphically on \(t \in \Delta\) with \(g_t(a) = a\) and \(g_0 = f\). Then \(t \mapsto dg_t(a)\) defines a holomorphic mapping \(\Delta \to \mathcal{L}(V)\) with \(\|dg_t(a)\| \leq 1\) for all \(t\). But \(dg_0(a) = \text{id}\) is an extreme point of the unit ball in the Banach algebra \(\mathcal{L}(V)\) by a result of Kakutani and hence \(dg_t(a) = \text{id}\) for all \(t \in \Delta\) — compare [9] p. 74 and p. 69. But then Cartan’s uniqueness theorem gives \(g_t = f\) for all \(t \in \Delta\).
For the open unit balls of Hilbert spaces Proposition 2.7 can be generalized, compare also 3.6.

**Example 2.8.** Suppose that $E \subset F$ are complex Hilbert spaces with open unit balls $B \subset D$. Then the canonical injection $f : B \hookrightarrow D$ is 2-rigid everywhere. Also, $f$ is infinitesimally 1-rigid everywhere.

**Proof.** Denote by $H$ the orthogonal complement of $E$ in $F$ and suppose that $\sigma_h(f, g) \geq 2$ for $g \in \text{Hol}(B, D)$ and some $a \in B$. Then there are holomorphic maps $\varphi : B \rightarrow B, h : B \rightarrow H$ with $g = \varphi + h$. From $\sigma_h(f, \varphi) \geq 2$ we derive $\varphi = f$. But then $\lim_{z \to \partial B} h(z) = 0$ implies $h = 0$. That $f$ is infinitesimally 1-rigid at $a = 0 \in B$ follows as in the proof of 2.7 since the canonical injection $E \hookrightarrow F$ is an extreme point of the unit ball in $\mathcal{L}(E, F)$. The statement for arbitrary $a \in B$ then is a consequence of 2.5 since there are $h \in \text{Aut}(B)$ and $g \in \text{Aut}(D)$ with $a = h(0)$ and $f = g \circ f \circ h^{-1}$.

For certain domains Cartan’s uniqueness theorem can be strengthened. The first part of the following statement is due to Harris [13], compare also the more general Proposition 6.8.

**Proposition 2.9.** Let $B$ be the open unit ball of the complex Banach space $E$ and let $f : B \rightarrow B$ be a holomorphic mapping with $df(0) = \text{id}$. Then also $f(0) = 0$ holds and hence $f$ is the identity on $B$. Furthermore, $f$ is infinitesimally 0-rigid everywhere on $B$.

**Proof.** Put $c := f(0)$ and start with the special case $B = \Delta$. Then Schwarz lemma applied to the function $g(z) := (f(z) - c)/(\overline{\sigma f(z)} - 1)$ in $\text{Hol}(\Delta, \Delta)$ gives $1 = (f'(0)) \leq (1 - c \overline{\lambda})$ and hence $c = 0$. In the general case choose $a \in \partial B$ and $\lambda \in E^*$ in such a way that $\|c\|a = c$ and $\|\lambda\| = 1 = \lambda(a)$ holds. Then $h \in \text{Hol}(\Delta, \Delta)$ defined by $h(z) = \lambda \circ f(za)$ satisfies $h'(0) = 1$ and hence $\|c\| = h(0) = 0$. Finally, suppose that $(g_t)$ is a family in $\text{Hol}(B, B)$ depending holomorphically on $t \in \Delta$ with $g_0 = f$. Then by Cauchy’s inequalities $\|dg_t(0)\| \leq 1$ holds and as before we derive $g_t = \text{id}$ for all $t$. But then also $g_t(0) = 0$ holds, i.e. $g_t = f$ for all $t$.

**Example 2.10.** Let $E$ be a complex Hilbert space of finite dimension $> 1$. Let $B \subset E$ be the open unit ball and let $S := \partial B$ be the unit sphere of $E$. Suppose that $f : S \rightarrow S$ is a holomorphic mapping and that $a \in S$ is a given point. Then it is known that $f$ extends to a holomorphic mapping $f : \overline{B} \rightarrow \overline{B}$. Therefore, if $f$ is not constant, its restriction to $B$ is a proper holomorphic map $B \rightarrow B$. But then by [1] $f|B$ is already an automorphism of $B$. But it is known that every $f \in \text{Aut}(\overline{B})$ is linear fractional and is uniquely determined by the three derivatives $f(a), df(a)$ and $d^2 f(a)$ within $\text{Aut}(\overline{B})$. From this we get: Either $f$ is constant, and then $f$ is 1-rigid on $S$ everywhere, or $f \in \text{Aut}(\overline{B})$, and then $f$ is 3-rigid everywhere—but not 2-rigid in any point of $S$. For more general situations of this type compare [23].
3. Rigid linear operators

Proposition 2.11. Suppose that the holomorphic mapping $f : U \to V$ is $m$-rigid at $a \in U$ and that $Q$ is an arbitrary domain. Then the mapping $g : U \times Q \to V$ defined by $g(z, w) = f(z)$ is $m$-rigid at every point $(a, q) \in U \times Q$, provided the bounded holomorphic functions on $U$ separate the points (this happens for instance if $U$ is a bounded domain).

Proof. Fix $q \in Q$ and assume that the holomorphic mapping $h : U \times Q \to V$ satisfies $\varphi(a, q)(h, g) \geq m$. We have to show that $g = h$. Consider

$$\Omega := \{(z, w) \in U \times Q : \exists \varphi \in \text{Hol}(U, Q) \text{ with } \varphi(a) = q, \varphi(z) = w\}$$

and fix $(z, w) \in \Omega$ together with a corresponding $\varphi$. Then $\gamma \in \text{Hol}(U, V)$ defined by $\gamma(t) = h(t, \varphi(t))$ satisfies $\varphi(a, q)(\gamma, g) \geq m$, i.e. $\gamma = f$ and hence $g(z, w) = h(z, w)$ for all $(z, w) \in \Omega$. Let $X \subset U$ be an open non-void subset that can be separated from $a$ via bounded holomorphic functions on $U$ (for instance $X = U \setminus \{a\}$). To every $x \in X$ there is a neighbourhood $Y \subset Q$ of $q$ with $(x, y) \in \Omega$ for all $y \in Y$. This implies by the identity theorem for holomorphic functions that $h$ coincides with $g$ on $X \times Q$ and hence on all of $U \times Q$.

The condition on $U$ in 2.11 cannot be omitted as the following counter example shows: Every holomorphic map $\mathbb{C} \to \Delta$ is constant and hence $m$-rigid everywhere while no constant map $\mathbb{C} \times \Delta \to \Delta$ is $m$-rigid at any point.

Corollary 2.12. Let $U$ be a bounded domain. Then for every domain $Q$ the canonical projection $U \times Q \to U$ is 2-rigid everywhere.

Corollary 2.13. Suppose that $U_i$ is a bounded domain and that the holomorphic mapping $f_i : U_i \to V_i$ is $m$-rigid at $a_i \in U_i$ for $i = 1, 2$. Then also $f_1 \times f_2 : U_1 \times U_2 \to V_1 \times V_2$ is $m$-rigid at $a = (a_1, a_2)$.

3. Rigid linear operators

In the following let $B \subset E$ and $D \subset F$ always be the open unit balls. We are mainly interested in holomorphic maps $f : B \to D$ with $f(0) = 0$. Then the derivative $L := df(0)$ satisfies $\|L\| \leq 1$. Therefore, if $f$ is 2-rigid at $0 \in B$ we must have $f = L|B$ and $\|L\| = 1$. It is clear that $f$ never can be 1-rigid at 0 although it may be infinitesimally 1-rigid at 0. This motivates the following definition for the linear case.

Definition 3.1. The linear operator $L \in \mathcal{L}(E, F)$ is called rigid if the induced map $B \to rD$ is 2-rigid at $0 \in B$ where $r = \|L\|$, that is, if for every holomorphic mapping $f : B \to rD$ with $f(0) = 0$ and $df(0) = L$ necessarily $f = L|B$ follows. We call $L$ strictly rigid if for every holomorphic $f : B \to rD$ the conclusion $f = L|B$ already from the only assumption $df(0) = L$ follows. In case $E \subset F$ is a subspace we call $E$ (strictly) rigid in $F$ if the canonical injection $E \hookrightarrow F$ has this property.
Clearly, the study of rigidity of linear operators $L$ always can be reduced to the case $\|L\| = 1$. The slightly more general situation in 3.1 avoids complicated constants sometimes. Rigidity in our sense is closely related to the vector valued Schwarz lemma in the following form: Suppose $f : B \to D$ with $f(0) = 0$ is holomorphic. Then the derivative $L = df(0)$ has norm $\leq 1$ and $\|f(z)\| \leq \|z\|$ holds for all $z \in B$. The question then is: Under what conditions can $f = L|B$ be concluded? Many authors studied this question under the additional assumption that $L$ is isometric. It is clear that in case $L$ isometric the Banach space $E$ can without loss of generality be identified with a subspace $E \subset F$ via $L$. To simplify notation we will frequently do so.

We start with the simple situation $\dim E = 1$, compare also 5.1 for a more quantitative statement and also 7.2 for a generalization to higher dimensional $E$.

**Lemma 3.2.** Let $L : \mathbb{C} \to F$ be a linear operator with $a : L(1) \in \partial D$. Then $L$ is rigid if and only if $a$ is a complex extremal boundary point of $D$. Also, $L$ is strictly rigid if and only if $a$ is a (real) extremal boundary point of $D$.

**Proof.** Put $f := L|B$ and assume that $a$ is not complex extremal. Then there is a vector $v \in F$ with $v \neq 0$ and $a + \Delta v \subset \partial D$. But then $g(z) = z(a + zv)$ defines a holomorphic map $g : \Delta \to D$ with $dg(0) = L$, i.e. $L$ is not rigid. Assume on the contrary that $a \in \partial B$ is complex extremal and that $g : \Delta \to D$ is a holomorphic mapping with $g(0) = 0$ and $dg(0) = L$. Then $h(z) := g(z)/z$ defines a holomorphic function $h : \Delta \to \overline{D}$ with $h(0) = a$, i.e. $h \equiv a$ and hence $f = g$, compare [9] p. 69 or [21]. Now assume that $a \in \partial B$ is not extremal. Then there is a vector $v \in E$ with $v \neq 0$ and $\|a \pm v\| = 1$. For every $\alpha, \beta \in \mathbb{C}$ then $2(\alpha a + \beta v) = (\alpha + \beta)(a + v) + (\alpha - \beta)(a - v)$ implies $2\|\alpha a + \beta v\| \leq |\alpha + \beta| + |\alpha - \beta|$. Define $g : \Delta \to F$ with $dg(0) = L$ by $2g(z) = 2za + (1 + z^2)v$. Then $\|4g(z)\| \leq |1 + z|^2 + |1 - z|^2 = 2(1 + z \overline{z}) < 4$ for all $z \in \Delta$ shows $g(\Delta) \subset D$, i.e. $L$ is not strictly rigid. It remains to show that $a \in \partial_c D$ implies strict rigidity of $L$. But this follows from [12] p. 27–28 and also from Théorème 3.6 in [18].

**Corollary 3.3.** Suppose that $E$ is arbitrary and that $A \subset \partial B$ is a set of determinacy in $E$. Then every $L \in \mathcal{L}(E, F)$ with $\|L\| = 1$ and $L(A) \subset \partial_c D$ is rigid. If in addition $L(a) \in \partial_c D$ holds for some $a \in A$ then $L$ is even strictly rigid. In particular, in case $\partial D = \partial_c D$ every linear isometry $L : E \to F$ is strictly rigid.

In case of $\dim E = 2$ the situation is already much more complicated as the following example indicates. In particular, rigid linear operators of norm 1 need not be isometric even if they are bijective. Notice that by 2.9 every surjective linear isometry between complex Banach spaces is strictly rigid.

**Example 3.4.** Let $I$ be an arbitrary set of cardinality $> 1$. For $1 \leq p < q \leq \infty$ fixed consider $E := \ell^p(I)$ and $F := \ell^q(I)$. Then the canonical injection $L : E \to F$ is not rigid if $q = \infty$. In case that $I$ is finite, the inverse operator
$L^{-1} : F \rightarrow E$ however always is rigid. In case $p > 1$ the operator $L^{-1}$ is strictly rigid.

**Proof.** Write every $z \in B$ as tuple $z = (z(i))$, fix $j \in J$ and let $v = e_j$ (i.e. $v(i) = \delta_{ij}$ for all $i \in I$) in the following. Choose furthermore an integer $m \geq 2$ together with a constant $c > 0$ such that $(1 - t^p) \leq (1 - ct^m)^p$ holds for all $t \in [0,1]$. Then $g(z) = L(z) + c(z(k))^{m}v$ defines a holomorphic mapping $g : B \rightarrow D$ with $c_0(f, g) = m$ for every $k \neq j$ in $I$ if $q = \infty$. Now suppose that $I$ is of finite cardinality $n$ and put $S := (rL)^{-1}$ for $r := n^{1/p - 1/q} = \|L^{-1}\|$. Then $A = \{z \in S : |z(j)| = |z(j)|$ for all $i \in I\}$ is a set of determinancy in $F$. By 3.3 therefore $L^{-1}$ is rigid since $S(A) \subset \partial_{cr}(B)$.

**Lemma 3.5.** Suppose that $E \subset F$ is a closed linear subspace and that $A$ is a set of determinancy in $E$. Suppose that for every $a \in A$ there is a closed linear subspace $E_a \subset E$ containing the point $a$ such that $E_a$ is rigid in $F$. Then also $E$ is rigid in $F$.

**Proof.** Let $f : B \rightarrow D$ be a holomorphic mapping with $c_0(f, g) \geq 2$ for the canonical injection $g : B \hookrightarrow D$. For every $a \in A$ then $f(a) = g(a)$ since $E_a$ is rigid in $F$. But then $f = g$ by 2.2.

**Proposition 3.6.** Let $H, K$ be complex Hilbert spaces and let $L \in \mathcal{L}(H, K)$ have norm $1$. Then $L$ is rigid if and only if $L$ is a (not necessarily surjective) isometry.

**Proof.** Assume that $L$ is rigid and let $L = V|L|$ be the polar decomposition, where $|L| = (L^*L)^{1/2}$ and $V \in \mathcal{L}(H, K)$ is a partial isometry with $\ker(V) = \ker(L)$. Let $[E(d\lambda)]$ be the spectral measure of $|L|$, so

$$|L| = \int_{[0,1]} \lambda E(d\lambda).$$

We claim that $|L| = \id$, that is $E([0, \alpha]) = 0$ for all $\alpha < 1$. Indeed, if $E([0, \beta]) \neq 0$ for some $\beta < 1$, choose a unit vector $a \in H$ fixed under the projection $P := E([0, \beta])$ and denote by $Q := \id - P$ the complementary projection. Define a holomorphic map $h : B \rightarrow B$ by $h(z) = (1 - \beta)(z|a)^2\bar{a}$ where $B \subset H$ is the open unit ball. Since $V^*V(a) = a$ holds and $V^*$ is an isometry on $V(H) = L(H)$ we have for all $z \in B$

$$\|L(z) + V(h(z))\|^2 = \||L|(z) + h(z)\|^2 \leq \|P|L|(z) + (1 - \beta)(P(z)a)^2\|^2 + \|Q|L|(z)\|^2$$

$$\leq \|P|L|(z)\|^2 + (1 - \beta)^2\|P(z)\|^4 + 2(1 - \beta)\|P|L|(z)\|\cdot \|P(z)\|\cdot \|Q|L|(z)\|^2 \leq (\beta^2 + (1 - \beta)^2)\|P(z)\|^2 + 2\beta(1 - \beta)\|P(z)\|^2 + \|Q(z)\|^2 = \|P(z)\|^2 + \|Q(z)\|^2 = \|z\|^2 < 1.$$
This is a contradiction to the rigidity of $L$ since $V(h(a)) \neq 0$. Therefore $|L| = \text{id}$ and $L = V$ is an isometry. The converse statement follows from 2.8.

**Corollary 3.7.** Suppose that $H, K$ are complex Hilbert spaces. Then a linear operator $L : H \to K$ is a surjective isometry if and only if $L$ and $L^*$ are rigid.

**Corollary 3.8.** Suppose that $E$ is a complex Hilbert space of finite dimension and that $a \in B$ is a given point. Then a holomorphic mapping $f : B \to B$ is rigid at $a$ if and only if $f \in \text{Aut}(B)$.

**Proof.** Suppose that $f$ is rigid at $a$. Since $\text{Aut}(B)$ acts transitively on $B$ we may assume that $a = 0$. But then $f = L|B$ for some linear isometry of $E$ by 3.6. Because of finite dimension $L$ must be surjective, i.e. $f \in \text{Aut}(B)$.

A projection $P \in L(F)$ is called contractive if $\|P\| \leq 1$ holds and $P$ is called bicontractive if in addition also $\text{id} - P$ is a contractive projection. Furthermore, a contractive projection $P$ from $F$ onto $E \subset F$ is called neutral if $\|P(z)\| = \|z\|$ always implies $z \in E$ for all $z \in F$. Neutrality is not invariant under $\ell^\infty$-sums, more precisely, suppose that $P_i$ is a neutral projection on the complex Banach space $F_i$ with range $E_i$ for $i = 1, 2$. Then $P := P_1 \times P_2$ is a contractive projection on $F = F_1 \oplus_\infty F_2$, but in general $P$ is not neutral. However, it is easily seen that $A := \{(x, y) \in E : \|x\| = \|y\|\}$ is a set of determinacy in $E$ and that $P$ is almost neutral in the following sense.

**Definition 3.9.** We call a contractive projection $P$ from $F$ onto $E$ almost neutral if there exists a set of determinacy $A$ in $E$ such that $z \in E$ for every $z \in F$ with $\|P(z)\| = \|z\|$ and $P(z) \in A$.

A nontrivial example for an almost neutral projection is obtained as follows (compare also 7.11). Let $H \subset K$ be complex Hilbert spaces with $H \neq K$ and $\text{dim}(H) \geq 2$. Let furthermore $p : K \to H$ be the orthogonal projection. Then $P(z) = p \circ z$ defines a contractive projection from $L(H, K)$ onto $L(H)$ that is not neutral. With $A \subset L(H)$ the unitary group we see that $P$ is almost neutral.

**Proposition 3.10.** Suppose that there exists an almost neutral projection $P$ from $F$ onto $E$. Then $E$ is rigid in $F$.

**Proof.** Choose $A \subset E$ as in definition 3.9. Then we may assume that $A$ is balanced and hence that $A \subset B$ holds by 2.3. Let $f : B \to D$ be a holomorphic mapping with $\alpha_0(f, g) \geq 2$ for the canonical injection $g : B \hookrightarrow D$. Then $g = P \circ f$ holds by Cartan’s uniqueness theorem. By Schwarz lemma we have for all $z \in B$ 

$$\|z\| = \|g(z)\| = \|Pf(z)\| \leq \|f(z)\| \leq \|z\|$$

and hence $\|Pf(z)\| = \|f(z)\|$, i.e. $f(z) \in E$ for all $z \in A$. This implies $f(z) \in E$ for all $z \in B$ by 2.2 and thus $f = g$. 


A little more can be said in 3.10 if the projection $P$ is strongly neutral in the following sense: For every sequence $(z_n)$ in $F$ with \( \lim \|P(z_n)\| = \lim \|z_n\| < +\infty \) always \( \lim \|z_n - P(z_n)\| = 0 \) holds. Then it is not difficult to see that the canonical injection $B \hookrightarrow D$ is rigid everywhere. As an example of this situation we may take for every $1 \leq p < \infty$ any $\ell^p$-sum $F = E \oplus_p W$ with the canonical projection $P : F \rightarrow E$ along $W$.

4. Quantitative study of rigidity

As before, let $E, F$ be complex Banach spaces with open unit balls $B, D$. We call a continuous mapping $f : E \rightarrow F$ a homogeneous polynomial of degree $n$ if there is a symmetric $n$-linear mapping $q : E^n \rightarrow F$ with $f(z) = q(z, z, \ldots, z)$ for all $z \in E$—or equivalently—if $f$ is holomorphic and satisfies $f(tz) = t^n f(z)$ for all $z \in E$ and all $t \in \mathbb{C}$. The $n$-linear map $q$ is uniquely determined by $f$ and can be recovered from $f$ with the polarization formula

$$q(z_1, z_2, \ldots, z_n) = (2^n n!)^{-1} \sum_{\varepsilon \in \{\pm 1\}^n} \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n f(\varepsilon_1 z_1 + \varepsilon_2 z_2 + \cdots + \varepsilon_n z_n).$$

Also $d^n f(a) = n! q$ holds for every $a \in E$. Put

$$\mathcal{P}_n := \mathcal{P}_n(E, F) := \{\text{homogeneous polynomials } E \rightarrow F \text{ of degree } n\}$$

and denote by $\mathcal{P}^n$ the Banach space of all bounded holomorphic functions $f : B \rightarrow F$ with norm $\|f\| = \sup_{z \in B} \|f(z)\|$. Then $\mathcal{P}_n$ as well as $\mathcal{P}^n$ can be considered as closed linear subspaces of $\mathcal{H}^\infty(B, F)$ for every $n \in \mathbb{N}$ and every $f \in \mathcal{H}^\infty(B, F)$ has a unique expansion

$$f = \sum_{n=0}^{\infty} f_n \quad \text{with } f_n \in \mathcal{P}_n \text{ for all } n \in \mathbb{N},$$

converging uniformly on every subball $sB \subset B, s \in \Delta$. Every $f_n : E \rightarrow F$ is given by

$$f_n(z) = \int_\mathbb{T} (rt)^{-n} f(tz) \, dt$$

with $z \in E$ and $r > 0$ satisfying $r_z \in B$ and $dt$ the (normalized) Haar measure on $\mathbb{T}$. In particular, $f \mapsto f_n$ defines a contractive projection $\mathcal{P}_n$ from $\mathcal{H}^\infty(B, F)$ onto $\mathcal{P}_n$ for every $n \in \mathbb{N}$.

In the following let $\mathcal{E} := \{ f \in \mathcal{H}^\infty(B, F) : f(0) = 0 \}$ and denote by $\mathbb{B}$ the closed unit ball of $\mathcal{E}$. For every $t \in \Delta$ define the commutation operator $C_t \in \mathcal{L}(\mathcal{E})$ by $f(z) \mapsto f(tz)/t$ if $t \neq 0$ and by $f \mapsto df(0) \in \mathcal{P}_1$ if $t = 0$.

**Lemma 4.1.** $t \mapsto C_t$ defines a semigroup homomorphism $\Delta \rightarrow \mathcal{L}(\mathcal{E})$ with
∥\mathcal{C}_t∥ = 1 for all \( t \). For every \( f = \sum_{n=1}^{\infty} f_n \in \mathcal{E} \) with \( f_n \in \mathcal{P}_n \) the family \( (g_t) \) in \( \mathcal{E} \) with
\[
g_t := \mathcal{C}_t(f) = \sum_{n=1}^{\infty} f_n^{n-1}
\]
depends holomorphically on \( t \in \overline{\mathbb{D}} \) and \( t \mapsto g_t \) defines a holomorphic curve \( \Delta \to \mathcal{E} \). In case \( ∥f∥ = ∥f_1∥ \) also \( ∥g_t∥ = ∥f_1∥ \) holds for all \( t \).

Proof. Fix \( f \in \mathcal{B} \). Then by Schwarz lemma we have \( ∥f(z)∥ = ∥z∥ \) for all \( z \in \mathcal{B} \) and hence \( ∥\mathcal{C}_t∥ \leq 1 \). From \( \mathcal{P} \subset \text{Fix}(\mathcal{C}_t) \) we thus get \( ∥\mathcal{C}_t∥ = 1 \). The last statement follows from \( g_1 = f \), \( g_0 = f_1 \) and \( ∥g_s∥ \leq ∥g_t∥ \) if \( |s| \leq |t| \).

Fix an operator \( L \in \mathcal{L}(\mathcal{E}, \mathcal{F}) \) with \( ∥L∥ = 1 \) in the following. We introduce some numerical invariants that measure the size of non-rigidity of \( L \): Let \( \mathcal{A} = \mathcal{A}(L) \) be the set of all \( f \in \mathcal{E} \) with \( df(0) = 0 \) and \( ∥L + f∥ \leq 1 \). Then \( L \) is rigid if and only if \( \mathcal{A} = 0 \). It is easily seen that \( \mathcal{A} \) is closed convex in \( \mathcal{E} \) and also is invariant under every operator \( \mathcal{C}_t \). From 4.1 we see that \( L + \mathcal{A} \) is contained in the boundary \( \partial \mathcal{B} \) of \( \mathcal{B} \). For every \( m \geq 2 \) let
\[
\mathcal{A}_m := \mathcal{A} \cap \mathcal{P}_m, \quad \alpha_m := \sup_{f \in \mathcal{A}_m} ∥f∥ \text{ and } \pi_m := \sup_{f \in \mathcal{A}} ∥P_m(f)∥,
\]
where \( P_m \) is the projection operator \( (\sum_n f_n) \mapsto f_m \) as defined above. Then clearly \( \alpha_m \leq \pi_m \leq 1 \) holds and every \( \mathcal{A}_m \) is a balanced subset of \( \mathcal{B} \), that is
\[
\mathcal{A}_m = \{ f \in \mathcal{P}_m : ∥L + tf∥ \leq 1 \text{ for all } t \in \overline{\mathbb{D}} \}.
\]
We may use this equation to define \( \mathcal{A}_m \) together with \( \alpha_m \) also for the remaining cases \( m = 0 \) and \( m = 1 \).

Lemma 4.2. \( \alpha_m \leq \alpha_{m+1} \) for all \( m \in \mathbb{N} \) and in particular, the limit \( \alpha_\infty := \lim_m \alpha_m \leq 1 \) exists.

Proof. Fix \( f \in \mathcal{A}_m \) and let \( \Lambda \) be the unit ball of \( \mathcal{E}^* \). For every \( \lambda \in \Lambda \), \( z \in \mathcal{B} \) and \( t \in \overline{\mathbb{D}} \) we have
\[
∥L(z) + t\lambda(z)f(z)∥ \leq 1,
\]
that is, \( \lambda \cdot f \) is contained in \( \mathcal{A}_{m+1} \). Then the Hahn–Banach theorem implies
\[
∥f∥ = \sup_{\lambda \in \Lambda} ∥\lambda \cdot f∥,
\]
which proves the statement.

The operators \( \mathcal{C}_t \) may be generalized in the following way. Let \( \mu \) be a regular complex Borel measure on \( \overline{\mathbb{D}} \) with finite total variation and put
\[
\hat{\mu}(k) := \int_{\overline{\mathbb{D}}} t^{-k} d\mu(t)
\]
for every integer \( k \leq 0 \). Then the operator \( C_\mu := \int_{\Delta} C_1 d\mu(t) \in \mathcal{L}(\mathcal{E}) \) satisfies \( \|C_\mu\| \leq \|\mu\| \) and for \( f = \sum_{n=1}^{\infty} f_n \) as before we have

\[
C_\mu(f) = \sum_{n=1}^{\infty} \hat{\mu}(1-n) f_n.
\]

In particular, if \( \mu \) is a probability measure on \( \Delta \), we have \( \|C_\mu\| = 1 = \hat{\mu}(0) \) and

\[
\|L + C_\mu(f)\| = \left| \int_{\Delta} C_1(L + f) d\mu(t) \right| \leq \int_{\Delta} \|L + f\| d\mu(t) \leq \int_{\Delta} d\mu(t) = 1
\]

for every \( f \in \mathcal{A} \), i.e. \( C_\mu \) maps the spaces \( \mathcal{A} \) and \( \mathcal{A}_m \) into themselves for every \( m \in \mathbb{N} \). In the following proposition we use measures \( \mu \) that are concentrated on \( \mathbb{T} \subset \Delta \).

**Proposition 4.3.** For every \( m \geq 2 \) and every \( s \in \mathbb{C} \) with \( |s| \leq 1/2 \) the operator \( sP_m \) maps \( \mathcal{A} \) into \( \mathcal{A}_m \). In particular, \( \alpha_m \leq \pi_m \leq 2\alpha_m \) holds and for every fixed \( m \geq 2 \) the factor 2 in this estimate is the best constant valid for all operators \( L \) uniformly.

**Proof.** Consider \( d\mu(t) = \text{Re}(1 + 2st^{1-m}) dt \), where \( dt \) is the Haar measure on \( \mathbb{T} \). Then \( \mu \) is a probability measure on \( \mathbb{T} \) with \( \hat{\mu}(1-n) = 0 \) for all \( n \geq 2 \) except \( \hat{\mu}(1-m) = s \). This implies \( sP_m(f) = C_\mu(f) \in \mathcal{A}_m \) for all \( f \in \mathcal{A} \). The last claim will be verified in example 5.2.

**Proposition 4.4.** The following conditions are equivalent for every \( L \in \mathcal{L}(E, F) \) with \( \|L\| = 1 \).

(i) \( L \) is a complex extreme point of the unit ball in \( H^\infty(B, F) \).

(ii) \( L \) is rigid.

(iii) \( \alpha_m = 0 \) for all \( m \) (i.e. \( \alpha_\infty = 0 \)).

**Proof.** (i) \( \implies \) (ii) Suppose that \( L \) is not rigid. Then there is \( f = \sum f_n \in B \) with \( f \neq f_1 = L \). Since \( g_t := C_t(f) \) depends holomorphically on \( t \in \Delta \) we get \( \|L + s(g_t - g_0)\| \leq 1 \) for all \( s \in \mathbb{C} \) and \( t \in \Delta \) with \( |2st| \leq 1 - |t| \) by [9] p. 68. Then \( g_t \neq g_0 = L \) for \( t \neq 0 \) implies that (i) does not hold.

(ii) \( \implies \) (iii) is trivial.

(iii) \( \implies \) (i) Suppose that (i) does not hold. Then there is a non-zero holomorphic map \( f : B \to D \) with \( \|L + tf\| \leq 1 \) for all \( t \in \Delta \). After replacing \( f \) by the function \( z \mapsto \lambda(z)^2 f(z) \) for a suitable \( \lambda \in E^* \) we may assume that \( f \in \mathcal{A} \). By 4.3 there is an \( m \geq 2 \) with \( \alpha_m > 0 \). Finally, Lemma 4.2 implies \( \alpha_\infty > 0 \).

The equivalence of (i) and (ii) in 4.4 is already contained in [12] p. 25, compare also [6] p. 75. In [12] it also has been shown that \( L \) (using our language) is strictly rigid if it is a (real) extreme point of the unit ball in \( H^\infty(B, F) \).
DEFINITION 4.5. For every $m \geq 1$ the linear operator $L \in \mathcal{L}(E, F)$ is called $m$-extreme if $L$ is an extreme point of the unit ball in $P^m(E, F)$. In case of a complex extreme point we call $L$ complex $m$-extreme.

LEMMA 4.6. For every $m \geq 2$ the conditions 'm-extreme', 'complex m-extreme' and $'\alpha_m = 0'$ are equivalent. Furthermore, $'\alpha_1 = 0'$ is equivalent to 'complex 1-extreme'.

The set $A \subset \mathcal{E}$ is convex and contains the origin. In general, $A$ it is not circular (compare 5.2 for an example).

LEMMA 4.7. Suppose that $A$ is circular. Then the projection $P_m$ maps $A$ onto $A_m$ and in particular, $\alpha_m = \pi_m$ holds for every $m \geq 2$.

Proof. Fix $f \in A$. Then

$$\|L(z) + t^{-m} f(tz)\| = \|L(tz) + t^{1-m} f(tz)\| \leq 1$$

holds for all $t \in T$ and $z \in B$. This implies

$$\|L(z) + P_m f(z)\| = \left\| \int_T (L(z) + t^{-m} f(tz)) \, dt \right\| \leq \int_T \|L(z) + t^{-m} f(tz)\| \, dt \leq 1$$

and hence $P_m(f) \in A_m$.

5. Some numerical estimates

In the following, for given $L \in \mathcal{L}(E, F)$ with $\|L\| = 1$, the spaces $A, A_m$ and the numerical invariants $\alpha_m, \pi_m$ have the same meaning as in the preceding section. We want to get estimates on these invariants in the special situation where one of the spaces $E, F$ has dimension 1. We start with the case $E = \mathbb{C}$ and a quantitative version of Lemma 3.2.

LEMMA 5.1. Let $L : \mathbb{C} \to F$ be a linear operator with $\|L\| = 1$. Then

$$\alpha_m = \alpha_0 = \sup \{ \|v\| : v \in F, \|a + tv\| \leq 1 \} \quad \text{for all } t \in \Delta$$

for all $m \in \mathbb{N}$ where $a := L(1) \in \partial D$.

Proof. Suppose that $f \in A_m$. Then $f(z) = z^m v$ for some $v \in F$ with $\|a + tv\| \leq 1$ for all $t \in T$. This shows $v \in A_0$ and thus $\alpha_m \leq \alpha_0$.

EXAMPLE 5.2. Let $K$ be a locally compact Hausdorff space and $F := C_0(K)$. Fix a function $a \in F$ with $\|a\| = 1$ and put $r := \|1 - |a|\|$. Let $L : \mathbb{C} \to F$ be defined by $L(z) = za$. Then by 3.2 the operator $L$ is rigid if and only if $r = 0$ holds. For $v \in F$ the condition $\|a + tv\| \leq 1$ for all $t \in \Delta$ is equivalent to $|a| + |v| \leq 1$ which implies $\alpha_m = 1 - r$ for all $m \in \mathbb{N}$. We claim that in general
the invariants $\pi_m$ differ from $\alpha_m$, i.e. the set $\mathcal{A}$ is not circular. To see this, define for every $c \in D$ the holomorphic mappings $g_c : \Delta \to D$ by
\[
g_c(z) := \frac{c + z}{1 + \overline{c}z} = c + (1 - \overline{c}c) \sum_{m=1}^{\infty} (-\overline{c})^{m-1} z^m,
\]
where every $z \in \mathbb{C}$ is identified with the constant function $\equiv z$ on $K$. Then for all $0 < s, t < 1$ and $z \in \Delta$ we have
\[
\|g_{sa}(z) - g_{ta}(z)\| = \left\|\frac{(s - t)(a - z^2 \overline{a})}{(1 + sz\overline{a})(1 + tz\overline{a})}\right\| \leq 2(1 - |z|)^{-2}|s - t|.
\]
This implies that the local uniform limit
\[
g := \lim_{s \to 1} g_{sa} \in \text{Hol}(\Delta, \overline{D})
\]
extists. Now consider the $F$-valued function $f$ on $\Delta$ defined by
\[
f(z) := zg(z) - L(z) = (1 - a\overline{a}) \sum_{m=2}^{\infty} (-\overline{a})^{m-2} z^m.
\]
Then $f$ is contained in $\mathcal{A}$ and hence
\[
\pi_m \geq r^{m-2}(1 - r^2) \quad \text{for all } m \geq 2.
\]
In particular, $\pi_2 > \alpha_2$ holds if $0 < r < 1$. Consequently, $\mathcal{A}$ is not circular in this case. Also, because of $\lim_{r \to 1} r^{m-2}(1 - r^2)/(1 - r) = 2$ the example shows that for every fixed $m \geq 2$ the estimate $\pi_m \leq 2\alpha_m$ in Proposition 4.3 cannot be improved with a universal constant $< 2$.

Next we consider the case $F = \mathbb{C}$, that is, when $L$ is a linear form on $E$. For every pair of vectors $a, v \in E$ with $\|a + tv\| \geq \|a\|$ for all $t \in \mathbb{C}$ put
\[
\delta_a(v) := \limsup_{t \to 0} \frac{\log(\|a + tv\| - \|a\|)}{\log |t|} \in [1, +\infty],
\]
where $t$ runs in $\mathbb{C}^*$. Then $\delta_a(sv) = \delta_a(v)$ holds for all $s \in \mathbb{C}^*$ and also $\delta_a(0) = +\infty$.

Now let $L : E \to \mathbb{C}$ be a linear form with $\|L\| = 1$ in the following. Assume that there exists a unit vector $a \in E$ with $L(a) = 1$. Then every $z \in E$ can be uniquely written as $z = (u, v)$ with $u = L(z)$ and $v = z - ua \in V := \ker(L)$. We would like to relate rigidity properties of $L$ to smoothness properties of the unit sphere of $E$ at the point $a = (1, 0)$.

**Proposition 5.3.** For every integer $m \geq 2$ and $\delta := \inf_{v \in V} \delta_a(v)$ we have
\[
(i) \quad L \text{ is } m\text{-extreme (i.e. } \alpha_m = 0) \text{ if } m < \delta,
\]
(ii) \( L \) is not \( m \)-extreme if \( m > \delta \) and \( V \) has the following property: To every \( w \in V \) there is a non-zero linear form \( \lambda \in V^* \) such that \( (u, v) \mapsto (u, \lambda(v)w) \) defines a linear operator of norm 1 on \( E \). This condition is always satisfied if \( E \) has dimension 2.

Proof. (i) Suppose that \( h \in \mathcal{A} \) is homogeneous of degree \( m \). For every \( (u, v) \in B \) and every \( t \in \mathbb{T} \) we have \(|u + th(u, v)| \leq \|(u, v)\|\) and hence

\[ |u| + |h(u, v)| \leq \|(u, v)\|. \]

Fix \( 0 < u < 1 \) in the following and put \( w := v/u \). Dividing by \( u \) then gives

\[ |u|^{m-1}h(1, w) \leq (1, w) - 1 \]

for all \( w \in V \) near \( 0 \in V \). Fix an arbitrary vector \( v \in V \) and put \( p(t) := u^{m-1}h(1, tv) \) for all \( t \in \mathbb{C} \). Then \( p \) is a polynomial of degree \( \leq m \) in \( t \). Choose \( \beta > m \) together with a sequence \( (t_n) \) in \( \Delta \setminus \{0\} \) satisfying

\[ \frac{\log(\|1, t_n v\| - 1)}{\log |t_n|} \geq \beta \quad \text{for all } n \]

and \( \lim t_n = 0 \). This implies

\[ \|(1, t_n v)\| - 1 \leq |t_n|^\beta \quad \text{and hence } \quad |p(t_n)| \leq |t_n|^\beta \]

for all \( n \) big enough. Since \( p \) has degree \( < \beta \) this implies \( p = 0 \) and hence \( h(1, v) = 0 \) for all \( v \in V \). But this means \( h = 0 \).

(ii) Because of \( m > \delta \) there exists \( r > 0 \) and a non-zero vector \( w \in V \) with \(|t|^m \leq \|(1, tw)\| - 1 \) for all \( t \in r \Delta \). Choose \( \lambda \in V^* \) as above. We may assume that \(|\lambda(v)| \leq 1\) holds for all \( (u, v) \in B \). There exists \( c > 0 \) such that

\[ 2c \leq 1 \quad \text{and } \quad 1 + c|t|^m \leq \|(1, tw)\| \quad \text{if } 0 \leq |t| \leq 2. \]

Consider the homogeneous polynomial \( h(u, v) = c\lambda(v)^m \) of degree \( m \) on \( E \). We claim that \( h \in \mathcal{A}_m \), i.e. \(|u + h(u, v)| \leq 1 \) for all \( (u, v) \in B \). Indeed, in case \(|\lambda(v)| < 2|u|\) we have

\[ |u + c\lambda(v)^m| \leq |u|(1 + c|u|^{m-1}\lambda(v/u)|^m) \leq |u|(1 + c|\lambda(v/u)|^m) \leq |u| \cdot \|(1, \lambda(v/u)w)\| = \|(v, \lambda(v)w)\| < 1. \]

In case \(|\lambda(v)| \geq 2|u|\) we have \(|u + c\lambda(v)^m| \leq |\lambda(v)/2| + c < 1 \), which proves the claim.

**Proposition 5.4.** Let \( V \) be a complex Banach space and put \( E := \mathbb{C} \oplus_p V \) for fixed \( 1 \leq p \leq +\infty \). Let \( L : E \to \mathbb{C} \) be the canonical projection and \( a = (1, 0) \). Then for every \( v \in V \subset E \) with \( v \neq 0 \) we have \( \delta_a(v) = p \). In particular, \( L \) is \( m \)-extreme if \( m < p \). Since \( V \) satisfies the condition in 5.3.ii, the operator \( L \) is not \( m \)-extreme if \( m > p \). Actually, we have \( \alpha_m \geq 1/p \) if \( m = p \) and \( \alpha_m = 1 \) if \( m > p \). In case \( V = \mathbb{C} \) and \( p = 2 \) the equality \( a_2 = 1/2 \) holds.
Proof. Fix a linear form $\lambda \in V^*$ with $\|\lambda\| = 1$ and define the homogeneous polynomial $h : E \to \mathbb{C}$ by $h(u, v) := c\lambda(v)^m$ for $c > 0$ to be determined. Then $\|h\| = c$ is clear. By elementary calculus we see: $h \in A_m$ if $m > p$ and $c = 1$ or if $m = p$ and $c = 1/p$. In case $V = \mathbb{C}$ and $p = 2$ one shows that $A_2 = \{(u, v) \mapsto ct^2 : |c| \leq 1/2\}$.

6. Tangent spaces

We start with an example that motivates the following definitions.

Example 6.1. Let $E$ be a complex Hilbert space with open unit ball $B$ and let $F := E \oplus E$ with norm

$$\|(z, w)\| = \sup_{t \in \mathbb{R}} \|((\cos t)z + (\sin t)w)\|.$$ 

Denote by $P$ the projection on $F$ defined by $P(z, w) = (z, 0)$ and identify $E$ with $P(F)$ in the obvious way. The projection $P$ is bicontractive but not almost neutral. Indeed, $\text{Re}(z|w) = 0$ and $\|w\| \leq \|z\|$ implies $\|(z, w)\| = \|z\|$. Our methods so far do not guarantee that $E \subset F$ is rigid. To get this, suppose that $f : E \to E$ is a homogeneous polynomial of degree $m \geq 2$ satisfying

$$\|(z, f(z))\| \leq 1 \quad \text{for all } z \in B.$$ 

This is easily seen to be equivalent to $\text{Re}(f(z)|z) = 0$ for all unit vectors $z \in E$ and hence for all $z \in E$ since $f$ is homogeneous. Geometrically this means that every vector $f(z)$ is tangent at $z$ to the sphere with radius $\|z\|$ about the origin.

The same holds for $if$ in place of $f$, i.e.

$$f(z)|z) = 0 \quad \text{for all } z \in E.$$ 

But then polarization gives $(f(z)|w) = 0$ for all $z, w \in E$, i.e. $f = 0$ and therefore $E$ is rigid in $F$ by Proposition 4.4. It can be shown that $F$ is isometrically isomorphic to the complex Banach space of all $\mathbb{R}$-linear operators $X \to E$, where $X$ is a real Hilbert space of real dimension 2.

Let $F$ be an arbitrary complex Banach space with open unit ball $D$. For every $a \in F$ denote by $S_a$ the set of all $\lambda \in F^*$ with $\lambda(a) = \|\lambda\| \cdot |a|$ and $\|\lambda\| = \|a\|$. Then $S_a$ is a non-void convex subset of $F^*$ with $S_a = \overline{tS_a}$ for all $t \in \mathbb{C}$ and hence also $S_a(v) := \{\lambda(v) : \lambda \in S_a\}$ is convex in $\mathbb{C}$ for every $v \in F$. Put

$$T_a^\mathbb{R} := \{v \in F : S_a(v) \subset i\mathbb{R}\} \quad \text{and} \quad T_a := \{v \in F : S_a(v) = \{0\}\}.$$ 

Then the $\mathbb{R}$-linear subspace $T_a^\mathbb{R}$ for $a \neq 0$ may be considered as the real tangent space at $a$ to the sphere $\{v \in F : \|v\| = \|a\|^\alpha\}$ and $T_a = T_a^\mathbb{R} \cap iT_a^\mathbb{R}$ is called the complex tangent space at $a$. We call $a \in F$ a smooth point if $S_a$ consists of a single functional or equivalently if $F = \mathbb{C}_a + T_a$. For every smooth $a \in F$ denote by $s_a$ the unique functional in $S_a$. For instance, if $F$ is a complex Hilbert space, then every $a \in F$ is smooth and $s_a(v) = (v|a)$ holds for all $a, v \in F$. 
Our definition of tangent space implies in particular $T_0 = F$ for the origin. This simplifies later notations and also means that for $T_a$ only the case $a \neq 0$ counts. The following characterization of tangent spaces in terms of differentiability conditions seems to be known.

**Remark 6.2.** For every $0 \neq a \in F$ the vector $v \in F$ is in $T^a_F$ (in $T_a$ respectively) if and only if
\[
\lim_{t \to 0} \frac{\|a + tv\| - \|a\|}{t} = 0
\]
holds where $t$ runs in $\mathbb{R}$ (in $\mathbb{C}$ respectively).

Simple examples show that $\{(a, v) \in F^2 : v \in T_a\}$ is not closed in $F^2$ in general. Therefore, denote by $C_a$ the set of all $v \in F$ such that there exist sequences $(a_n), (v_n)$ in $F$ with $a = \lim a_n, v = \lim v_n$ and $v_n \in T_{a_n}$ for all $n$. Then $C_a$ is a closed complex cone in $F$ with $T_a \subset C_a$. For every subset $A \subset F$ we put
\[
T_A := \bigcap_{a \in A} T_a \quad \text{and} \quad C_A := \bigcap_{a \in A} C_a.
\]
To indicate the dependence on $F$ we also write $T_a(F), T_A(F)$ and $C_A(F)$ instead of $T_a, T_A$ and $C_A$. For arbitrary complex Banach spaces $E \subset F \subset R$ the identity $T_E(F) = F \cap T_E(R)$ is clear by the Hahn–Banach theorem. For every contractive projection $P$ from $F$ onto $E$ and every $a \in E$ we have $P(T_a(F)) = T_a(E)$. Also, for every $a \in F$ and every skew-hermitian operator $\delta \in L(F)$, i.e. $\|\exp(t\delta)\| = 1$ for all $t \in \mathbb{R}$, the vector $\delta(a)$ belongs to the tangent space $T^a_F$.

**Lemma 6.3.** For every closed linear subspace $E \subset F$ the space $T_E$ is a closed linear subspace of $F$ with $E \cap T_E = 0$. Furthermore, $R := E + T_E$ is closed in $F$ and the projection $R \to E$ along $T_E$ is contractive.

**Proof.** For every $a \in E$ and $v \in T_a$ we have $\|a + v\| \geq \|a\|$. In particular, $a = 0$ if $a + v = 0$, i.e. $E \cap T_E = 0$. The projection $P : R \to E$ along $T_E$ is contractive. This implies for every Cauchy sequence $(z_n)$ in $R$ that also $(Pz_n)$ and $(z_n - Pz_n)$ are Cauchy sequences in $E$ and $T_E$, respectively. Therefore $(z_n)$ converges in $R$ and thus $R$ is closed in $F$.

We call the linear subspace $E \subset F$ smooth in $F$ if $F = E + T_E$.

**Example 6.4.** Let $K$ be a locally compact Hausdorff space and $F := C_0(K)$. For every unit vector $a \in F$ put
\[
\Sigma_a := \{s \in K : |a(s)| = 1\}.
\]
Then $S_a$ is the space of all linear forms
\[
f \mapsto \int_{\Sigma_a} f(s)\overline{a(s)} \, d\mu(s)
\]
where \( \mu \geq 0 \) is a regular Borel measure on \( \Sigma_a \) with \( \mu(\Sigma_a) = 1 \). In particular, if \( E \subset F \) is a closed linear subspace and

\[
\Omega := \{ s \in K : |a(s)| = 1 \text{ for some unit vector } a \in E \}
\]
then \( T_E(F) = \{ f \in F : f|\Omega = 0 \} \). Therefore, \( T_E(F) = 0 \) if and only if \( \Omega \) is dense in \( K \). Also, \( a \in E \) is smooth if and only if \( T_a = C_a \) holds. The subspace \( E \) is smooth in \( F \) if and only if the restriction operator \( E \to C_0(\Omega) \) is surjective and then in particular \( E \) has to separate the points of \( \Omega \).

**Definition 6.5.** The complex Banach space \( E \) is called a JB*-triple if the group \( \text{Aut}(B) \) acts transitively on the unit ball \( B \subset E \). More generally, for an arbitrary complex Banach space \( F \) with open unit ball \( D \) a closed linear subspace \( E \subset F \) is called a JB*-subtriple of \( F \) if the group \( \{ g \in \text{Aut}(D) : g(B) = B \} \) acts transitively on \( B \). Then clearly \( E \) is a JB*-triple by itself.

The name JB*-triple comes from the fact that for every JB*-subtriple \( E \subset F \) there is a natural triple product mapping \( \{ \} : F \times E \times F \to F \) such that \( \{zaw\} \) is symmetric bilinear in \( (z, w) \in F^2 \), antilinear in \( a \in E \) and such that for every \( a \in E \) the polynomial \( a - [az] \in \mathcal{T}^2(F, F) \) is a complete holomorphic vector field on \( D \) tangent to the subspace \( E \subset F \), compare [3] and [15].

**Example 6.6.** Let \( F \) be a C*-algebra or more generally a JB*-algebra with unit \( e \), compare [11]. Then \( F \) is also a JB*-triple. Furthermore, the self-adjoint part \( J := \{ z \in F : z^* = z \} \) is a JB-algebra and \( T_E^{R}(F) = iJ = \{ z \in F : z^* = -z \} \) (compare 7.8). In particular, \( T_E(F) = 0 \) holds for every closed linear subspace \( E \subset F \) with \( e \in E \).

**Definition 6.7.** We say that the pair \( E \subset F \) of complex Banach spaces satisfies

(i) \( \text{Property } P \) if \( f(E) \subset T_E(F) \) holds for every holomorphic mapping \( f : E \to F \) satisfying \( f(z) \in T_z(F) \) for all \( z \in E \).

(ii) \( \text{Property } Q \) if there exists a complex Banach space \( R \supset F \) such that \( E \) is a JB*-subtriple of \( R \).

The assumptions in 6.7 may be weakened in several ways: From the power series expansion of holomorphic functions it is clear that in 6.7.i instead of arbitrary holomorphic mappings \( f \) only homogeneous polynomials \( f \) have to be checked. Also, the condition \( f(z) \in T_z \) only has to be assumed for all \( z \in A \) where \( A \) is some set of determinacy in \( E \). Clearly, \( E \) is a JB*-triple if Property \( Q \) holds.

**Proposition 6.8.** Suppose that \( E \subset F \) are complex Banach spaces and denote by \( L : E \hookrightarrow F \) the canonical injection. Suppose furthermore that \( f : B \to D \) is a holomorphic mapping with \( df(0) = L \). Then \( f(0) \in T_E(F) \) and \( g(z) \in T_z(F) \) holds for all \( z \in B \) and \( g := f - L \). In particular, if \( T_E(F) = 0 \)
and $E \subset F$ satisfies Property $P$, we have $f = L|B$ and hence $E$ is strictly rigid in $F$.

Proof. For every $a \in \partial B$ and $\lambda \in S_a$ consider the function $h \in \text{Hol}(\Delta, \Delta)$ defined by $h(t) = \lambda \circ f(ta)$. Then $h'(0) = 1$ implies $h(t) = t$ and hence $\lambda(g(ta)) = 0$ for all $t$. This means $g(z) \in T_z$ for all $z \in B$ and in particular $f(0) = g(0) \in T_E(F)$. Now suppose that $T_E(F) = 0$ holds and that $E$ is not rigid in $F$. Then we may assume that $g \neq 0$ is a homogeneous polynomial of degree $m \geq 2$ by Proposition 4.3. Clearly, $g(z) \in T_z$ holds for all $z \in E$ by homogeneity and hence Property $P$ cannot be satisfied.

Obviously every linear subspace $E \subset F$ of dimension 1 satisfies Property $P$. Further examples are obtained in the following way.

LEMMA 6.9. Suppose that $P$ is a contractive projection from $F$ onto $E$. Let $f \in \text{Hol}(E, F)$ satisfy $f(z) \in T_z(F)$ for all $z \in E$. Then $f(E) \subset \ker P$.

Proof. $g := P \circ f$ satisfies $g(z) \in T_z(E)$ for all $z \in E$ which implies $g = 0$ by the special case $E = F$ of the following proposition.

PROPOSITION 6.10. The pair $E \subset F$ satisfies Property $P$ if $E$ is smooth in $F$.

Proof. Because of 6.3 and 6.9 we only have to consider the special case $E = F$. Fix a holomorphic map $f : E \to E$ with $f(a) \in T_a$ for all $a \in \partial B$. Then $f$ is a complete holomorphic vector field on the unit ball $B \subset E$ by [19], compare also [20], p. 28. But $f$ has the same property and hence is also complete on $B$, i.e. $f = 0$.

COROLLARY 6.11. Let $E$, $W$ be arbitrary complex Banach spaces and $F = E \oplus_p W$ the $\ell^p$-sum for $1 \leq p \leq \infty$. Then the pair $E \subset F$ satisfies Property $P$.

Proof. Let $f \in \text{Hol}(E, F)$ satisfy $f(z) \in T_z(F)$ for all $z \in E$ and denote by $P : F \to E$ the canonical projection along $W$. In case $p = 1$ we have $T_z(F) = T_z(E)$ for all $z \in E$ and the statement follows by 6.10. In case $p > 1$ the subspace $E$ is smooth in $F$.

For every measure space $(X, \mu)$ and every $p$ with $2 \leq p < \infty$ proposition 6.10 applied to $E = F = L^p(X, \mu)$ gives $f = 0$ for every holomorphic function $f : E \to E$ satisfying

$$\int_X |f(z)|^p |z|^{p-2} d\mu = 0$$

for all $z \in E$. In case $p = 2$ this is trivial (compare the reasoning in 6.1) and for $p > 2$ a direct proof also can be obtained by taking real derivatives with respect to $z$ and then considering their complex linear as well as their complex antilinear parts. Notice that every $a \in E$ is smooth and that for $a \neq 0$ the corresponding
supporting functional $s_a$ is given by

$$s_a(v) = \|a\|^{2-p} \int_X v|a|^{p-2} \, d\mu.$$ 

All these considerations remain valid for $1 < p < 2$ if for every $a \in E$ the function $|a|^{p-2}$ on $X$ is interpreted in an appropriate way.

Property $P$ does not always hold.

**Example 6.12.** Let $E$, $W$ be complex Banach spaces and let $A \in \mathcal{L}(E)$ with $\|A\| = 1$ be an operator such that $E_1 := \{z \in E : \|Az\| = \|z\|\}$ is a linear subspace with $0 \neq E_1 \neq E$. Let $F$ be the Banach space $E \oplus W$ with norm given by

$$\|(z, w)\| = \max(\|z\| \cdot \|Az\| + \|w\|)$$

for all $z \in E$ and $w \in W$. Then for all $z \in E \subset F$ we have

$$T_z(F) = \begin{cases} T_z(E) & \text{if } z \in E_1 \setminus \{0\} \\ T_z(E) \oplus W & \text{otherwise.} \end{cases}$$

Therefore, if $0 \neq \lambda \in E^*$ satisfies $\lambda(E_1) = 0$ and $0 \neq v \in W$ is a given vector, then $f(z) := \lambda(z)v \in T_z(F)$ defines a holomorphic map $f : E \to F$ with $f(E) \not\subset T_E(F) = \{0\}$, but clearly $f(E) \subset C_E(F)$ holds.

**Proposition 6.13.** Suppose that $E \subset F$ satisfies Property $Q$ and that $f : B \to F$ is a holomorphic mapping with

$$\lim_{z \to a} \lambda \circ f(z) = 0$$

for every $a \in \partial B$, $\lambda \in S_a$ and $z$ running over the open unit ball $B$ of $E$. Then $f(B) \subset T_E(F)$.

**Proof.** Because of $T_E(F) = F \cap T_E(R)$ for every JB*-triple $R \supset F$ we may assume without loss of generality that $F$ is a JB*-triple containing $E$ as a subtriple. Fix $a \in \partial B$ and $\lambda \in S_a$. Then we also have $\lim_{z \to a} \lambda \circ f(tz) = 0$ for all $t \in \mathbb{T}$. Therefore, if we put $g(s) := \lambda \circ f(sa)$ for $s \in \Delta$, the holomorphic function $g : \Delta \to \mathbb{C}$ satisfies $\lim_{|s| \to 1} g(s) = 0$, i.e. $g \equiv 0$ and hence $f(sa) \in T_a$ for all $s \in \Delta$. This shows $f(0) \in T_E := T_E(F)$. Fix an arbitrary point $c \in B$. Then there exists a complete holomorphic vector field $X^a := (a - az)$ on the open unit ball $D$ of $F$ with $g(0) = c$, $g(B) = B$, $d_g(0) = \exp(L)$ and $L(E) \subset E$ for $g := \exp(X^a) \in \text{Aut}(D)$ and a certain hermitian operator $L \in \mathcal{L}(F)$, compare [17] Proposition 2.6. For every real $t$ the isometry $\exp(itL) \in \text{GL}(F)$ leaves the subspaces $E$ and $T_E$ invariant. This implies that also $L$ and consequently also $dg(0)$ leaves $T_E$ invariant. Define the holomorphic mapping $\tilde{f} : B \to F$ by $f(g(w)) := dg(w)\tilde{f}(w)$ for all $w \in B$. Because of $f(c) = dg(0)\tilde{f}(0)$ we only have to show that also $\tilde{f}$ satisfies the assumptions of the proposition since then $\tilde{f}(0) \in T_E$ by the above reasoning. For this fix $b \in \partial B$ and $\mu \in S_b$. By [16] $g$ extends to a biholomorphic mapping $g : U \to V$ for suitable open neighbourhoods
U, V of \( \overline{D} \) in \( F \). Consider \( a := g(b) \in \partial B \) and \( \lambda := \mu \circ dg(b)^{-1} \in F^* \). Then \( S := \{g(z) : z \in U, \mu(z) = 1\} \) is a complex-analytic hypersurface of \( V \) with \( S \cap D = \emptyset \). Therefore also the corresponding tangent hyperplane \( \{z \in F : \lambda(z) = \lambda(a)\} \) at \( a \in S \) does not intersect \( D \), i.e. \( \lambda \in CS \), and thus

\[
\lim_{w \to b} \mu \circ f(w) = \lim_{w \to b} \mu \circ dg(w)^{-1} \circ f(g(w)) = \lim_{z \to a} \lambda \circ f(z) = 0.
\]

**Corollary 6.14.** Property Q implies Property P.

**Theorem 6.15.** Suppose that \( E \subseteq F \) are complex Banach spaces satisfying Property P and \( T_E(F) = 0 \). For \( B \), the open unit ball of \( E \), let \( F \) be the space of all holomorphic mappings \( f : B \to F \) with \( f(a) \in T_{\mathbb{C}}^n(F) \) for every \( a \in \partial B \). Then \( F \subseteq \text{Hol}(\overline{B}, F) \) is an \( \mathbb{R} \)-linear subspace with \( F \cap iF = 0 \) and every \( f \in F \) is a polynomial of degree at most 2. Every \( f \) is uniquely determined in \( F \) by \( f(0) \) and \( df(0) \).

**Proof.** Fix \( f \in F \) and expand it on \( \overline{B} \) into the uniformly convergent series

\[
f = \sum f_n \text{ with } f_n \in \mathcal{P}_n(E, F) \text{ for every } n \in \mathbb{N}.
\]

Fix \( a \in \partial B, \lambda \in S \), and define \( c_n := \lambda \circ f_n(a) \in \mathbb{C} \) for all \( n \). Then we also have \( \text{Re}(\lambda \circ f(ta)) = 0 \) for every \( t \in \mathbb{T} \) and hence

\[
\sum_{n=0}^{\infty} (t^{n-1}c_n + t^{-n}e_n) = 2 \text{Re}(\lambda \circ f(ta)) = 0
\]

for all \( t \in \mathbb{T} \). Since the coefficients of a Fourier series are uniquely determined we get

\[
(*) \quad c_0 + e_2 = e_1 + e_1 = 0 \quad \text{and} \quad c_n = 0 \quad \text{for all } n \geq 3.
\]

On the other hand, for every \( f \in \text{Hol}(\overline{B}, F) \), condition (\( * \)) for every \( a \in \partial B, \lambda \in S \), and \( c_k = \lambda \circ f_k(a) \) is also sufficient for \( f \) to be in \( F \), i.e. \( f_n = 0 \) for all \( n \geq 3 \) as a consequence of Property P and hence \( f = f_0 + f_1 + f_2 \) is a polynomial of degree at most 2. In particular \( F \cap iF = 0 \) follows. In case \( f_0 = 0 \) the quadratic function \( f_2 \) is in \( F \cap iF \), i.e. every \( f \in F \) is uniquely determined by \( f_0 \) and \( f_1 \).

Suppose that \( E \subseteq F \) is a JB*-subtriple. For every \( a \in E \) the polynomial \( h : F \to F \) defined by \( h(z) = a - |z|z \) is a complete holomorphic vector field on \( D \) and therefore the restriction \( f = h|\overline{B} \) is in the space \( F \). The proof of Proposition 6.15 therefore gives the decomposition \( F = K \oplus P \) where

\[
K = \{ f \in F : f(0) = 0 \} = F \cap \mathcal{L}(E, F) \quad \text{and} \quad P = \{ f \in F : df(0) = 0 \} = \{ z \mapsto a - |z|z : a \in E \}.
\]

**7. The JB*-triple case**

Property P together with \( T_E(F) = 0 \) is sufficient but by no means necessary for the rigidity of \( E \subseteq F \). For instance, if \( F \) is a complex Hilbert space and \( E \neq F \) is
an arbitrary closed linear subspace, then $E$ is rigid in $F$ as a consequence of 3.3 or of 3.10. On the other hand, $E \subset F$ satisfies Property $\mathcal{P}$ as a consequence of 6.14 and $T_E(F) \neq 0$ is the orthogonal complement of $E$ in the Hilbert space $F$. Therefore, the tangent spaces $T_a(F)$ and $T_E(F)$ still seem to big for some rigidity questions.

For every $a \in F$ denote by $\Theta_a = \Theta_a(F) \subset F$ the smallest closed linear subspace containing every $v \in F$ with $\|a + tv\| = \|a\|$ for all $t \in \mathbb{C}$ with $|t| \leq \|a\|$. Then $\Theta_a$ is a linear subspace of $T_a$ with $\Theta_0 = F$ and $\Theta_s = \Theta_a$ for all $s \in \mathbb{C}^*$. Clearly, $a \in \partial D$ is a complex extreme boundary point of $D$ if and only $\Theta_a = 0$. In case $E \subset F$ also $\Theta_a(E) = \Theta_a(F) \cap E$ holds for all $a \in E$. The following result is well known, compare also Théorème 3.1 in [18].

**LEMMA 7.1.** Let $U$ be a domain in a complex Banach space and suppose that $f : U \rightarrow F$ is a holomorphic mapping with $f(U) \subset \overline{D}$. Then $f(U)$ is contained in the affine subspace $(a + \Theta_a)$ for every $a \in f(U) \cap \partial D$.

**Proof.** We may assume that $U = \Delta$ and $a = f(0) \in \partial D$. Fix an arbitrary $c \in \Delta \setminus \{0\}$ and consider

$$v := \frac{1 - |c|}{2|c|}(f(c) - a) \in F.$$ 

Then [9], p. 68, implies $a + \Delta v \subset \overline{D}$ and hence $a + \Delta v \subset \partial D$, i.e. $(f(c) - a) \in \Theta_a$.

**PROPOSITION 7.2.** Let $E \subset F$ be arbitrary complex Banach spaces and suppose that the balanced set

$$A := \{a \in E : \Theta_a(E) = \Theta_a(F) = \Theta_a(F) or a = 0\}$$

is a set of determinacy in $E$. Then $E$ is rigid in $F$.

**Proof.** Let $f : B \rightarrow D$ be a holomorphic mapping with $f(0) = 0$ and $df(0) : E \hookrightarrow F$ the canonical injection. Fix an arbitrary unit vector $a \in A$ and consider $h(t) = f(ta)/t \in \overline{D}$ for all $t \in \Delta$. Then $h(t) \in (a + \Theta_a(F)) \subset E$ for all $t \neq 0$ by 7.1, i.e. $f(z) \in E$ for all $z \in A \cap B$. Since $A$ is balanced in $E$ we derive $f(B) \subset B$ by 2.3 and 2.2. But then Cartan’s uniqueness theorem implies that $f$ is linear.

Proposition 7.2 for the special case of finite dimensions and $A$ dense in $E$ essentially already occurs in [22], compare Théorème 5.2. The proof is different from ours and does not extend to infinite dimensions. In the following we want to get rigidity also in cases where the set $A$ in Proposition 7.2 is not a set of determinacy (even where $A = \{0\}$, compare the discussion at the end of this section).

As before in the case of the tangent spaces we put

$$\Theta_E := \Theta_E(F) := \bigcap_{a \in E} \Theta_a(F)$$
for every closed linear subspace \( E \subset F \). Then \( \Theta_E(F) \subset T_E(F) \) is a closed linear subspace and the following analogue of Proposition 6.13 holds. The proof is similar to the one of 6.13.

**Proposition 7.3.** Suppose that \( E \subset F \) satisfy Property Q. Let \( f : B \to F \) be a holomorphic mapping with

\[
\lim_{z \to a} \lambda \circ f(z) = 0
\]

for every \( a \in \partial B \) and every \( \lambda \in F^* \) with \( \lambda(\Theta_a) = 0 \). Then \( f(B) \subset \Theta_E(F) \).

**Corollary 7.4.** Let \( f : E \to F \) be a holomorphic mapping with \( f(a) \in \Theta_a \) for all \( a \in E \). Then \( f(E) \subset \Theta_E(F) \) if \( E \subset F \) satisfy Property Q.

For the rest of the section let \( F \) be a JB*-triple with triple product \([abc]\). By the symmetry in the outer variables the triple product is uniquely determined by all triple products of the form \([aba]\). For every \( a, b \in F \) denote by \( a\Box b \in \mathcal{L}(F) \) the operator \( z \mapsto [abz] \). Then \( \Box \) can be understood as an operator-valued positive-definite hermitian product on \( F \), compare [15] for details. In particular, we write \( a \perp b \) if \( a\Box b = 0 \) or—equivalently—if \( b\Box a = 0 \). For every subset \( A \subset F \) call \( A^\perp := \{z \in F : z \perp A\} \) the annihilator of \( A \) in \( F \).

Examples of JB*-triples are for instance all Hilbert spaces with triple product given by \([za\bar{z}] = (\zbar a)z \) or more generally all spaces \( \mathcal{L}(H, K) \) with triple product \([za\bar{z}] = z\bar{a}z \) where \( H, K \) are arbitrary complex Hilbert spaces and \( * \) is the usual adjoint of operators. The class of subtriples of all \( \mathcal{L}(H, K) \) includes in particular the class of all C*-algebras.

For every JB*-triple \( F \) and every \( a \in F \) the smallest closed subtriple of \( F \) containing \( a \) is isometrically isomorphic to a space \( C_0(K) \) with \( K \subset (0, \infty) \subset \mathbb{R} \) and \( K \cup \{0\} \) compact. For the study of rigidity and tangent spaces in JB*-triples therefore the following example is helpful.

**Example 7.5.** Let \( F = C_0(K) \), the linear subspace \( E \subset F \) and \( \Omega \subset K \) be as in Example 6.4. Then \( F \) is a JB*-triple and it is seen easily that \( \Theta_E(F) \) is the closure of \( \{f \in F : \Omega \cap \text{support}(f) = \emptyset\} \) in \( F \), i.e. \( \Theta_E(F) = T_E(F) \) by Stone–Weierstraß.

**Proposition 7.6.** For every closed linear subspace \( E \subset F \) and every unit vector \( a \in F \) we have

(i) \( (u - [aua] + [vwa] - [wva]) \in T_a^E(F) \) for all \( u, v, w \in F \).

(ii) \( E^\perp \subset \Theta_E(F) \).

Proof. (i) The polynomial \( f(z) = u - [zu\bar{z}] + [vz\bar{w}] - [wz\bar{v}] \) is a complete holomorphic vector field on the open unit ball \( D \) of \( F \), compare [15]. Therefore the solution \( g : \mathbb{R} \to F \) of the initial value problem \( g'(t) = f(g(t)) \), \( g(0) = a \)
satisfies $g(\mathbb{R}) \subset \partial D$ and hence $\text{Re}(\lambda \circ g)$ has a critical value in 0 for every $\lambda \in S_0$, i.e. $g'(0) = f(a) \in T^R_d$.

(ii) Follows from the fact that $v \perp w$ implies $\|v + w\| = \max(\|v\|, \|w\|)$.

**Corollary 7.7.** \( \ker(E) := \{z \in F : Q(E)z = 0\} \subset T_E(F) \) for every linear subspace $E \subset F$.

The element $e \in F$ is called a **tripotent** if \([eee] = e\) holds. Every tripotent $e$ induces a direct sum decomposition $F_1 \oplus F_{1/2} \oplus F_0$ (the Peirce decomposition with respect to $e$), where $F_k = F_k(e)$ is the $k$-eigenspace of the operator $e \otimes e$ in $F$.

Every Peirce space $F_k$ is a $J^*$-subtriple and the canonical projection $F \to F_k$ is contractive. For $k = 1/2$ the Peirce projection is even bicontractive.

A special rôle is played by the conjugate linear operators $Q(a)$ on $F$ defined by $z \mapsto [aza]$. These satisfy the fundamental formula $Q(Q(a)b) = Q(a)Q(b)Q(b)$.

For every tripotent $e \in F$ the operator $Q(e)$ splits $F$ into real subspaces $F = F^1 \oplus F^{-1} \oplus F^0$ where $F^k$ is the $k$-eigenspace of $Q(e)$. Then $F_1 = F^1 \oplus F^{-1}$, $F^{-1} = iF^1$ and $F^0 = F_{1/2} \oplus F_0$.

**Lemma 7.8.** \( T_{e}(F) = F^{-1} \oplus F^0, T_{c}(F) = F^0 \) and $\Theta_c(F) = F_0$ for every tripotent $e \in F$.

**Proof.** Proposition 7.6 implies $F_0 \subset \Theta_e$, $F^{-1} \oplus F_{1/2} = \{[eve] - [vee] : v \in F\} \subset T_e$ and hence also $F^{-1} \oplus F^0 \subset T^R_e$. Now consider a vector $v \in T^R_e \cap F^1$ and denote by $V \subset F$ the closed (complex) subtriple generated by $v$ and $e$. Then $V$ coincides in the $J^*$-algebra $F_1(e)$ with the closed complex subalgebra generated by the unit $e$ and the self-adjoint element $v$. In particular, $V$ is a unital associative $J^*$-algebra and hence isometrically isomorphic to $C(K)$ for some compact subset $K \subset \mathbb{R}$ in such a way that $e(s) = 1$ and $v(s) = s \geq 0$ for all $s \in K$. But then $\int_V v(s) \, d\mu(s) = 0$ for every Borel measure $\mu \geq 0$ implies $K = \{0\}$ and hence $T_{e} = F^{-1} \oplus F^0$ as well as $\Theta_e \subset T_e = F^0$. The proof will be finished if we show that $w \in F_0$ for all $w \in \Theta_e$. For this we may assume that $w \in F_{1/2}$ and $e + \Delta w \subset \partial D$ holds. Consider the complete holomorphic vector field $f(z) = [zez] - e$ on $D$ and denote by $t \mapsto g_t(z)$ for every $z \in \overline{D}$ the solution of $\partial g_t(z)/\partial t = f(g_t(z))$ to the initial value $g_0(z) = z$. Then $g_t \in \text{Hol}(\overline{D}, D)$ for all $t \in \mathbb{R}$. Furthermore, $f(e) = 0$ and $df(e) = 2e \otimes e$ imply $g_t(e) = e$ and $dg_t(e) = \exp(t(e \otimes e))$ for all $t$, compare [17] p. 210. For every $i$ define $h_i \in \text{Hol}(\Delta, F)$ by $h_i(s) = g_i(e + sw) \in \partial D$. Then $[h_i : t \in \mathbb{R}]$ is a bounded family of holomorphic mappings. Therefore also the set of all derivatives $[h'_i(t) = e^t w : t \in \mathbb{R}]$ must be bounded in $F$, i.e. $w = 0$.

**Lemma 7.9.** For every tripotent $e \in F$ and every closed linear subspace $E \subset F_1(e)$ we have (i) $F^0(e) \subset T_E(F)$, (ii) $F_0(e) \subset T_E(F)$.

**Proof.** For every unit vector $a \in E$ and every $u \in F^0(e)$ we have $u = u - [aau] \in F^0(e)$ by 7.6.i, proving (i). The second statement follows from $E \perp F_0$. 
LEMMA 7.10. Suppose that $E \subset F$ is a $JB^*$-subtriple with the following property: To every $v \neq 0$ in $F$ there exists a tripotent $e \in E$ with $\{eve\} \neq 0$ (i.e. the Peirce-1-component of $v$ with respect to the tripotent $e$ does not vanish). Then $T_E(F) = 0$ and hence $E$ is rigid in $F$.

Proof. Suppose that $v \neq 0$ for a vector $v \in T_E(F)$. Choose a tripotent $e \in E$ with $\{eve\} \neq 0$. Then $v \in F^0(e)$ by 7.8, a contradiction.

As an application of 7.10 we see for instance that for $F = \mathcal{L}(H, K)$ the subspace $\kappa(H, K)$ of all compact operators is a rigid subspace of $F$ for every pair of complex Hilbert spaces $H, K$.

PROPOSITION 7.11. Let $e$ be a tripotent in the $JB^*$-triple $F$ and denote by $P = Q(e)^2$ the Peirce projection from $F$ onto $E := F_1(e)$. Then the following conditions are equivalent.

(i) $F_0(e) = 0$.
(ii) $P$ is almost neutral,
(iii) $E$ is rigid in $F$.

Proof. (i) $\implies$ (ii) $E$ is a $JB^*$-algebra with unit $e$ in the product $a \circ b = \{aeb\}$ and the involution $a^* = \{ae\}$. The selfadjoint part $V := F^1(e)$ is a $JB$-algebra and $\Omega := \exp(V)$ is an open convex cone in $V$. The generalized unit circle $A := \exp(iV)$ is a set of determinacy in $E$. This follows from the fact that the real analytic mapping $\varphi : V \to E$ defined by $v \mapsto \exp(iv)$ has real differential $d\varphi(0) : V \to E$ given by $v \mapsto iv$. Every $a \in A$ is a tripotent with $F_{1/2}(a) = W := F_{1/2}(e)$. Indeed, $a = \exp(2iv)$ holds for some $v \in V$ and $\lambda := \exp(h(v\Box e + e\Box v)) \in \text{GL}(F)$ is a triple automorphism with $\lambda(e) = a$ and $\lambda(W) = W$. Therefore it is enough to show that $\|e + w\| = 1$ for $w \in W$ implies $w = 0$. Suppose on the contrary that $w \neq 0$ holds. Then $c := \{ewu\} \in \overline{\Omega}$ and $c \neq 0$, compare [16] p. 183. But this is not possible—the closed real subalgebra of $V$ generated by $e$ and $c$ can be realized as $\mathcal{L}(K, \mathbb{R})$ in such a way that $e$ is the function $\equiv 1$ on the compact space $K$ and $c \geq 0$.

(ii) $\implies$ (iii) Follows from Proposition 3.10.

(iii) $\implies$ (i) Is trivial because of $e^\perp = F_0(e)$.

For every tripotent $e \in F$ the Peirce spaces $F_1(e)$ and $F_0(e)$ are inner ideals of $F$—by definition, a closed linear subspace $J \subset F$ is called an inner ideal if $\{JFJ\} \subset J$ holds. Every inner ideal $J \subset F$ is a subtriple of $F$ and with every tripotent $e \in J$ also the whole Peirce space $F_1(e)$ is contained in $J$. By [7] the inner ideals of $F$ can be uniquely characterized in the class of all closed subtriples $E \subset F$ by the unique norm preserving extension property of linear functionals:

To every $\lambda \in E^*$ there exists a unique $\sigma \in F^*$ with $\|\lambda\| = \|\sigma\|$ and $\sigma|E = \lambda$.

The following proposition is a characterization of inner ideals within the bigger class.
of all closed linear subspaces of $F$ in terms of holomorphic automorphisms of the open unit ball $D$ of $F$.

**Proposition 7.12.** Let $E$ with open unit ball $B$ be a closed linear subspace of the JB*-triple $F$ with open unit ball $D$. Then $E$ is an inner ideal of $F$ if and only if $g(B)$ is convex in $F$ for every $g \in \text{Aut}(D)$.

**Proof.** Fix an arbitrary $c \in D$. Then $\|z \circ c\| < 1$ holds for all $z \in D$ and there exists an automorphism $g \in \text{Aut}(D)$ such that

\[
g(z) = c + \lambda(1 + z \circ c)^{-1}z
\]

for $\lambda = dg(0) \in GL(F)$ and all $z \in D$, compare [15] p. 132. Therefore, if $E$ is an inner ideal in $F$, the function $f(z) := (1 + z \circ c)^{-1}z$ maps $B$ into $E$ and hence $g(B) \subset (A \cap D)$ for the affine subspace $A := c + \lambda(E)$. By the implicit function theorem $g(B)$ is a neighbourhood of $c$ in $A$, therefore $g^{-1}$ maps the domain $(A \cap D) \subset A$ into $E$, i.e. $g(B) = A \cap D$. In particular, $g(B)$ is convex in $F$. Any other automorphism $\tilde{g} \in \text{Aut}(D)$ with $\tilde{g}(0) = c$ is of the form $\tilde{g} = gk$ for some $k \in \text{Aut}(D) \cap GL(F)$. Then $k$ respects the triple product and hence $\tilde{E} := k(E)$ is also an inner ideal of $F$, i.e. also $\tilde{g}(B)$ is convex. On the contrary, suppose that $g(B)$ is convex in $F$ for all $g \in \text{Aut}(D)$. Fix an arbitrary $c \in D$ and choose $g$ as in (1). Then $f(z) = \lambda^{-1}(g(z) - c) = z - [zc] + o(\|z\|^2)$ defines a holomorphic mapping $f : D \to F$ with $f(0) = 0$, $df(0) = 1d$ and $f(B)$ convex. By the implicit function theorem $f(B)$ must be contained in $E$. This implies $[zc] = -\lim_{t \to 0} t^{-2}(f(tz) - tz) \in E$ for all $z \in B$ and all $c \in D$. Therefore $E$ is an inner ideal in $F$.

A JB*-triple $F$ is called a JBW*-triple if $F$ as a Banach space is the dual of another Banach space, compare [14] and [10]. This predual is uniquely determined by $F$ and is denoted by $F_\ast$. It is known [2] that the triple product on every JBW*-triple is separately $w^*$-continuous. For every JB*-triple $E$ the bidual $E^{**}$ is a JBW*-triple with triple product extending the one of $E \subset E^{**}$, compare [5]. The advantage of JBW*-triples is that they contain many tripotents. By [8] a linear subspace $E$ of the JBW*-triple $F$ is a $w^*$-closed inner ideal if and only if $E$ is the range of a structural projection $P$ in $F$ (structural means $Q(P(a)) = P \circ Q(a) \circ P$ for all $a \in F$ — such a projection is automatically contractive and $w^*$-continuous).

**Example 7.13.** Let $F := \mathcal{L}(H, K)$ for complex Hilbert spaces $H, K_1, K_2$ and $K := K_1 \oplus_2 K_2$. Then every $z \in F$ can be realized as a pair $z = (z_1, z_2)$ with $z_k \in \mathcal{L}(H, K_k)$ for $k = 1, 2$ and $P(z_1, z_2) = (z_1, 0)$ defines a structural projection onto a $w^*$-closed inner ideal $E$ with $E^\perp = 0$. In general, the projection $P$ is not almost neutral and also $E$ is not the Peirce-1-space of a tripotent, compare the matrix example at the end.

**Theorem 7.14.** Let $E$ be a $w^*$-closed ideal in the JBW*-triple $F$ and let $P$
be the corresponding structural projection from \( F \) onto \( E \). Then
\[
T_E(F) = \ker(P) = \{ z \in F : Q(E)z = 0 \} \quad \text{and} \quad \Theta_E(F) = E^\perp = \{ z \in F : (E \square E)z = 0 \}.
\]
Furthermore, \( E \) is rigid in \( F \) if and only if \( \Theta_E(F) = 0 \).

**Proof.** Fix \( v \in T_E(F) \) and \( a \in E \). Then there is a tripotent \( e \in E \) with \( a \in F_1(e) \). This implies \( Q(a)v = 0 \) because of \( v \in \overline{F}(e) \) and hence \( T_E(F) \subset \ker(E) \). The opposite inclusion follows with 7.7. But \( \ker(E) = \ker(P) \), compare [8]. The statement concerning \( \Theta_E(F) \) follows by a similar argument. Finally, \( E^\perp = 0 \) is necessary for \( E \) being rigid in \( F \). Let us therefore assume conversely, that \( E^\perp = 0 \) holds. Consider a homogeneous polynomial \( f : E \to F \) of degree \( m \geq 2 \) with \( z + f(z) \in D \) for all \( z \in B \) and fix \( c \in B \). We have to show that \( f(c) = 0 \). Because of 7.4 it is enough to show that \( f(c) \in \Theta_c(F) \). Choose a tripotent \( e \in E \) with \( c \in U := F_1(e) \) and put \( Z := F_0(e) \). Then \( \|e + tf(e)\| \leq 1 \) for all \( t \in \Delta \) implies \( f(e) \in Z \) by 7.1. Now let the selfadjoint part \( V \) of the JB*-algebra \( U \) and \( A = \exp(iV) \) be as in the proof of Proposition 7.11. Every \( a \in A \) has the same Peirce spaces as \( e \) and therefore also \( f(a) \in Z \) by the above reasoning. Since \( A \) is a set of determinacy in \( U \) this implies \( f(c) \in Z \subset \Theta_c(F) \) as a consequence of 2.2 and 7.9.

As an example consider the case of an arbitrary \( W^* \)-algebra \( F \). Then \( F \) is also a JBW*-triple and a \( W^* \)-closed linear subspace \( E \subset F \) is an inner ideal if and only if \( E = eFc \) for (Hermitian) projections \( e, c \in F \) having the same central support, compare [8] p. 59. Then \( T_E(F) = (1 - e)F + F(1 - c) \) and \( \Theta_E(F) = (1 - e)F(1 - c) \).

We close with a finite dimensional illustration of Theorem 7.14: For fixed integers \( 1 \leq p \leq n \) and \( 1 \leq q \leq m \) with \( n \leq m \) the matrix space \( F := \mathbb{C}^{n \times m} = \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m) \) is a JBW*-triple of dimension \( nm \) and rank \( n \). Write every matrix \( z \in F \) in block form \( z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), where \( a, b, c, d \) are rectangular matrices of sizes \( p \times q, p \times (m - q), (n - p) \times q \) and \( (n - p) \times (m - q) \) respectively. \( P(z) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) defines a structural projection onto an inner ideal \( E \) of \( F \). Then \( E \approx \mathbb{C}^{p \times d} \) is the Peirce-1-space of a tripotent \( e \in F \) if and only if \( E \) has square size, i.e. \( p = q \). Under the assumption \( E \neq F \) the projection \( P \) is neutral if and only if \( n = 1 \), that is, if \( F \) is a Hilbert space. Furthermore, \( P \) is neutral if and only if \( q \geq p = n \) holds, that is, if \( E \) and \( F \) have the same rank. \( E \) is rigid in \( F \) if and only if \( p = n \) or \( q = m \). Finally, for the set \( A \) of Proposition 7.2 the following conditions are equivalent: (i) \( A \neq \{0\} \), (ii) \( A \) is a set of determinacy in \( E \), (iii) \( E \) and \( F \) have the same rank. Also, \( A \) is dense in \( E \) if and only if \( E = F \) or \( F \) is a complex Hilbert space. In particular, \( E \) may be rigid in \( F \) in the case \( A = \{0\} \).
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