# Holomorphic isometries and related problems

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# PROJECTION PRINCIPLE

$\mathbf{E}$ Banach space, $\mathbf{B} := B(\mathbf{E})$ open unit ball

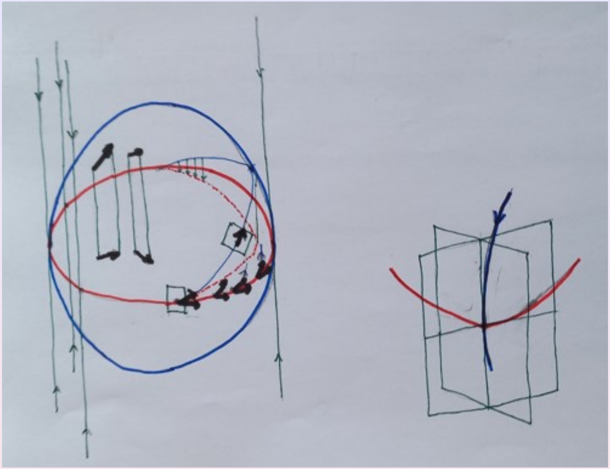
$V : \mathbf{E} \to \mathbf{E}$ loc.Lip vector field

$V$ is *complete* in a figure $\mathbf{K} \subset \mathbf{E}$ if
the maximal solution of $\dot{x} = V(x)$ starting from any point $p \in \mathbf{K}$
is defined on the whole $\mathbb{R}$ and ranges in $\mathbf{K}$

THEOREM. *Assume* $P = P^2 \in \mathcal{L}(\mathbf{E})$, $\|P\| = 1$ *is a contractive projection and let* $\mathbf{E}_0 := P\mathbf{E}$, $\mathbf{B}_0 := P\mathbf{B}$,

$$V : x \mapsto a - Q(x,x), \quad Q : \mathbf{E}^2 \to \mathbf{E} \text{ cont. symm. bilin. map.}$$

*If* $V$ *is complete in* $\mathbf{B}$ *then* $PV$ *is complete in* $\mathbf{B}_0$.



$\mathbf{S} := \{p \in \mathbf{E} : \|p\| = 1\}$, $\mathbf{S}_0 := \mathbf{S} \cap \mathbf{E}_0$ unit spheres

$\mathrm{Supp}(\mathbf{B}, p) := \big\{\Phi \in \mathbf{E}^* : \langle \Phi, p\rangle = \|\Phi\|\big\}$, $p \in \mathbf{S}$ support funct.s

$\mathrm{Tan}(\mathbf{S}, p) := \big\{v \in \mathbf{E} : \frac{d}{d\tau}\big|_0 \|p + \tau v\| = 0\big\} = \bigcap \mathrm{Supp}(\mathbf{B}, p)$, tangent vect.s

# PROOF (preliminaries)

1) The max. sol. of $\dot{x} = PV(x)$ ranges in $\mathbf{E}_0$ if $x(0) \in \mathbf{E}_0$

2) $V$ is complete in $\mathbf{B}$ $\iff$ $V$ is complete in $\mathbf{S}$
$PV$ is complete in $\mathbf{B}_0$ $\iff$ $PV$ is complete in $\mathbf{S}_0$

3) $p \in \mathbf{S}_0, \Rightarrow \mathrm{Supp}_{\mathbf{E}_0}(\mathbf{B}_0, p) = \{\Phi|\mathbf{E}_0 : \Phi \in \mathrm{Supp}(\mathbf{B}, p)\}$
$P^*\mathrm{Supp}(\mathbf{B}, p) = \{\Phi \circ P : \Phi \in \mathrm{Supp}(\mathbf{B}, p)\} \subset \mathrm{Supp}(\mathbf{B}, p)$,

4*) $V$ BDED on $\mathbf{S}$ $\Rightarrow$ Enough to see:

*Max.sol. of* $\dot{x} = PV(x)$, $x(0) \in \mathbf{S}_0$ *ranges in* $\mathbf{S}_0$

# PROOF

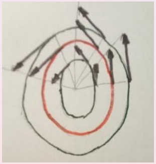
Let $p \in \mathbf{S}_0$, $[t \mapsto w(t)]$ max.sol. of

$$\dot{x} = W(x) = \|x\| Pa - \frac{1}{\|x\|} PQ(x,x), \quad x(0) = p$$

$PV, W$ coincide on $\mathbf{S}_0$, $t \mapsto \|w(t)\|$ Lipschitzian $\Rightarrow$ *Enough to see*

$$\|w(t)\|^\bullet := \frac{d}{dt}\|x(t)\| = 0 \text{ almost everywhere in } t$$

If $x \in \mathbf{S}_0$,
$W(x) = \|x\| PV\big(\frac{x}{\|x\|}, \frac{x}{\|x\|}\big)$,
$W(\varrho x) = \varrho W(x) = \varrho V(x) \in \mathrm{Tan}(\mathbf{S}_0, x)$.
Sol. $w_\varrho$ of $\dot{x} = W(x)$, $x(0) = \varrho p$
satisfies $w_\varrho(t) = \varrho w_1(t) = \varrho v(t) \in \varrho \mathbf{S}_0$.



$\|\cdot\|$ Liapunov for $\dot{x} = W(x)$

SCUOLA NORMALE SUPERIORE

Tesi di perfezionamento

HOLOMORPHIC MAPS AND FIXED POINTS

Candidato
Laslo Stacho

Relatore
Prof. Edoardo Vesentini

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the fact that $\psi^*(t)\in\partial D$. □

Corollary 5. $v\in\log^*\mathrm{Aut}\,B(E)$ if and only if (12") holds.

Proof. By the triangle inequality, the norm function ($\equiv$gauge $B(E)$) is Lipschitzian. □

At this point we are prepared to establish the following basic relation between the biholomorphic automorphism groups of Banach space domains and those of their sections with linear subspaces:

Theorem 8. (Projection principle). Let E denote a Banach space, D a bounded balanced domain in E whose gauge function is locally Lipschitzian and has a non-empty subgradient at every point of E. Assume that P is such a continuous projection of E onto a subspace $E_1$ of E that maps D onto $E_1\cap D$. Then for all $v\in\log^*\mathrm{Aut}\,D$ we have $[E_1\ni f\mapsto Pv(f)]\in\log^*\mathrm{Aut}(E_1\cap D)$[9]. Moreover, $(\mathrm{Aut}(E_1\cap D))\{0\}\supset P((\mathrm{Aut}\,D)\{0\})$.

Proof. Set $p(.)\equiv$gauge $D(.)$. Observe that gauge $E_1\cap D=p|_{E_1}$ whence the function gauge $D|_{E_1}$ is locally Lipschitzian and has non-empty subradient everywhere on $E_1$. Therefore, by Lemma 14, $Pv|_{E_1}\in\log^*\mathrm{Aut}(E_1\cap D)$ if and only if $\mathrm{Re}\langle P\ell(f_1),\phi_1\rangle=0$

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9) $E_1\cap D$ being considered as a domain in $E_1$.

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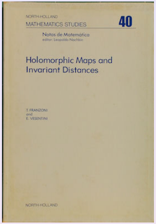
# ADVANCED VERSIONS






2.1. Theorem. *Let $M$ be a complex Banach manifold, $M'$ a (complex) submanifold of $M$ and $v$ a complete holomorphic vector field on $M$. Suppose $P$ is a holomorphic mapping of $M$ onto $M'$ such that $P|_{M'} = \mathrm{id}_{M'}$ (the identity mapping on $M'$).*

*Suppose there exists a differential Finsler metric $\delta$ on $M'$ such that*

(i) *the vector field $P'v|_{M'}$ is $\delta$-bounded (i.e. $\sup_{x\in M} \delta(x, P'(x)\,v(x)) < \infty$)*

*and by writing $d$ for the intrinsic distance generated by $\delta$ on $M'$,*

(ii) *the topology of the metric $d$ is finer than that of $M'$,*

(iii) *for any sequence $x_1, x_2, \ldots \in M'$ which is a Cauchy sequence with respect to $d$ but which is not convergent in $M'$ we have $d_{M'}(x_1, x_n) \to \infty$ $(n \to \infty)$.*

*Then the vector field $P'v$ is complete in $M'$.*

# Real version

**3.1 Theorem.** *Let $E$ be a Banach space, $D, D_0$ open $\subset E$ such that $\overline{D} \subset D_0$ and $\partial D$ is a Lipschitzian submanifold of codimension 1 in $D_0$. Assume $X : D_0 \times \mathbb{R} \to E$ is a locally Lipschitzian bounded vector field which is complete in $D$ and let $P : D_0 \to D_0$ be a twice continuously differentiable projection such that $\mathrm{ran}P$ is a $C^2$-submanifold of $D_0$. Then the projected vector field $Y(a,t) := P'(a)X(a,t) \quad (a \in D_0,\ t \in \mathbb{R})$ is also complete in $D \cap \mathrm{ran}P$.*

[**Stachó 1999**] *On nonlinear projections of vector fields*,
Josai Mathematical Monograps 1, 47-54

COMPLEX SETTING: $\mathbf{D} \subset \mathbf{E}$ bded dom with Lip boundary,
$P = P^2 \in \mathcal{L}(\mathbf{E})$, $\mathbf{D}_0 := P\mathbf{D} \subset \mathbf{D}$, $V$ hol. vect.field on a nbh of $\overline{D}$,
$V$ complete in $\mathbf{D} \Rightarrow V$ complete in $P\mathbf{D}$

# APPLICATIONS (background)

Henceforth COMPLEX SPACES

$\mathrm{Aut}(\mathbf{B}) := \{$ holom. $\mathbf{B} \leftrightarrow \mathbf{B}$ maps $\}$ Holomorphic automorphisms

$\mathbf{B}_{sym} := \{F(0) : F \in \mathrm{Aut}(\mathbf{B})\}$ Symmetric part of ball

$\mathbf{E}_{sym} := \mathbb{C}\mathbf{B}_{sym} = \{\zeta F(0) : \zeta \in \mathbb{C},\ F \in \mathrm{Aut}(\mathbf{B})\}$ Symmetric part of space

[**Kaup-Upmeier 1976**] $\mathbf{E}_{sym}$ closed subspace of $\mathbf{E}$, $\mathbf{B}_{sym} = \mathbf{E}_{sym} \cap \mathbf{B}$

$$F \in \mathrm{Aut}(\mathbf{B}) \iff F = \exp(V_a) \circ U$$

$\exists\ a \in \mathbf{E}_{sym},\ U \in \mathcal{U}(\mathbf{E}),\ V_a : x \mapsto a - Q_a(x,x)$ complete vfield in $\mathbf{B}$

$\mathcal{U}(\mathbf{E}) = \{$surj. $\mathbf{E}$-isom.$\}$

$\exp(V) : p \mapsto \left[x(1) \text{ with sol. of } \dot{x} = V(x), x(0) = p\right]$

# APPLICATIONS (elementary)

**Rigidity**: If **B** has only linear automorphisms

$\{P_j : j \in J\}$ complete syst. of contr. proj. with rigid $P_j\mathbf{B}$, $\Longrightarrow$ **B** is rigid.

Application with low-dim $P_j\mathbf{B}$: rigid balls are

$B\big(L^p(\mu)\big),\ p\neq 2,\infty;$
$B\big(\mathcal{L}^n[\mathbf{H}_1\times\cdots\times\mathbf{H}_n\to\mathbb{C}]\big),\ n>2, \dim(H_k)>1$

**New structure approach** to $\mathrm{Aut}(\mathbf{B})$ in finite dim.

**[Stachó-Zalar 2003]** Unit ball of $\mathcal{C}(\Omega)$ with $\|f\|\geq\|g\|$ if $|f|\geq|g|$
beautiful topological mixture of finite dim Hilbert balls

**[Stachó 2007]** Banach-Stone type thm + disproof of a conjecture of Sunada: linear isometry $\not\Leftarrow$ positive surj. isometry.

# SYMMETRY

**D** $\subset$ **E** bded dom. symmetric: every $p \in \mathbf{D}$ admits a hol. $p$-reflection

$$S_p \in \mathrm{Aut}(\mathbf{D}), \quad S_p \circ S_p = \mathrm{Id}, \quad S_p(p) = p, \; S'(p) = -\mathrm{Id}$$

**[E. Cartan 1933] D** $\subset \mathbb{C}^N$, $\mathrm{Aut}(\mathbf{D})$ *Lie group, its Lie alg.*

$$\mathrm{aut}(\mathbf{D}) = \big[\text{compl. hol. vect.fields on } \mathbf{D}\big]$$

$$[F,G] : x \mapsto F'(x)G(x) - G'(x)F(x), \quad F'(x)v := \frac{d}{d\tau}\Big|_{\tau=0} F\big(x+\tau v\big)$$

Classification of semisimple Lie algebras;
**[H. Cartan, Harish-Chandra 1955]** In $\mathbb{C}^N$, with modern interpretation, the Lie algebra of compl.hol.vfields of a symm.dom. is isomorphic to a direct sum by 6 types appearing in the cases of unit balls of

$$\mathbb{C}^{m\times n}, \;\; \mathrm{Sym}\big(\mathbb{C}^{n\times n}\big), \;\; \mathrm{Antisym}\big(\mathbb{C}^{n\times n}\big), \;\; \mathrm{Spin}\big(\mathbb{C}^{n}\big), \;\; \mathbb{O}^{2\times 1}, \;\; \mathcal{H}\big(\mathbb{O}^3\big).$$

# SYMMETRY ($\infty$ dim)

**[Vigué 1976]** Bded.symm.doms. are hol.equiv. to
bded homogeneous circular doms containing 0

Homogenity: $x, y \in \mathbf{D} \Rightarrow \exists\ F \in \mathrm{Aut}(\mathbf{D}) \ \ y = F(x)$
Circularity: $x \in \mathbf{D} \Rightarrow e^{i\tau} x \in \mathbf{D} \ \ (\tau \in \mathbb{R})$

A circular domain $\mathbf{D}$ with $0 \in \mathbf{D}$ is symm. $\iff \forall\ v \in \mathbf{E} \ \ \exists$ complete hol.vectfield $V : \mathbf{D} \to \mathbf{E}$ with $V(0) = 0$.

In particular
$(*)$ $\mathbf{B} = \mathbf{B}(\mathbf{E})$ symmetric $\iff \mathbf{E}_{sym} = \mathbf{E}$.

Projection Principle $\Longrightarrow$ If $\mathbf{B}$ is symmetric then $P\mathbf{B}$ with a contactive projection $P \in \mathcal{L}(\mathbf{E})$ is also symmetric

2.3. Theorem. *If $E$ is a Banach space and $P: E \to E$ is a contractive linear projection then* $P[\log^* \operatorname{Aut} B(E)]|_{PE} \subset \log^* \operatorname{Aut} B(PE)$.

Proof. Let $u \in \log^* \operatorname{Aut} B(E)$ be arbitrarily fixed. We have to show that the vector field $Pu|_{B(PE)}$ is complete in $B(PE)$. As in the proof of Lemma 2.2b, let us consider the manifolds $M \equiv B(E)$, $M' \equiv B(PE)$, the projection $P|_{B(E)}$ of $M$ onto $M'$ and the vector field $v \equiv u|_{B(E)}$ which is by definition complete in $M$. Take the differential Finsler metric $\delta(x, w) \equiv \|w\|$ $(x \in B(PE), w \in PE)$ on $M'$ whose generated intrinsic distance is obviously $d(x, y) \equiv \|x - y\|$ $(x, y \in B(PE))$. To complete the proof, we need only to verify (i), (ii), (iii).

(i): For $x \in B(PE)$ we have $P'(x)v(x) = Pu(x)$ whence by a theorem of Kaup—Upmeier [8],

$$\delta(x, P'v(x)) = \|Pu(x)\| \leqq \|u(x)\| = \left\| u(0) + u'(0)x + \frac{1}{2} u''(0)(x, x) \right\| \leqq$$

$$\leqq \|u(0)\| + \|u'(0)\|_{\mathscr{L}(E,E)} + \left\| \frac{1}{2} u''(0) \right\|_{\{\text{bilin } E \times E \to E\}}.$$

(ii): Trivial.

(iii): Assume $x_1, x_2, \ldots$ is a Cauchy sequence with respect to the metric $d$ without a limit in $M'$. Then for some unit vector $f \in PE$, $\|x_n - f\| \to 0$ $(n \to \infty)$ i.e. $\|x_n\| \to 1$. Therefore, by Lemma 2.2c, $d_{M'}(x_1, x_n) = d_{B(PE)}(x_1, x_n) \geqq d_{B(PE)}(x_n, 0) - d_{B(PE)}(x_1, 0) = \operatorname{areath} \|x_n\| = \operatorname{areath} \|x_1\| \to \infty$.

2.4. Corollary. *If $E$ is a Banach space and $P: E \to E$ is a contractive linear projection then $P(E_0) \subset (PE)_0$. In particular, if $B(E)$ is a symmetric manifold then so is $B(PE)$, too.*

2.5. Corollary. *Let $E$ be a Banach space. If one can find a family $\mathscr{P}$ of contractive linear projections $E \to E$ such that for every $P \in \mathscr{P}$,* $\operatorname{Aut} B(PE)$ *consists only of linear transformations and* $\bigcap_{P \in \mathscr{P}} \ker P = \{0\}$ *then all the elements of* $\operatorname{Aut} B(E)$ *are also linear.*

Proof. If $v \in \log^* \operatorname{Aut} B(E)$ then $Pv(0) = 0$ $\forall P \in \mathscr{P}$ whence $v(0) = 0$ i.e. the

# 1981-84


**Santiago de Compostela, fall 1981.** 14 lectures on the previous topics

Starting lecture notes for North Holland [suggested by *L. Nachbin*]

> Stacho's joint book with Isidro is not only a good introduction to the study of known results but it contains useful$^2$ original material.

*Seán Dineen, Dublin*
*DSc opponent 2011*

# PHYSICS, Jordan triples

**Quantum Mechanics.** Self-adjoint operators for measurments

**[J. von Neumann 1932]** $\mathcal{C}^*$-algebras with predual, spectral resolution

**[P. Jordan- J.v. Neumann- E. Wigner 1934]**, $A \diamondsuit B := \frac{1}{2}(AB + BA)$,
Abstract approach, octonions, subsystems by contractive projections

**Conjecture.** The contr. proj. image of a symm. unit ball is symm.

**Problem.** Natural algebra of symmetric unit balls.

**[M. Koecher 1967]** Triple product $\{AB^*C\} := \frac{1}{2}\big(AB^*C + CB^*A\big)$,
AXIOMATIC approach in finite dim.

# CIRCULARITY

Henceforth $\mathbf{D} \subset \mathbf{E}$ bded circular domain, $\quad B(\mathbf{E}, \|\cdot\|) = \mathrm{Conv}(\mathbf{D})$,
$\mathrm{Her}(\mathbf{E}) := \{L \in \mathcal{L}(\mathbf{E}) : \|\exp(i\tau L)\| = 1 \ (\tau \in \mathbb{R})\}$

**[Kaup 1970] IF** $\mathrm{Aut}(\mathbf{D})$ Banach-Lie group $\Rightarrow$

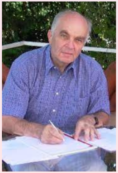
$\exists\ \mathbf{E}_D \subset \mathbf{E}$ closed compl. subsp, $\mathbf{L}_D \subset \mathrm{Her}(\mathbf{E})$ closed real subsp,

$Q_a : (x, y) \mapsto Q_a(x, y)$ cont. symm. 2-lin. $\mathbf{E} \times \mathbf{E}$ maps $\quad (a \in \mathbf{E}_D)$

such that, with $V_a + iL : x \mapsto a - Q_a(x, x) + iLx$,

$\mathrm{aut}(\mathbf{D}) = \{V_a + iL : a \in \mathbf{E}_D,\ L \in \mathbf{L}_D\}$

**[Upmeier 1975]** Topology for B-L group structure of Banach manifolds with Aut-invariant Finsler metr. [$\not\equiv$ loc.unif.conv. on generic bded.dom.]



# Triple product

**[KAUP 1977]** Koecher's triple product

$\{xa^*y\} := Q_a(x,y) \quad a \in \mathbf{E}_D,\ x,y \in \mathbf{E}$

works for circular domains with $\quad L(a,b) : \mathbf{E} \ni x \mapsto \{ab^*x\}$ :

(J1) $\{xa^*y\}$ cont. $\mathbf{E} \times \mathbf{E}_D \times \mathbf{E} \to \mathbf{E}$ symm. 2-lin. in $x, y$,
conj.lin. in $a$, $\quad \{\mathbf{E}_D[\mathbf{E}_D]^*\mathbf{E}_D\} \subset \mathbf{E}_D$

(J2) $\{ab^*\{xc^*y\}\} = \{\{ab^*x\}c^*y\} - \{x\{ba^*c\}y\} + \{xc^*\{ab^*y\}\}$

(J2$^*$) $\delta_a := iL(a) = iL(a,a)$ derivation of $\{.^*.\}$:
$\{\delta_a\{xc^*y\} = \{(\delta_a x)c^*y\} + \{x(\delta_a c)^*y\} + \{xc^*(\delta_a y)\}$

(J3) $\delta_a \in \mathbf{L}_D \subset \mathrm{Her}(\mathbf{E}, \|\cdot\|_D)$, $\quad \mathbf{E}_D \cap \mathbf{D}$ symmetric.

# $\mathbf{JB^*}$-TRIPLES

**[Kaup 1984]** If $0 \in \mathbf{D}$ symmetric, then $\mathbf{D} \cap \mathbf{E}_D$ convex.
Hence $\mathbf{D} = B(\mathbf{E}, \|\cdot\|_D)$ and

(J3) $\mathrm{Sp}(\delta_a) \subset \frac{1}{2}\mathrm{Sp}(\delta_a|\mathcal{J}(a)) + \frac{1}{2}\mathrm{Sp}(\delta_a|\mathcal{J}(a)) \subset [0, \|\delta_a\|]$
where $\mathcal{J}(a) :=$ [ closed subtriple gen.by $a$ ]

(J4) $\|\{aa^*a\}\| = \|a\|^3$ " $C^*$-axiom"

**Def.** $(\mathbf{E}, \{..^*.\})$ $JB^*$-triple if (J1-J4) hold with $\mathbf{D} = B(\mathbf{E})$

**Example.** $\mathcal{C}^*$-algebra with $\{ab^*c\} := [ab^*c + cb^*a]/2$,
$\mathrm{Her}(\mathbf{E}) = \{[x \mapsto Ax + xB] : A, B \text{ self-adj.}\}$

Contained in Chapters VI-X in [Book Isidro-Stachó]

No PP in [Book Isidro-Stachó]

# BIDUALIZATION

**[S. Dineen 1984]** PP with $\mathbf{D} := B(\mathbf{E})$, $\mathbf{E} \subset \mathbf{E}^{**}$

$\exists\,\Omega$ abstr.set, $\exists\,\mathcal{U}$ ultrafilter on $\Omega$,
$\exists\, P : \mathbf{E}^{\mathcal{U}} \to \mathbf{E}^{\mathcal{U}}$ cont.proj.
$\exists\, I : \mathbf{E}^{**} \to \mathbf{E}^{\mathcal{U}}$ lin. isom. embed. with
$\mathrm{range}(P) = \big(I(\mathbf{E}^{**})\big)$.
Hence $\mathbf{B}$ symm $\Rightarrow^{triv}$ $B\big(\mathbf{E}^{\mathcal{U}}\big)$ symm $\Rightarrow^{PP}$
$P\big(B(\mathbf{E}^{\mathcal{U}})\big)$ symm $\Rightarrow B\big(\mathbf{E}^{**}\big) = I^{-1}P\big(B(\mathbf{E}^{\mathcal{U}})\big)$ symm

$\mathbf{E}^{\mathcal{U}} := \mathcal{B}(\Omega, \mathbf{E})/\|\cdot\|_{\mathcal{U}} = \big\{\widetilde{\mathbf{b}} : \mathbf{b} \in \mathcal{B}(\Omega, \mathbf{E})\big\}$ where
$\mathcal{B}(\Omega, \mathbf{E}) := \Big\{\text{bded } \Omega \to \mathbf{E} \text{ funct.}\Big\}$, $\|\mathbf{b}\|_{\mathcal{U}} := \lim_{\mathcal{U}} \|\mathbf{b}(\omega)\|$ $\big(\mathbf{b} \in \mathcal{B}(\Omega, \mathbf{E})\big)$
$\widetilde{\mathbf{b}} := \big\{\mathbf{a} : \|\mathbf{a} - \mathbf{b}\|_{\mathcal{U}} = 0\big\}$ eqiuv. classes wrt. seminorm

**[R. Barton- R. Timononey 1985]**
If $(\mathbf{E}, \{..^*.\})$ $JB^*$-triple, $\{..^*.\}_{\mathbf{E}^{**}}$ is *weak*$^*$-cont extension of $\{..^*.\}$

# GELFAND-NEUMARK TYPE THM

Bidual enbedding into second dual $\longrightarrow$ Sakai type structure theory of $JBW^*$-triples ($JB^*$-triple with predual)

Tripotents $\{ee^*e\} = e$ vs. ort. proj. $e = e^* = e^2$

**[E. Neher 1985]** Description of covering GRIDS (generalization of the system of matrices with a unique non-zero entry) of tripotents in $JBW^*$-triples.

**[Y. Friedman- B. Russo 1985]** GEOMETRY of ball faces grid theory $\to$ Any $JB^*$-triple is isometrically isomorphic to a closed subtriple of some $\ell^\infty$-direct sum of Cartan factors

$$\mathcal{L}(\mathbf{H},\mathbf{K}),\ \ \mathcal{L}_{\mathrm{SYM}}(\mathbf{H},\bar{\cdot}),\ \ \mathcal{L}_{\mathrm{ANTISYM}}(\mathbf{H},\bar{\cdot}),\ \ \mathrm{Spin}(\mathbf{H},\bar{\cdot}),\ \ \mathbb{O}^{2\times 1},\ \ \mathcal{H}(\mathbb{O}^3).$$

# My CONTRIBUTIONS (circular dom.s)

$\mathbf{D} \subset \mathbf{E}$ bded. circ.dom, $0 \in \mathbf{D}$

$\mathbf{E}_D := \{\varrho F(0) : F \in \mathrm{Aut}(\mathbf{D}),\ \varrho \geq 0\}$ compl. subsp. in $\mathbf{E}$,

$\mathbf{D}_0 := \mathbf{D} \cap \mathbf{E}_D$ symm.dom., $\mathbf{D}_0 = B(\mathbf{E}_D)$, $\mathrm{Conv}(\mathbf{D}) = B(\mathbf{E})$ w.loss.gen.

$$\mathrm{aut}(\mathbf{D}) = \{\text{complete hol. vfieds in } \mathbf{D}\} =$$
$$= \Big\{ \big[x \mapsto a - Q_a(x,x) + iLx\big] : a \in \mathbf{E}_D,\ L \in \mathcal{L}(\mathbf{E}),\ \exp(iL)\mathbf{D} = \mathbf{D} \Big\}$$

Triple product: $\{xa^*y\} := Q_a(x,y)$ satisfies Jordan identities

$\big(\mathbf{E}, \{..^*.\}\big)$ partial $JB^*$-triple

**Remark.** Contained implicitly in W. Kaups's works 1970-76.
Unified approach in our book [Isidro-Stachó 1985]
Main stream concentrates to symm. domains $\rightarrow$ $JB^*$-triples

# Partial $JB^*$-triples

$\mathbf{E}_0 \subset \mathbf{E}$ closed subsp, $\{..^*.\} : \mathbf{E} \times \mathbf{E}_0 \times \mathbf{E} \to \mathbf{E}$

(J0) $\big(\mathbf{E}_0, \{..^*.\}\big)$ $JB^*$-triple:
$\mathbf{D}_0 := B(\mathbf{E}_0)$ symmetric domain

(J1) $\{xa^*y\}$ symm 2-lin in $x, y$, conj.lin in $a$, $\|\{xa^*y\}\| \leq M\,\|x\|\,\|a\|\,\|y\|$

(J2) $L(a,b) : y \mapsto \{ab^*y\}$, $\delta_a := iL(a,a)$ satisfy

$$\delta_a\big\{xc^*x\big\} = 2\big\{[\delta_a x]c^*x\big\} + \big\{x[\delta_a c]^*x\big\}:$$

$$\big\{ab^*\{xc^*y\}\big\} = \big\{\{ab^*x\}c^*y\big\} - \big\{x\{ba^*c\}^*y\big\} + \big\{xc^*\{ab^*y\}\big\}$$

**[W. Kaup 1983]** In case of $\mathbf{E}_0 = \mathbf{E}_D$, (J0) is consequence of

(J3) $L(a,a)|\mathbf{E}_0 \in \mathrm{Her}(\mathbf{D} \cap \mathbf{E}_0)$,

$$\mathcal{J}(a) = \mathrm{Gen}_{\{..^*\}}\{a\} \text{ isometr.isomorphic to } \mathcal{C}(\Omega_a)$$

$$\text{with } \Omega_a := \mathrm{Sp}\big(L(a,a)|\mathcal{J}(a) \subset \mathbb{R}_+,$$

$$\mathrm{Sp}\big(L(a,a)|\mathbf{E}_0\big) \subset \tfrac{1}{2}[\Omega_a \cup \{0\}] + \tfrac{1}{2}[\Omega_a \cup \{0\}]$$

(J4) $\|\{aa^*a\}\| = \|a\|^3 \quad (a \in \mathbf{E}_0)$.

**[D. Panou 1989]** $0 \in \mathbf{D} \subset \mathbb{C}^N$ bded circ.dom. $\Rightarrow \mathrm{Sp}L(a,a) \geq 0$,

Inner derivations in $\mathrm{aut}(\mathbf{D}_0)$ extend uniquely to inn.der. in $\mathrm{aut}(\mathbf{D})$:

(Ext) $\Big\| \sum_n \alpha_n L(a_n,a_n) \Big\| \leq M \Big\| \sum_n \alpha_n L(a_n,a_n)\big|\mathbf{E}_D \Big\| \quad (a_n \in \mathbf{E}_D,\ \alpha_n \in \mathbb{R})$

# GEOMETRIC partial JB*-triples

Panou's proofs heavily finite dim. $\leftarrow$ compact Lie Algebras

Shape of bicircular doms, $\mathbf{D} \subset \mathbb{C}^K \times \mathbb{C}^L$
$D \cap \left(\mathbb{C}^K \times \{0\}\right)$ symm., $[x, y] \in \mathbf{D} \Rightarrow \left[e^{i\tau}x, e^{i\sigma}y\right] \in \mathbf{D}$ $(\sigma, \tau \in \mathbb{R})$

**[Stachó 1990]** In general: **D** circular $\Rightarrow \mathrm{Sp}\big(L(a, a)\big) \geq 0$ $\big(a \in \mathbf{E}_D\big)$

Ultraproduct techniques ( $\not\sim$ Dineen, Panou)

**[Stachó 1991]** Assume (J1),(J2),(J4) for $\big(\mathbf{E}, \mathbf{E}_0, \|\cdot\|, \{..^*.\}\big)$. EQIUV:
(i) $\exists\, \mathbf{D} \subset \mathbf{E}$ bded circ.dom with $\mathbf{E}_0 \subset \mathbf{E}_D$;
(ii) also (J3*),(J5) holds where

(J3*) $L(a, a) \in \mathrm{Her}_+(\mathbf{E}, \|\cdot\|) := \big\{A \in \mathcal{L}(\mathbf{E}) : \|\exp(\zeta A)\| \leq 1 \text{ if } \mathrm{Re}\,\zeta \leq 0\big\}$,

(J5) $\big\{za^*\{zb^*z\}\big\} = \big\{\{za^*z\}b^*z\big\}$ $\big(a, b \in \mathbf{E}_0,\ z \in \mathbf{E}\big)$ weak assoc.

NEW even in finite dim

# PROBLEMS

Techniques: Banach-Lie alg. of automorphism germs on $\mathbf{E}_0$
decomposition of **E**-complete orbits

**Remark.** Explicit construction of some suitable domains, but not all.

**Feasable projects.**

a) Classification in finite dim. shape integration;

b) The (Ext) property in $\infty$-dims: partial results **[Stachó 1995-99]** with WEIGHTED GRIDS

c) Elaborate weighted grid theory, continuation of **[Stachó 2010]**, application to a)

d) Direct approach to the B-L topology of $\mathrm{aut}(\mathbf{D})$ for circ. dom.s, short alternative for Chapter VI in **[book Isidro-Stachó 1985]**

# Theory with Strongly Cont. Semigroups

Aim: $\mathcal{C}_0$-sgr of isometries wrt. to $\mathrm{Aut}(\mathbf{D})$-inv. distances, $\mathrm{Iso}(d_{\mathbf{D}})$

**Franzoni-Vesentini 1980:** Holomorphic Maps and Invariant Distances
Last chapter: Hilbert ball

Visits of R. Nagel in Pisa $\to$ linear models, Hille-Yosida theory ?

$\Phi \mapsto \left[f(\in \mathrm{Hol}(\mathbf{D},\mathbf{E})) \mapsto f \circ \Phi\right]$ goes beyond H-Y,
Unsuitable estimates in Fréchet setting as far (future hopes?)

Alternative approach in $\infty$-dim. reflexive Cartan factors
$\mathbf{E}$ refl, $\mathbf{D} = B(\mathbf{E})$ symm $\to$ $\mathbf{E} = \left[\text{finite } \ell^\infty\text{-sum of refl. C-factors}\right]$
$\mathrm{Aut}(\mathbf{D})$ reduced by the above decomposition
$\infty$-dim parts $\simeq$ $\mathcal{L}(\mathbf{H},\mathbf{K}), \mathrm{Spin}(\mathbf{H},\bar{\cdot})$, $\mathbf{H},\mathbf{K}$ Hilbert, $\dim(\mathbf{K}) < \infty$.

**Remark.** $\mathcal{L}(\mathbf{H},\mathbf{K})$ and $\mathrm{Spin}(\mathbf{H})$ are motived by Physics, $\mathbf{H} \simeq \mathcal{L}(\mathbf{H},\mathbb{C})$
In general, $\mathrm{Iso}(d_{\mathbf{B}})$ is also reduced by the factor decomposition
[Apazoglou-Peralta, Stachó 2016]

Steps:

1) Generalize Hierzbruch's finite dim. matrix representations

for $\mathrm{Aut}(B(\mathbf{E}))$, $\mathbf{E} = \mathcal{L}(\mathbf{H}, \mathbf{K})$, $\mathrm{Spin}(\mathbf{H})$

Tools: Carathéodory dist., gen. Möbius trf.,
direct descr. of lin. isometries (with Franzoni),
Cartan uniqueness thm

Required features: lin. $C_0$ repr. $\longrightarrow$ $C_0$ sgr.

2) Characterize non-cont. lin. operators corresponsponding to infinitesimal generators of a $C_0$-sgr. in Hierzbruch type repr.

Tools: lin. Hille-Yosida theory (before [Engel-Nagel])

3) Describe an integration process for the calculated inf. gen.-s.

**Remark.** Missing for a "perfect closed" theory:
explicit final formulas, non-continous lin. repr. may represent $C_0$ sgr-s

# ADJUSTED CONTINUITY

**Conjecture.** (Botelho-Jamison 2008)
If $\mathcal{B} := \{\text{bded } N\text{-lin maps } \mathbf{X}_1 \ldots, \mathbf{X}_N \to \mathbb{C}\}$ with
$[\Lambda^t : t \in \mathbb{R}]$ $\mathcal{C}_0$-group of surj lin isometries $\mathcal{B} \to \mathcal{B}$
then

$$\Lambda^t = \underbrace{U_1^t \otimes \cdots \otimes U_N^t}_{\varphi \mapsto \varphi(U_1^t x_1, \ldots, U_N^t x_N)} \qquad t \mapsto U_k^t \quad \textit{str.cont. 1-par.grp. of surj.lin.isom. } X_k \to X_k$$

**Problem.** $U_1 \otimes \cdots \otimes U_N = [\kappa_1 U_1] \otimes \cdots \otimes [\kappa_N V_N]$ if $\prod_j \kappa_j = 1, \kappa_j \in \mathbb{T}$

**Stachó** [JMAA 2010]: Proof for Hilbert sp $\mathbf{X}_k$, probabilistic argument

$U_1^t \otimes \cdots \otimes U_N^t = [\kappa_{1,t} U_1^t] \otimes \cdots \otimes [\kappa_{N,t} U_N^t]$ with $[\kappa_{j,t} U_j^t : t \in \mathbb{R}]$ $\mathcal{C}_0$-group

# HILBERT BALL BY VESENTINI

$\mathbf{H}, \langle\cdot|\cdot\rangle$ Hilbert space, $\quad a^* := [x \mapsto \langle x|a\rangle]$

$x \oplus \xi \equiv \left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right]$, $\quad \mathcal{L}(\mathbf{H} \oplus \mathbb{C}) \equiv \left\{ \left[\begin{smallmatrix} A & b \\ c^* & d \end{smallmatrix}\right] : \; A \in \mathcal{L}(\mathbf{H}),\, b, c \in \mathbf{H},\, d \in \mathbb{C} \right\}$

$\mathbf{B} = B(\mathbf{H}), \;\; I = \mathrm{Id}_{\mathbf{H}} \quad \mathcal{J} = \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix}$

Fractional linear trf.:

$$F \begin{bmatrix} A & b \\ c^* & d \end{bmatrix} : x \mapsto \frac{Ax + b}{c^* x + d} \;\; (c^* x + d \neq 0), \qquad \begin{matrix} F(\mathcal{G}) = F(\lambda\mathcal{G}), \\ \lambda \in \mathbb{C} \setminus \{0\} \end{matrix}$$

Matrix repr: $\quad \mathfrak{G} = \{\mathcal{G} : \mathcal{G}^* \mathcal{J} \mathcal{G} = \mathcal{J}\}$

$\Phi \in \mathrm{Iso}(d_{\mathbf{B}}) \iff \Phi = F(\mathcal{G})$ with $\mathcal{G} \in \mathfrak{G}$

$F(\mathcal{G}_1\mathcal{G}_2) = F(\mathcal{G}_1) \circ F(\mathcal{G}_2)$ on a neighborhood of $\overline{\mathbf{B}}$ if $\mathcal{G}_1, \mathcal{G}_2 \in \mathfrak{G}$

Ambiguity: $F(\mathcal{G}) = F(\kappa\mathcal{G}),\; \kappa\mathcal{G} \in \mathfrak{G}$ $\quad$ if $\quad |\kappa| = 1,\; \mathcal{G} \in \mathfrak{G}$

Möbius decomposition

$$\Phi^t = \Theta_{a_t} \circ U_t, \;\; a_t = \Phi^t(0), \;\; U_t \;\mathbf{H}\text{-isom}, \;\; \Theta_{a_t} = F(\mathcal{M}_{a_t}), \;\; U_t = F(\mathcal{U}_t)$$

$$\mathcal{M}_a = \begin{bmatrix} \mathrm{Id}_{\mathbf{H}} - aa^* & 0 \\ 0 & 1 - a^*a \end{bmatrix}^{-1/2} \begin{bmatrix} 1 & a \\ a^* & 1 \end{bmatrix}, \quad \mathcal{U}_t = \begin{bmatrix} U_t & 0 \\ 0 & 1 \end{bmatrix}$$

norm cont. in $\mathcal{M}_{a_t}$ and str. cont. in $\mathcal{U}_t$ iff $t \mapsto \Phi^t$ is str. cont.

**Remark.** Even in 1 dim, $[\mathcal{M}_{a_t}\mathcal{U}_t : t \in \mathbb{R}_+]$ no sgr. in general.
Example: $\Phi^t(\zeta) = \frac{1-it\zeta}{1+it\zeta} \cdot M_{it/(1-it)}(\zeta)$

**Theorem.** [Ves87, Section 2].

Let $\mathcal{G}^t \in \mathfrak{G}$ $(t \geq 0)$ and define $\Phi^t = F(\mathcal{G}^t)\big|_{\mathbf{H}}$.

If $\left[\mathcal{G}^t : t \in \mathbb{R}_+\right]$ is a $\mathcal{C}_0$-sgr in $\mathcal{L}(\mathbf{H} \oplus \mathbb{C})$

then $[\Phi^t : t \in \mathbb{R}_+]$ is a $\mathcal{C}_0$-sgr of holomorphic $d_{\mathbf{B}}$-isometries whose generator $\Phi'$ is densely defined (and $\Phi^t$-invariant) in $\mathbf{B}$

**Remark.** $\Phi^t = F\big(\kappa_t \mathcal{G}^t\big)\big|_{\mathbf{H}}$ with any function $t \mapsto \kappa_t \in \mathbb{T} = \{\text{unit circle}\}$

**Theorem.** [Ves87, Thms. V+VI with Prop.5.3; corrected in Ves94].

$\mathcal{A} : \mathrm{dom}(\mathcal{A}) \overset{\text{lin}}{\to} \mathbf{H} \oplus \mathbb{C}$ generates of a $\mathcal{C}_0$-sgr $[\mathcal{G}^t : t \in \mathbb{R}_+]$ in $\mathcal{L}(\mathbf{H} \oplus \mathbb{C})$ giving rise to the $\mathcal{C}_0$-sgr $[\Phi^t : t \in \mathbb{R}_+]$, $\Phi^t = F(\mathcal{G}^t)$ of $d_{\mathbf{B}}$-isometries such that $0 \in \mathrm{dom}(\Phi')$ if and only if

$$\mathcal{A} = \begin{bmatrix} iA + \nu I & b \\ b^* & \nu \end{bmatrix} \qquad \begin{array}{c} \nu \in \mathbb{C}, \quad b \in \mathbf{H} \\ iA = \left[ \text{gen. of a } \mathcal{C}_0\text{-sgr of } \mathbf{H}\text{-isometries} \right] \end{array}$$

In this case,

$\mathrm{dom}(\Phi') = \left\{ x \in \mathbf{B} : \ x \oplus 1 \in \mathrm{dom}(\mathcal{A}) \right\}$,

$$\Phi'(\mathbf{x}) = \frac{d}{dt}\Big|_{t=0+} F(\mathcal{G}^t)(x) = \frac{d}{dt}\Big|_{t=0+} \frac{\left[\mathcal{G}^t x \oplus 1\right]_{\mathbf{H}}}{\left[\mathcal{G}^t x \oplus 1\right]_{\mathbb{C}}} =$$

$$= b - \langle x | b \rangle x + iAx = b - \{x b^* x\} + iAx$$

NO EXPLICIT FORMULAS. $\leftarrow$ no Jordan-algebra

treating the Dyson-Philips series for a sgr with gen $\begin{bmatrix} iA & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b^* & 0 \end{bmatrix}$.

# FIXED POINTS

**Vesentini 1987-94:**

1) **E** reflexive JB*-triple, $[\Phi^t : t \in \mathbb{R}_+]$ $\mathcal{C}_0$-sgr in $\mathrm{Iso}(d_{B(\mathbf{E})})$

$\Phi^t \longrightarrow \overline{\Phi}^t$ weak*-cont extension to $\overline{\mathbf{B}}$,

$[\overline{\Phi}^t : t \in \mathbb{R}_+]$ $\mathcal{C}_0$-sgr wrt. $\|\cdot\|$, $\bigcap_{t\in\mathbb{R}_+} \mathrm{Fix}(\overline{\Phi}^t) \neq \emptyset$

2) **F** refl. Cartan factor, $[\Psi^t : t \in \mathbb{R}_+]$ $\mathcal{C}_0$-sgr in $\mathrm{Iso}(d_{B(\mathbf{F})})$,

$a \in \mathrm{Fix}[\Psi^t : t \in \mathbb{R}_+]$ with $\|a\| < 1 \ \Rightarrow$

$\Psi^t = \Theta_a \circ U^t \circ \Theta_{-a}$ with $[U^t : t \in \mathbb{R}_+]$ $\mathcal{C}_0$-sgr of lin **F**-isometries.

**Question.** Is $U^t$ linear for generic JB*-triple **E** ? (NO: end of talk)

# ADJUSTMENT WITH FIXED POINTS

**Stachó** [JMAA 2017]

$\mathbf{E} = \mathbf{H}$ Hilbert sp., $\quad e \in \bigcap_t \mathrm{Fix}\overline{\Phi}^t, \ \|e\| = 1, \quad 0 \in \mathrm{dom}(\Phi')$

**Thm.** $\Phi^t = F(\mathcal{G}^t), \ \mathcal{G}^t = \begin{bmatrix} A_t & b_t \\ c_t^* & d_t \end{bmatrix} \quad \Rightarrow$

(1) $\mathcal{G}^t \begin{bmatrix} e \\ 1 \end{bmatrix} = \lambda_t \begin{bmatrix} e \\ 1 \end{bmatrix} \qquad \lambda_t = \left[\mathcal{G}^t \begin{bmatrix} e \\ 1 \end{bmatrix}\right]_{\mathbb{C}} = \langle e | c_t \rangle + d_t$

(2) $\mathcal{K}^t := \lambda_t^{-1} \mathcal{G}^t \quad (t \in \mathbb{R}_+) \ \ \mathcal{C}_0$-sgr, $\ F(\mathcal{K}^t) = F(\mathcal{G}^t) = \Phi^t$

$\Phi'(x) = b - \langle x | b \rangle x + iAx, \ e \in \mathrm{dom}(\Phi') = \mathrm{dom}(A), \ \ \mathcal{K}^t \begin{bmatrix} e \\ 1 \end{bmatrix} = e^{\nu t} \begin{bmatrix} e \\ 1 \end{bmatrix}$

**Corollary.** $\mathcal{A} = \mathcal{K}' = \begin{bmatrix} iA & b \\ b^* & 0 \end{bmatrix}, \quad \mathrm{dom}(\mathcal{A}) = \mathrm{dom}(A) \oplus \mathbb{C},$

$iA = U', \quad [U^t : t \in \mathbb{R}_+]$ $\mathcal{C}_0$-sgr of lin $\mathbf{H}$-isom

$$\mathcal{T} = \begin{bmatrix} \mathrm{Id}_{\mathbf{H}} & e \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & \mathrm{Id}_{\mathbf{H}_0} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{H} = \mathbf{H}_0 \oplus \mathbb{C}e$$

$$\mathcal{M}^t = \mathcal{T}^{-1}\mathcal{K}^t\mathcal{T}, \quad \mathcal{M}' = \mathcal{T}^{-1}\mathcal{A}\mathcal{T} = \begin{bmatrix} -\overline{\nu} & 0 & 0 \\ -b_0 & iA_0 & 0 \\ \nu & b_0^* & \nu \end{bmatrix}, \qquad \begin{matrix} b_0 = P_{\mathbf{H}_0} b, \\ \nu = \langle e | b \rangle \end{matrix}$$

$A_0 = P_{\mathbf{H}_0} A|_{\mathbf{H}_0}$, $iA_0 = U_0'$ with $[U_0^t : t \in \mathbb{R}_+]$ $\mathcal{C}_0$-sgr of $\mathbf{H}_0$ isom

**Thm**. There exist constants $\lambda, \mu \in \mathbb{R}$ such that, by setting

$$S = A - i\mu I, \quad S_0 = P_{\mathbf{H}_0} S \big| \mathbf{H}_0, \quad V_0^t = e^{i\mu t} U_0^t, \quad \widetilde{b}_0 = P_{\mathbf{H}_0}(iSb),$$

$$P_e \Phi^t(x_0 \oplus \zeta e) = \Big[\varphi_{\lambda,\mu}(t,x,\xi)^{-1}(\xi - 1)e^{-2\lambda t} + 1\Big] e,$$

$$P_{\mathbf{H}_0}\Phi^t(x_0 \oplus \zeta e) = \frac{1}{\varphi_{\lambda,\mu}(t,x_0,\xi)} \Big[e^{-\lambda t} V_0^t x_0 - (\xi - 1)e^{-2\lambda t}\Big(\textstyle\int_0^t e^{\lambda\tau} V_0^\tau \, d\tau\Big)\widetilde{b}_0\Big],$$

$$\varphi_{\lambda,\mu}(t,z,\xi) := \Big\langle \Big(\textstyle\int_0^t e^{-\lambda\tau} V_0^\tau d\tau\Big) z \Big| \widetilde{b}_0 \Big\rangle + (\xi - 1)\big(\lambda + i\mu\big) e^{-2\lambda t} + 1 -$$

$$-(\xi - 1)\Big\langle \Big(\textstyle\int_0^t e^{-2\lambda\tau} \int_0^\tau e^{\lambda\sigma} V_0^\sigma d\sigma d\tau\Big) \widetilde{b}_0 \Big| \widetilde{b}_0 \Big\rangle .$$

**Corollary.** Each str.cont. 1-prsg $[\Psi^t : t \in \mathbb{R}_+]$ of hol. Carathéodory **B**-isometries with exactly two joint boundary fixed points is Möbius equivalent to some semigroup $[\Phi^t : t \in \mathbb{R}_+]$ of the form

$$P_e \Phi^t(x_0 \oplus \zeta e) = \frac{(1-\lambda)(1-\xi)e^{-2\lambda t} + 1}{1-(1-\xi)\lambda e^{-2\lambda t}} e,$$

$$P_{\mathbf{H}_0} \Phi^t(x_0 \oplus \zeta e) = \frac{e^{-\lambda t}}{1-(1-\xi)\lambda e^{-2\lambda t}} V_0^t x_0 \ .$$

**Dilation Thm.** $\exists\, [\widehat{\Phi}^t : t \in \mathbb{R}]$ $\mathcal{C}_0$**-group in** $\mathrm{Aut}\big(B(\widehat{\mathbf{H}})\big)$ **with some Hilbert space** $\widehat{\mathbf{H}} \supset \mathbf{H}$ **(as subspace) such that** $\Phi^t = \widehat{\Phi}^t \big| \mathbf{B}$ $\ (t \in \mathbb{R}_+)$.

**Proof:** $\exists$ **unitary dilation** $[\widehat{V}_0^t : r \in \mathbb{R}]$ **of the isom sgr** $[V_0^t : t \in \mathbb{R}_+]$**:**

# TRO CASE: $\mathbf{E} = \mathcal{L}(\mathbf{H}_1, \mathbf{H}_2)$, $\dim(\mathbf{H}_2) < \infty$

Projective representation:

$$F \begin{bmatrix} A & B \\ C & D \end{bmatrix} : X \mapsto (AX + B)(CX + D)^{-1} \qquad \big(\mathbf{H}_1 \oplus \mathbf{H}_2\text{-matrices}\big)$$

**Vesentini 1994** $+$ **Khatskevich-Reich-Shoikhet 2001**

If $[\kappa_t \mathcal{G}^t : t \in \mathbb{R}_+]$ $\mathcal{C}_0$-sgr in $\mathcal{L}(\mathbf{H}_1 \leftarrow \mathbf{H}_2)$ with $\Phi^t = F(\kappa_t \mathcal{G}_t) \in \mathrm{Iso}(d_{\mathbf{B}})$

then $[\kappa \mathcal{G}]' = \begin{bmatrix} U_1' + \nu I_1 & B \\ B^* & U_2' + \nu I_2 \end{bmatrix}$ with $[U_k^t : t \in \mathbb{R}_+]$ $\mathcal{C}_0$-sgr in $\mathcal{L}(\mathbf{H}_k)$

Adjustment $\kappa_t$: JORDAN theoretic approach **[Stachó 2012]**

# STRUCTURE

**Thm.** [Stachó 2016-7] Up to Möbius-equiv, $\exists\, E$ tripotent, $0 \in \mathrm{dom}(\Phi')$ ,

$$\Phi^t(X) = E + W^t(X-E)\Big[\int_0^t S^{t-h} B^* W^h(X-E) + S^t\Big]^{-1}$$

**Corollary.** Triang. wrt. $\mathbf{H}_1 = \mathbf{H}_{10} \oplus \mathbf{H}_{11}$, $\mathbf{H}_{10} = \mathrm{range}(E)$

$$W^t = \begin{bmatrix} W_0^t & 0 \\ \int_0^t W_1^{t-s}\big[P_{11}(U' - EB^*)P_{10}\big] W_0^s ds & W_1^t \end{bmatrix}, \quad \begin{matrix} P_{10} = P_{\mathrm{ran}(E)} = EE^* \\ P_{11} = \mathrm{Id}_{\mathbf{H}_1} - P_{10}, \end{matrix}$$

$W_0^t = \exp\big(tP_{10}[U' - EB^*]P_{10}\big)$ finite dim,

$[W_1^t : t \in \mathbb{R}_+] \subset \mathcal{L}\big(\mathbf{H}_{11}\big)$, $\mathcal{C}_0$-sgr of isometries, $W_1' = P_{11}U'|\mathbf{H}_{11}$.

# SPIN FACTORS

$(\mathbf{H}, \langle\cdot|\cdot\rangle)$ Hilbert space, $x \mapsto \overline{x}$ conjugation, $\langle x|y\rangle^- = \langle\overline{x}|\overline{y}\rangle$

$\mathcal{S} := \mathcal{S}(\mathbf{H}, \bar{\cdot})$ JB*-triple

$$\{xa^*y\} = \langle x|a\rangle y + \langle y|a\rangle x - \underbrace{\langle x|\overline{y}\rangle}_{\langle y|\overline{x}\rangle} \overline{a}$$

$e = \{eee\}$ TRIPOTENT:

(1) $e = \lambda v, \quad \lambda \in \mathbb{T}, \quad v \in \mathrm{Re}(\mathbf{H}), \quad \|v\| = 1;$

(2) $e = \lambda u + iv, \quad \lambda \in \mathbb{T}, \quad u, v \in \mathrm{Re}(\mathbf{H}), u \perp v, \quad \|u\| = \|v\| = \frac{1}{2}.$

$\mathcal{S}$-unitary op.s: $U_t = \kappa_t V_t$: $\quad V_t : \mathrm{Re}(\mathcal{S}) \to \mathrm{Re}(\mathcal{S})$ $\langle\cdot|\cdot\rangle$-unitary, $\kappa_t \in \mathbb{T}$.

**History.** Pauli matrices $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\longrightarrow$ $\mathbb{S}$ cl. selfadj. subs.$\subset \mathcal{L}(\mathbf{H})$, $s^2 \in \mathbb{C}\mathrm{Id}$ $(s \in \mathbb{S})$.

Fractional lin. form for some $\mathcal{C}_0$-groups of inner automorphisms.

**Vesentini 1989.** No fr.lin. form for gen. $\mathbf{\Phi}$ in $\mathbb{S}$-setting

**Hierzbruch 1965.** finite dim. $\rightarrow$ **Vesentini 1992.**

$$\Phi^t = R\big(G^t\big), \qquad [G^t : t \in \mathbb{R}_+] \quad \mathcal{C}_0 \text{ sgr. in } \mathcal{L}(\mathbf{H} \oplus \mathbb{C}^2).$$

**Remark.** Continuity adjustment HARMLESS.

$$G^t = \begin{bmatrix} M_t & B_t \\ C_t^{\mathrm{T}} & E^t \end{bmatrix} \qquad B_t = [b_1^t, b_2^t] \in \mathbf{H}^2, \; C_t^{\mathrm{T}} = \begin{bmatrix} \overline{c_1^t}^* \\ \overline{c_2^t}^* \end{bmatrix}, \; E = \big[E_{k\ell}\big]_{k,\ell=1}^2$$

MATRIX REPRESENTATION:

$$\Phi^t(x) = R(G^t)(x) = F^t(x)/\varphi^t(x)$$

$$F^t(x) = (b_1^t - ib_2^t) + 2M_t x + (x^{\mathrm{T}}x)(b_1^t + ib_2^t)$$

$$\varphi^t(x) = (E_{11}^t + E_{22}^t - iE_{12}^t + iE_{21}^t) + 2(c_1^t + ic_2^t)^{\mathrm{T}} x + (E_{11}^t - E_{22}^t + iE_{12}^t + iE_{21}^t) x^{\mathrm{T}} x$$

Alg. constrains:

$$[G^t]^* \mathrm{diag}(I, -I_2) G^t = \mathrm{diag}(I, -I_2), \quad \det(E^t) > 0 \qquad (t \in \mathbb{R}_+),$$

$$C_t E^t = M_t^{\mathrm{T}} B_t, \quad M_t^{\mathrm{T}} = I + C_t C_t^{\mathrm{T}}, \quad [E^t]^{\mathrm{T}} E^t = I_2 + B_t^{\mathrm{T}} B_t.$$

**Proposition.** If (up-to Möbius-equiv) $0 \in \mathrm{dom}(\Phi')$ then

$\Phi'(x) = a + iAx - \{xa^*x\}$ is of Kaup's type and

$$G' = \begin{bmatrix} iA - i\varepsilon I & 2\,\mathrm{Re}(a) & -2\,\mathrm{Im}(a) \\ 2\,\mathrm{Re}(a)^{\mathrm{T}} & 0 & -\varepsilon \\ -2\,\mathrm{Im}(a)^{\mathrm{T}} & \varepsilon & 0 \end{bmatrix}$$

$\varepsilon \in \mathbb{R}$, $\quad iA = U'$ with $[U^t : t \in \mathbb{R}_+]$ $\mathcal{C}_0$-sgr of real **H**-isom

# TRIANGULARIZATION WITH FIXED POINTS

$0 \neq e \in \bigcap_t \mathrm{Fix}(\overline{\Phi}^t)$ common fixed point

**[Stachó 2020]** Up to Möbius equiv: $e$ TRIPOTENT

$$\Phi'(x) = a + iAx - \{xa^*x\} =$$
$$\left(\tfrac{1}{2}b_1 - \tfrac{i}{2}b_2\right) + M'x + i\varepsilon x - \langle x|b_1 - ib_2\rangle x + \langle x|\overline{x}\rangle\left(\tfrac{1}{2}b_1 + \tfrac{i}{2}b_2\right),$$

$$G' = \begin{bmatrix} M' & b_1 & b_2 \\ b_1^{\mathrm{T}} & 0 & -\varepsilon \\ b_2^{\mathrm{T}} & \varepsilon & 0 \end{bmatrix} \quad \text{where} \quad \begin{array}{l} b_1 := 2\mathrm{Re}(a), b_2 := -2\mathrm{Im}(a), \\ M' = \overline{M'} = -[M']^{\mathrm{T}}, \ \ \varepsilon \in \mathbb{R}. \end{array}$$

**Cases up to lin. equiv.**

1) $e = \overline{e}$, $\langle e|e\rangle = 1$ (real extreme point),

2) $e \perp \overline{e}$, $\langle e|e\rangle = \frac{1}{2}$ (face middle point).

**Case (1)**: $\mathbf{H} \oplus \mathbb{C}^2 = [\mathbb{C}e] \oplus \mathbf{H}_0 \oplus \mathbb{C} \oplus \mathbb{C}$ matrix decomposition

$$G' = \begin{bmatrix} M' & b_1 & b_2 \\ b_1^{\mathrm{T}} & 0 & -\varepsilon \\ b_2^{\mathrm{T}} & \varepsilon & 0 \end{bmatrix} = \begin{bmatrix} 0 & x_1^{\mathrm{T}} & \rho_1 & -\varepsilon \\ -x_1 & M_1' & x_1 & x_2 \\ \rho_1 & x_1^{\mathrm{T}} & 0 & -\varepsilon \\ -\varepsilon & x_2^{\mathrm{T}} & \varepsilon & 0 \end{bmatrix}$$

**Quasi-triangular form**

$$T := \begin{bmatrix} 1/2 & 0 & 0 & 1 \\ 0 & I_1 & 0 & 0 \\ -1/2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad T^{-1} G' T = \begin{bmatrix} -\rho_1 & 0 & 0 & 0 \\ -x_1 & M_1' & x_2 & 0 \\ -\varepsilon & x_2^{\mathrm{T}} & 0 & 0 \\ 0 & x_1^{\mathrm{T}} & -\varepsilon & \rho_1 \end{bmatrix}.$$

**Remark.** (1) $M_1' = U_0'$ with $[U_1^t : t \in \mathbb{R}_+]$ $\mathcal{C}_0$-sgr.

(2) $G'$ triang if $y = \langle z|y\rangle z - M_1' z$ has solution $z \in e^{\perp}$, e.g. if $y \in \mathrm{range}(M_1')$

**Case (2):** $e = \frac{1}{2}u + \frac{i}{2}v, \quad u \perp v, \;\; u, v \in \mathrm{Re}(\mathbf{H}), \;\; \langle u\rangle^2 = \langle v\rangle^2 = 1.$

$$0 = \Phi'(e) = \left(\tfrac{1}{2}b_1 - \tfrac{i}{2}b_2\right) + M'e + i\varepsilon e - \langle e|b_1 - ib_2\rangle e.$$

$$G' = \begin{bmatrix} M' & b_1 & b_2 \\ b_1^{\mathrm{T}} & 0 & -\varepsilon \\ b_2^{\mathrm{T}} & \varepsilon & 0 \end{bmatrix} = \left[\begin{array}{cc|c|cc} 0 & 2\rho_2-\varepsilon & x_1^{\mathrm{T}} & \rho_1 & \rho_2 \\ \varepsilon-2\rho_2 & 0 & -x_2^{\mathrm{T}} & \rho_2 & -\rho_1 \\ \hline -x_1 & x_2 & M_2' & x_1 & x_2 \\ \hline \rho_1 & \rho_2 & x_1^{\mathrm{T}} & 0 & -\varepsilon \\ \rho_2 & -\rho_1 & x_2^{\mathrm{T}} & \varepsilon & 0 \end{array}\right] \begin{array}{l} \}\mathbb{C}u \oplus \mathbb{C}v \\ \leftarrow \{u, v\}^{\perp} \\ \}\mathbb{C} \oplus \mathbb{C} \end{array}$$

**Quasi-triangular** $\mathbb{C}^2 \oplus \mathbf{H}_0 \oplus \mathbb{C}^2$**-form** $T^{-1}G'T$

$$\left[\begin{array}{ccccc} -\rho_1 & \varepsilon-\rho_2 & 0 & 2\varepsilon & 0 \\ \rho_2-\varepsilon & -\rho_1 & 0 & 0 & 2\varepsilon \\ x_2 & -x_1 & M_2' & 0 & 0 \\ \rho_2 & \rho_1 & x_1^{\mathrm{T}} & \rho_1 & -\varepsilon-\rho_2 \\ -\rho_1 & \rho_2 & x_2^{\mathrm{T}} & \rho_2+\varepsilon & \rho_1 \end{array}\right], \; T = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & I_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

**Conclusion.**
$[G^t : t \in \mathbb{R}]$ can be expressed with finite formulas
which are fractional lin. terms of

some $\mathcal{C}_0$-sgr. of Hilbert space isometries,

Hilbert space operators of rank 1,

some classical special functions,

the solution of a Volterra type scalar convolution equation

admitting closed form with Laplace- and inverse Laplace transforms.

**Dilation.**
Using a Deddens type $\mathcal{C}_0$-group dilation (with enlarged Hilbert space),
we can construct a $\mathcal{C}_0$-group dilation for $[\Phi^t : t \in \mathbb{R}_+]$
on the unit ball of a suitable covering spin factor.

**Open problem.** Simplify the procedure with Laplace transform

# HOLOMORPHIC HILLE-YOSIDA THEORY

[Stachó, Rev. Roum. Acad. Sci. 2018]

**Basic $\mathcal{C}_0$-principles** $\sim$ [Engel-Nagel, Ch. 2]

$D \subset E$ domain, $[\Phi^t : t \in \mathbb{R}_+]$ $\mathcal{C}_0$-sgr in $\mathrm{Hol}(\mathbf{D})$, generic type

$x \in \mathrm{dom}(\Phi') \iff t \mapsto \Phi^t(x)$ differentiable.

$\mathrm{dom}(\Phi')$ is $\Phi^t$-invariant, $\Phi'\big(\Phi^t(x)\big) = \Phi^t\big(\Phi'(x)\big)$;

$\mathrm{graph}(\Phi')$ is rel.closed in $\mathbf{D} \times \mathbf{E}$;

$\Phi^t$ is unambigously defd on $\overline{\mathrm{dom}(\Phi')}$.

**Proof.** With Cauchy estimates $\not\approx$ linear argument

**Open problem.** $\exists$? $[\Phi^t : t \in \mathbb{R}_+]$ nowhere diff. in $t$?

**Remark.** $\exists$ real non-linear dyn. system without inf.gen.

# 0-FIXING $d_{B(\mathbf{E})}$-ISOMETRIES

$\mathbf{D} = \mathbf{B} = B(\mathbf{E})$ unit ball

**Question.** Cartan's Linearity Thm. with holomoric $d_{\mathbf{B}}$-isometries?

**Counter-ex.** [Vesentini 1992]:

$\mathbf{E} := c_0, \ \ \Phi(\zeta_0, \zeta_1, \zeta_2 \dots) := (\zeta_0^2, \zeta_0, \zeta_1, \zeta_2, \dots)$

Not suited directly for constructing non-lin $\mathcal{C}_0$-sgr counter-ex.

**Proposition.** If $\Phi(0) = 0$ then $\Phi$ differs from its linear part only with vectors of tangential directions.

**New counter-ex with $\mathcal{C}_0$-sgr:** with $\mathbf{E} = C_0(\mathbb{R}_+, \mathbb{C})$,

$$\Phi^t(x) : \mathbb{R}_+ \ni \tau \mapsto \left[ \frac{2x(0)}{\left(1 - e^{2(t-\tau)}\right) x(0) + 2e^{2(t-\tau)}} \text{ if } \tau \leq t, \quad x(\tau - t) \text{ if } \tau \geq t \right]$$

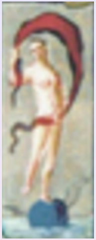
# COMMON FIXED POINTS IN JB*-TRIPLES

**Setting:** **E** JB*-triple,
$M_a =$ [Kaups' Möbius trf $0 \mapsto a$],
$[\Phi^t : t \in \mathbb{R}_+]$ $\mathcal{C}_0$-sgr in $\mathrm{Iso}(d_{B(\mathbf{D})})$.

**Thm.** Assume (i) $\Phi^t = M_{a(t)} \circ U_t \ (t \in \mathbb{R}_+)$, (ii) $\bigcap_t \mathrm{Fix}(\overline{\Phi}^t) \neq \emptyset$.
Then either $\mathrm{dom}(\Phi')$ dense in $B(\mathbf{D})$,
or $\mathrm{dom}(\Phi') = \emptyset$.

Proof: $\widetilde{M}_a : \|a\|^{-1}\mathbf{B} \to \mathbf{E}$ well-def. hol. extension for $M_a$ [Kaup 1983],
Jordan calculations with the Fréchet derivatives
$\Lambda_t = \widetilde{M}^{[1]}_{a(t)}(e)$, $a(t) = \Phi^t(0)$.

# My coat of arms



AB AQUA
AD TERRAM

# Thanks for your attention

# THANKS
# Szőkefalvi-Nagy Béla,
# Edoardo Vesentini,
# Wilhelm Kaup