J. Math. Anal. Appl. 398 (2013) 707–714

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Global dynamics in a commodity market model

Eduardo Liz [a,*], Gergely Röst [b]

[a] Departamento de Matemática Aplicada II, Universidade de Vigo, 36310 Vigo, Spain

[b] Bolyai Institute, University of Szeged, Aradi vértanúk tere 1., H-6720 Szeged, Hungary

ARTICLE INFO

Article history:
Received 24 May 2012
Available online 19 September 2012
Submitted by Juan J. Nieto

Keywords:
Delay–differential equation
Commodity market model
Global attractor
Absolute stability
One-dimensional map

ABSTRACT

We study the global behavior of the price dynamics in a commodity market governed by a balance between demand and supply. While the dependence of demand on price is considered instantaneous, the supply term contains a delay, leading to a delay–differential equation. A discrete model is naturally defined as a limit case of this equation. We provide a thorough study of the discrete case, and use these results to get new sufficient conditions for the global convergence of the solutions to the positive equilibrium in the continuous case. For when the equilibrium is unstable, we provide some bounds for the amplitude of the oscillations that are quite sharp when the delay is large.

1. Introduction

Production delays are responsible for generating fluctuations in economic indicators, and a good mathematical setting in which to study these models is provided by a delay–differential equation. Mackey [1] suggested that the relative variations in market price $P(t)$ are governed by a simple balance between demand and supply:

$$\frac{P'(t)}{P(t)} = D(P(t)) - S(P_S(t)), \tag{1.1}$$

where D and S denote the demand and supply functions for the commodity, respectively. While it is assumed that the consumers base their buying decisions on the current market price, for most commodities there is a finite time τ that elapses before a change in production occurs. This time lag can be affected by several factors, such as the time necessary for increasing production, buying new machines, hiring workers or building factories; in the case of agricultural commodities, there are also natural constraints that affect the delay. Assuming that only the market price at a time $t-\tau$ has an effect on the current supply price $P_S(t)$, we get $P_S(t) = P(t-\tau)$ (see [1] for more details). A more general model was proposed in [2].

Another simplification suggested in [1] consists of assuming that the demand is a decreasing function of the price, and the supply is increasing. An interpretation is that higher price leads to less buying, while industry reacts to higher prices by increasing production [3]. Under these assumptions, there is a unique positive equilibrium P^* of (1.1), which satisfies $D(P^*) = S(P^*)$, that is, demand equals supply.

Mackey found the necessary and sufficient conditions for the asymptotic stability of P^*. In particular, he proved that P^* is asymptotically stable if

$$S'(P^*) \le -D'(P^*), \tag{1.2}$$

* Corresponding author.
E-mail addresses: eliz@dma.uvigo.es (E. Liz), rost@math.u-szeged.hu (G. Röst).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2012.09.024

and, if $S'(P^*) > -D'(P^*)$, then there is a critical value τ^* of the delay such that P^* is asymptotically stable if $\tau < \tau^*$ and unstable if $\tau > \tau^*$. At this critical value, Eq. (1.1) undergoes a Hopf bifurcation, which leads to stable oscillations in the supercritical case. However, it was not discussed whether or not nontrivial oscillations can appear when the positive equilibrium is asymptotically stable. This question is far from being trivial for nonlinear equations, and it leads to the study of the global dynamics of the equation rather than only the dynamics near the equilibrium point.

Assuming that the delay τ is constant, we can set $\tau = 1$ after making the change of variables $t = \tau s$. When the demand and supply relaxation times are very short, then price adjustment is quite rapid, and $P'(t)/P(t) \approx 0$ (see [2,1]), and a limit case of model (1.1) is the difference equation

$$D(x_k) = S(x_{k-1}), \quad k \geq 1, \tag{1.3}$$

where x_k stands for $P(k)$, $k = 1, 2, \ldots$.

As noticed by Mackey, the equilibrium P^* of (1.3) is asymptotically stable if condition (1.2) holds, thus providing an agreement between the continuous and the discrete models. However, it is stated in [2,1] that the correspondence between the global behaviors of (1.1) and (1.3) may be severely limited.

Our aim in this note is to establish a further relationship between the discrete and continuous models, showing that Eq. (1.3) provides useful information about the global dynamics of (1.1). We get, as an application, some new results for the global convergence of the solutions to the positive equilibrium in a particular case of (1.1) recently investigated in [4,5,3].

2. The discrete model

2.1. General results

In this section we consider the difference Eq. (1.3) under the following assumptions:

(A1) $D : [0, \infty) \to [0, \infty)$ and $S : [0, \infty) \to [0, \infty)$ are smooth functions; $D(x) > 0, S(x) > 0$ for all $x > 0, D(0) > 0, S(0) = 0$.
(A2) $D'(x) < 0$ and $S'(x) > 0$ for all $x > 0$.
(A3) There exists $S(\infty) = \lim_{x\to\infty} S(x)$, and $S(\infty) < D(0)$.

Conditions (A1) and (A2) have been discussed in the introduction. Condition (A3) means that the maximum supply is limited by the maximum demand. Under conditions (A1)–(A3), for each $x_0 > 0$, there exists a unique solution $\{x_k\}_{k\geq 0}$ to Eq. (1.3). Moreover, the function $F = D^{-1} \circ S$ is well defined and (1.3) is equivalent to

$$x_k = F(x_{k-1}), \quad k \geq 1. \tag{2.1}$$

We recall some basic definitions. We say that the positive fixed point P^* of (2.1) is globally attracting if it is asymptotically stable and

$$\lim_{k\to\infty} F^k(x) = P^* \tag{2.2}$$

for all $x > 0$.

A set $X \subset (0, \infty)$ is called invariant if $F(X) \subset X$. An invariant subset X is attracting if there is a neighbourhood U of X such that $\bigcap_{k=0}^{\infty} f^k(U) \subset X$.

A set $\{\alpha, \beta\}$ such that $F(\alpha) = \beta, F(\beta) = \alpha$ is called a cycle of period 2 of F (or a 2-cycle). We say that the 2-cycle $\{\alpha, \beta\}$ is globally attracting if $F^k(x) \to \{\alpha, \beta\}$ as $k \to \infty$ for all $x \in (0, \infty) \setminus \{P^*\}$.

We recall that the Schwarzian derivative of a C^3 map f is defined by the relation

$$(\mathbf{S}f)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2,$$

whenever $f'(x) \neq 0$.

From (A2) it is clear that $F'(x) < 0$ for all $x > 0$. Thus, we have the following properties:

Proposition 2.1. *Assume that conditions* (A1)–(A3) *hold. Then:*

(i) *F has a compact invariant and attracting interval $[A, B]$, where A, B are the solutions of equations $D(A) = S(\infty)$, $D(B) = S(A)$, respectively.*
(ii) *The positive equilibrium P^* of* (2.1) *is globally attracting if and only if F does not have any 2-cycle different from P^*.*

Proof. To prove (i), notice that by the definition of A it is clear that $F(x) \geq A$ for all $x > 0$. On the other hand, since F is decreasing, it follows that $F(x) \leq F(A) = B$ for all $x \geq A$; see Fig. 1.

Next, since the interval $[A, B]$ is invariant and attracting, to study the asymptotic properties of the solutions of (2.1), it is enough to consider the restriction of F to $[A, B]$. Hence, statement (ii) follows from the main theorem in [6]. □

If the Schwarzian derivative of F is negative, then we can get more information. See, e.g., [7, Theorem 1].

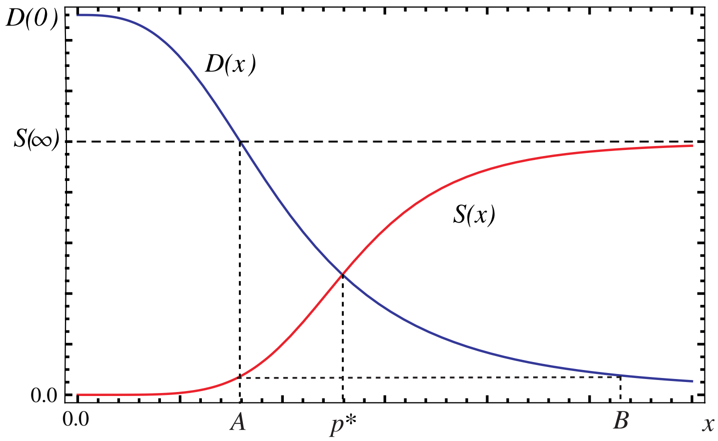

Fig. 1. Graphs of maps $D(x)$ and $S(x)$ in a case where (A1)–(A3) hold. We show the equilibrium P^* and the attracting invariant interval $[A, B]$ indicated in Proposition 2.1.

Proposition 2.2. *Assume that conditions* (A1)–(A3) *hold and*

(A4) $F : (0, \infty) \to (0, \infty)$ *is a* C^3 *map and* $(\mathbf{S}F)(x) < 0$ *for all* $x > 0$.

Then the following dichotomy holds:

(i) *The positive equilibrium* P^* *of* (2.1) *is globally attracting if and only if condition* (1.2) *is satisfied.*
(ii) *If* (1.2) *does not hold then* F *has a globally attracting* 2*-periodic orbit.*

2.2. An application

In this subsection, we apply the previous results to the discrete version of a particular case of (1.1) studied in [4,5,3], namely

$$\frac{P'(t)}{P(t)} = \frac{a}{b + P^n(t)} - \frac{cP^m(t-\tau)}{d + P^m(t-\tau)}, \tag{2.3}$$

where $a, b, c, d, m, \tau > 0$, $n \geq 1$.

The associated difference equation is

$$\frac{a}{b + x_k^n} = \frac{cx_{k-1}^m}{d + x_{k-1}^m}, \quad k = 1, 2, \ldots. \tag{2.4}$$

It is clear that $D(x) = a/(b+x^n)$ and $S(x) = cx^m/(d+x^m)$ satisfy (A1) and (A2). On the other hand, $S(\infty) = c$, $D(0) = a/b$, and therefore (A3) holds if

$$bc < a. \tag{2.5}$$

In the following we assume that condition (2.5) holds, and hence we can define $F : (0, \infty) \to (0, \infty)$ by

$$F(x) = D^{-1}(S(x)) = \left(-b + \frac{a(d + x^m)}{cx^m}\right)^{1/n}. \tag{2.6}$$

Lemma 2.3. *Condition* (1.2) *is equivalent to*

$$(bc - a)\left(\frac{adm}{cn}\right)^{m/(m+n)} \leq ad\left(1 - \frac{m}{n}\right). \tag{2.7}$$

Proof. According to Remark 3.3 in [3], condition (1.2) is equivalent to $adm \leq cn(P^*)^{n+m}$, that is, to the inequality

$$P^* \geq \left(\frac{adm}{cn}\right)^{1/(n+m)} := \gamma. \tag{2.8}$$

On the other hand, P^* is the only positive solution of equation

$$0 = H(x) = D(x) - S(x) = \frac{a}{b + x^n} - \frac{cx^m}{d + x^m}.$$

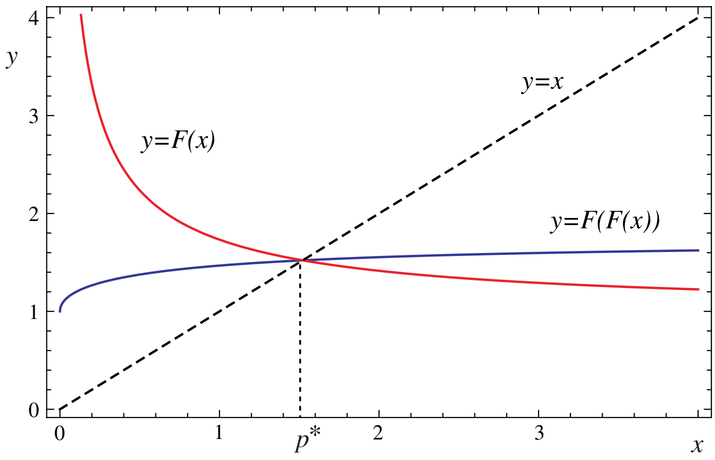

Fig. 2. Graphs of maps $F(x)$, $F(F(x))$, and the line $y = x$ for (2.6) with $a = 2$, $b = c = d = 1$, and $m = 1 < 2 = n$. It follows from Lemma 2.4 that $F(F(x)) = x$ does not have any positive solution in $(0, P^*)$, and thus P^* is a global attractor of (2.4).

Since $H(0) = a/b > 0$ and $H(\infty) = -c < 0$, (2.8) holds if and only if $H(\gamma) \geq 0$. Direct computations show that inequality $H(\gamma) \geq 0$ is equivalent to (2.7). $\square$

Lemma 2.4. *If $m \leq n$ then F does not have any 2-cycle different from P^*.*

Proof. Define $G(x) = F^2(x) = (F \circ F)(x)$. A pair $\{\alpha, \beta\}$ is a 2-cycle of F if and only if $G(\alpha) = \alpha$ and $G(\beta) = \beta$. We shall prove that if $G(x) = x$ for some $x > 0$, then $G'(x) < 1$. This fact clearly implies that G can have at most one fixed point, which must be the positive equilibrium P^*.

Indeed, an easy computation leads to

$$F'(x) = \frac{-admF(x)}{nx\left((a-bc)x^m + ad\right)}.$$

Thus,

$$G'(x) = F'(F(x))F'(x) = \left(\frac{-admF(F(x))}{nF(x)\left((a-bc)(F(x))^m + ad\right)}\right)\left(\frac{-admF(x)}{nx\left((a-bc)x^m + ad\right)}\right).$$

At a fixed point of G, we have $F(F(x)) = x$, and therefore

$$G'(x) = \frac{a^2d^2m^2xF(x)}{n^2xF(x)\left((a-bc)(F(x))^m + ad\right)\left((a-bc)x^m + ad\right)}.$$

Taking into account that $a - bc > 0$, it follows that

$$G'(x) < \frac{m^2}{n^2} \leq 1. \quad \square$$

In Fig. 2, we plot the graphs of F and F^2 in a situation where $m < n$, and therefore the equilibrium is the only fixed point of F^2.

Remark 1. If $a = bc$, then the map F is well defined, and the previous proof shows that F does not have any 2-cycle different from P^* if $m < n$. If $a = bc$ and $m = n$ the result is no longer valid; actually, in this case, $F(x) = k/x$, with $k = bd$, and therefore $F(F(x)) = x$ holds for all $x > 0$.

Lemma 2.5. *If $m > n$ then $(\mathbf{S}F)(x) < 0$ for all $x > 0$.*

Proof. After some computations, we arrive at the expression

$$(\mathbf{S}F)(x) = \frac{a^2d^2(n^2 - m^2) - (a-bc)(m^2-1)n^2x^m\left((a-bc)x^m + 2ad\right)}{2n^2x^2\left((bc-a)x^m - ad\right)^2}.$$

Taking into account that $a - bc > 0$, it is clear that this expression is negative for all $x > 0$ if $n^2 - m^2 < 0$ and $m^2 - 1 \geq 0$, that is, if $m > n \geq 1$. $\square$

Using the previous lemmas and our discussion above, we get the following result for Eq. (2.1) with F defined in (2.6).

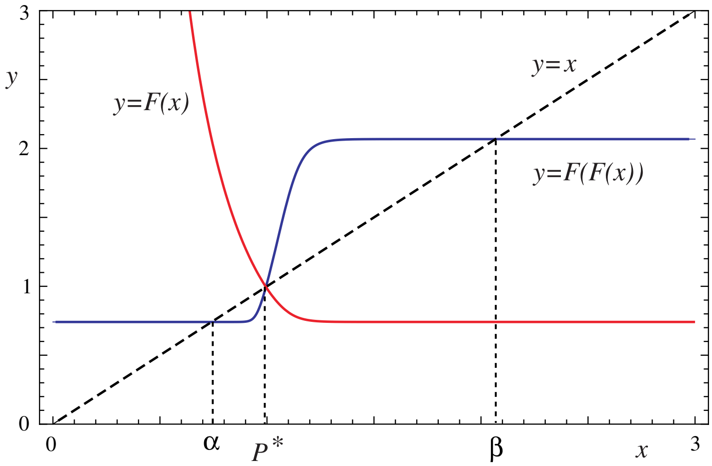

Fig. 3. Graphs of maps $F(x)$, $F(F(x))$, and the line $y = x$ for (2.6) with $a = 0.4$, $b = 0.75$, $c = 0.5$, $d = 1$, $m = 25$, and $n = 10$. Since $m > n$, and (2.7) does not hold, there is a nontrivial 2-cycle $[\alpha, \beta]$, which attracts all positive solutions of (2.4) except the equilibrium P^*.

Theorem 2.6. *Assume that condition (2.5) holds. The following claims are true:*

1. *Eq. (2.4) has a compact invariant and attracting interval $[A, B]$ defined by*

$$A = \left(\frac{a}{c} - b\right)^{1/n}, \qquad B = F(A) = \left(-b + \frac{a(d + A^m)}{cA^m}\right)^{1/n}. \tag{2.9}$$

2. *If $m \le n$ then the positive equilibrium P^* is a global attractor of all positive solutions of (2.4).*
3. *If $m > n$ then the positive equilibrium P^* is a global attractor of all positive solutions of (2.4) if and only if (2.7) holds, that is, whenever it is asymptotically stable.*
4. *If $m > n$, and (2.7) does not hold, then (2.4) has a globally attracting 2-cycle $[\alpha, \beta]$ such that $A < \alpha < P^* < \beta < B$.*

In Fig. 3, we plot the graphs of F and F^2 in a situation where the equilibrium is not attracting.

Remark 2. From Remark 1, and using that F is strictly decreasing, it is not difficult to prove that in the limit case $a = bc$, the positive equilibrium is a global attractor of all positive solutions of (2.4) if $m < n$. Note that in this case $F(x) = (bd)^{1/n}x^{-m/n}$.

3. Global dynamics of the continuous model

Let us consider again the continuous model defined by the delay–differential equation (1.1), that can be rewritten in the form

$$P'(t) = P(t)(D(P(t)) - S(P(t - \tau))). \tag{3.1}$$

We shall apply some results from [8], which is devoted to studying the class of delay–differential equations

$$x'(t) = f_1(x(t - \tau))g_2(x(t)) - f_2(x(t - \tau))g_1(x(t)). \tag{3.2}$$

It is clear that (3.1) belongs to this class with $f_1(x) = 1$, $f_2(x) = S(x)$, $g_1(x) = x$, and $g_2(x) = xD(x)$.

Under conditions (A1)–(A3), Eq. (3.1) meets almost all of the assumptions (H0)–(H2) in [8]. We need to impose the additional condition

(A5) $\lim_{x\to\infty} D(x) = 0$.

Notice that condition (A5) is quite natural because it means that the demand tends to zero if the price is too high.

Defining $f(x) = f_1(x)/f_2(x)$ and $g(x) = g_1(x)/g_2(x)$, the only remark as regards using the results in [8] is that the condition $g(0^+) = 0$ required there can be replaced by $g(0) < f(\infty)$. Indeed, $g(0^+) = 0$ is used in [8, Section 2.1] to ensure that every solution of (3.2) with strictly positive initial data is strictly positive for all $t > 0$. It is easy to see that the arguments are still valid if $f(x) > g(0)$, for all $x > 0$. Since $f(x) = 1/S(x)$ is decreasing, it is enough to require that $g(0) < f(\infty)$, which follows from (A3), since $g(0) = 1/D(0) < 1/S(\infty) = f(\infty)$.

As observed in [8], Eq. (3.2) is equivalent, via the change of variables $t = \tau s$, to the equation

$$\frac{1}{\tau}x'(t) = f_1(x(t - 1))g_2(x(t)) - f_2(x(t - 1))g_1(x(t)).$$

Thus, its limit case as $\tau \to \infty$ gives the difference equation with continuous argument

$$f_1(x(t-1))g_2(x(t)) = f_2(x(t-1))g_1(x(t)),$$

whose dynamics is mainly governed by the difference equation

$$f_1(x_{k-1})g_2(x_k) = f_2(x_{k-1})g_1(x_k), \quad k = 1, 2, \ldots,$$

that is,

$$g(x_k) = f(x_{k-1}), \quad k = 1, 2, \ldots. \tag{3.3}$$

Clearly, for Eq. (3.1), the difference Eq. (3.3) is equivalent to (1.3). That is to say, the limit case of (1.1) when the price adjustment is quite rapid coincides with the limit case where the time necessary for making a decision to alter production is large.

The advantage of this equivalence is that we can use the properties of (1.3) stated in Section 2 to get information about the global dynamics of (3.1). Indeed, the following result is a consequence of Theorems 2 and 3 in [8], and Propositions 2.1 and 2.2 in this paper.

Theorem 3.1. *Assume that conditions* (A1), (A2), (A3) *and* (A5) *hold. Then:*

(a) *Let $J = [A, B]$ the invariant and attracting interval defined in Proposition 2.1. Then any positive solution $P(t)$ of (3.1) satisfies the inequalities*

$$A \le \liminf_{t\to\infty} P(t) \le \limsup_{t\to\infty} P(t) \le B.$$

(b) *If the map $F(x) = D^{-1}(S(x))$ does not have any 2-cycle different from the positive equilibrium P^*, then all positive solutions of (3.1) converge to P^*.*
(c) *If* (A4) *and (1.2) hold, then all positive solutions of (3.1) converge to P^*.*
(d) *Let* (A4) *hold and assume that $S'(P^*) > -D'(P^*)$. If $\{\alpha, \beta\}$ is the globally attracting 2-periodic orbit of F ensured by Proposition 2.2, then any positive solution $P(t)$ of (3.1) satisfies the inequalities*

$$\alpha \le \liminf_{t\to\infty} P(t) \le \limsup_{t\to\infty} P(t) \le \beta.$$

Next we apply this result to Eq. (2.3), for which (A5) trivially holds. The following theorem is a consequence of Theorems 2.6 and 3.1.

Theorem 3.2. *Consider Eq. (2.3) with $a, b, c, d, m, \tau > 0$, $n \ge 1$, and $bc < a$. The following claims are true:*

(a) *All positive solutions of (2.3) satisfy the inequalities*

$$A \le \liminf_{t\to\infty} P(t) \le \limsup_{t\to\infty} P(t) \le B,$$

where A, B are defined in (2.9).
(b) *If $m \le n$ then all positive solutions of (2.3) converge to P^*.*
(c) *If $m > n$ and condition (2.7) holds, then all positive solutions of (2.3) converge to P^*.*
(d) *If $m > n$, and (2.7) does not hold, then all positive solutions of (2.3) satisfy the inequalities*

$$\alpha \le \liminf_{t\to\infty} P(t) \le \limsup_{t\to\infty} P(t) \le \beta,$$

where $\{\alpha, \beta\}$ is the only solution of system $D(\alpha) = S(\beta)$, $D(\beta) = S(\alpha)$ such that $0 < \alpha < P^* < \beta$.

Remark 3. In view of Remark 2, it is possible to ensure that all positive solutions of (2.3) converge to P^* if $m < n$ and $a = bc$. This situation includes the example $a = b = c = d = 1$, $n = 100$, $m = 2$ considered in [3]. Our results show that in this case all positive solutions of (2.3) converge to the positive equilibrium $P^* = 1$, regardless of the value of $\tau > 0$.

4. Numerical results and discussion

From our results, we know that the positive equilibrium of Eq. (2.3), under condition (2.5), is globally attracting if $m \le n$. Thus, price oscillations can only occur when $m > n$ and (2.7) does not hold. In this case, there is a critical value $\tau^* > 0$ such that P^* is unstable if $\tau > \tau^*$ (see [1,3]). The information that we get in this situation from Theorem 3.2 is a bound for the amplitude of the oscillations. Our simulations show that this bound is sharp for large values of the delay τ; see Fig. 4. Moreover, it seems that there is an attracting periodic orbit for all $\tau > \tau^*$, and its profile converges to the "square wave" $p(t)$ defined by the unique 2-cycle $\{\alpha, \beta\}$ of F, that is, $p(t) = \alpha$ for $t \in [0, 1)$, $p(t) = \beta$ for $t \in [1, 2)$, $p(t) = p(t+2)$ for all t (after re-scaling the delay). That is, we conjecture that the global bifurcation results from [9,10] apply to Eq. (2.3); for related results and further discussion, see [7].

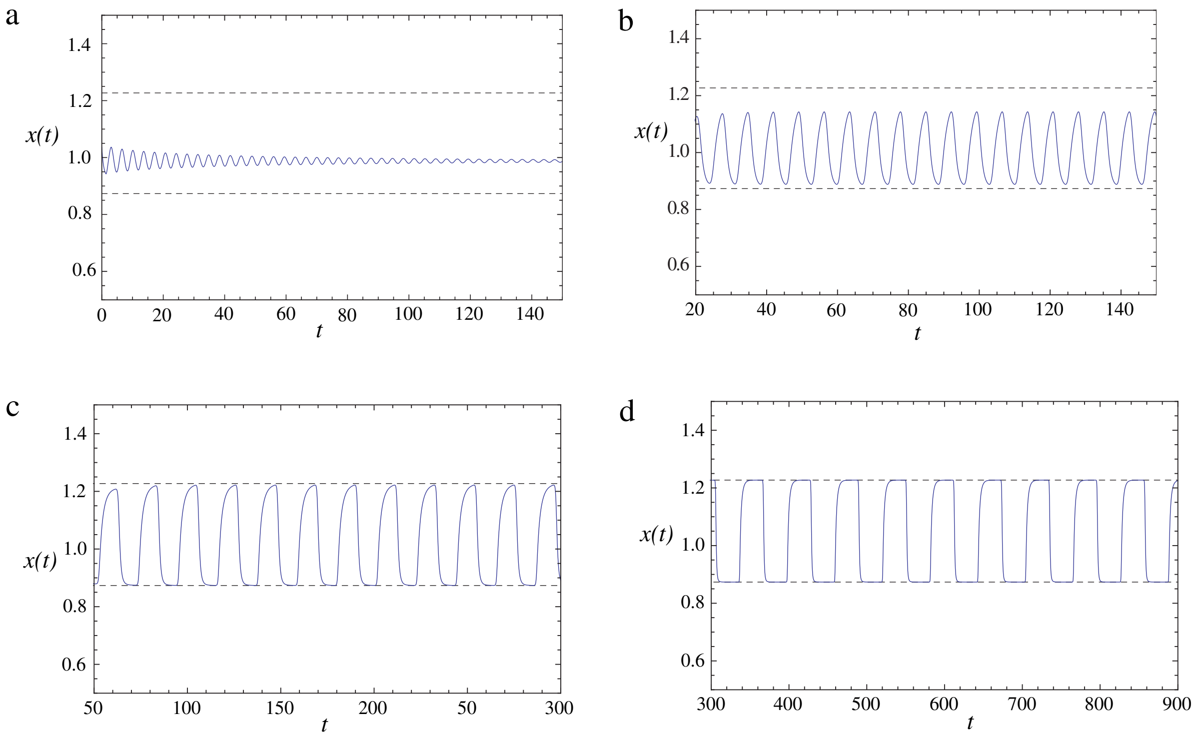

Fig. 4. Solutions $P(t)$ of Eq. (2.3) with $a = 0.5$, $b = 0.75$, $c = 0.5$, $d = 0.5$, $m = 20$, $n = 10$ for different values of τ ((a) $\tau_1 = 1.3$, (b) $\tau_2 = 3$, (c) $\tau_3 = 10$, (d) $\tau_4 = 30$). The horizontal dashed lines represent the bounds given by Theorem 3.2; the 2-cycle $\{\alpha, \beta\}$ of F is defined by $\alpha = 0.873413$, $\beta = 1.22711$. The equilibrium $P^* = 0.988012$ is locally asymptotically stable for $\tau < \tau^* = 1.37273$.

Thus, our results, together with numerical simulations, show that there is a close relation between the continuous model (1.1) and its discrete version (1.3). Let us mention that the idea of relating delay–differential equations with its formal limit given by a difference equation goes back at least to a paper of May [11]; the interested reader is referred to [12,8,13–16,7,9,10,17,18] for more results, including equations with distributed delays, equations with infinite delay, and delayed partial differential equations.

We emphasize that Theorem 3.2 provides new results for the global stability of the equilibrium and uniform bounds for the positive solutions of (2.3). Although some sufficient conditions for the global convergence to the equilibrium were recently given in [5,3], our results are independent of the delay parameter τ, providing us in this way with a set of parameter values of the model for which sustained oscillations cannot occur regardless of the time delay in the supply term.

Acknowledgments

E. Liz was supported in part by the Spanish Ministry of Science and Innovation and FEDER, grant MTM2010–14837. G. Röst was supported in part by the Hungarian Scientific Research Fund OTKA K75517, European Research Council Starting Investigator Grant No. 259559, and the Bolyai Scholarship of the Hungarian Academy of Sciences. EL is grateful for the partial support and kind hospitality enjoyed during his visit to Bolyai Institute. We acknowledge the comments of two anonymous reviewers.

References

[1] M.C. Mackey, Commodity price fluctuations: price dependent delays and nonlinearities as explanatory factors, J. Econom. Theory 48 (1989) 497–509.
[2] J. Bélair, M.C. Mackey, Consumer memory and price fluctuations in commodity markets: an integrodifferential model, J. Dynam. Differential Equations 1 (1989) 299–325.
[3] G. Röst, Global convergence and uniform bounds of fluctuating prices in a single commodity market model of Bélair and Mackey, Electron. J. Qual. Theory Differ. Equ. (26) (2012) p. 9.
[4] A.M. Farahani, E.A. Grove, A simple model for price fluctuations in a single commodity market, in: Oscillation and Dynamics in Delay Equations, San Francisco, CA, 1991, in: Contemp. Math., vol. 129, Amer. Math. Soc., Providence, RI, 1992, pp. 97–103.
[5] C. Qian, Global attractivity in a delay differential equation with application in a commodity model, Appl. Math. Lett. 24 (2011) 116–121.
[6] W.A. Coppel, The solution of equations by iteration, Proc. Cambridge Philos. Soc. 51 (1955) 41–43.
[7] E. Liz, G. Röst, Dichotomy results for delay differential equations with negative Schwarzian derivative, Nonlinear Anal. RWA 11 (2010) 1422–1430.
[8] A.F. Ivanov, E. Liz, S. Trofimchuk, Global stability of a class of scalar nonlinear delay differential equations, Differential Equations Dynam. Systems 11 (2003) 33–54.

[9] J. Mallet-Paret, R. Nussbaum, Global continuation and asymptotic behaviour for periodic solutions of a differential–delay equation, Ann. Mat. Pura Appl. 145 (1986) 33–128.
[10] J. Mallet-Paret, R. Nussbaum, A differential–delay equation arising in optics and physiology, SIAM J. Math. Anal. 20 (1989) 249–292.
[11] R.M. May, Nonlinear phenomena in ecology and epidemiology, Ann. New York Acad. Sci. 357 (1980) 267–281.
[12] E. Braverman, S. Zhukovskiy, Absolute and delay-dependent stability of equations with a distributed delay, Discrete Contin. Dyn. Syst. 32 (2012) 2041–2061.
[13] A.F. Ivanov, M.A. Mammadov, Global asymptotic stability in a class of nonlinear differential delay equations, Discrete Contin. Dyn. Syst., Suppl. (2011) 727–736.
[14] A.F. Ivanov, A.N. Sharkovsky, Oscillations in singularly perturbed delay equations, Dynam. Report. Expositions Dynam. Systems (N.S.) 1 (1992) 164–224.
[15] E. Liz, C. Martínez, S. Trofimchuk, Attractivity properties of infinite delay Mackey–Glass type equations, Differential Integral Equations 15 (2002) 875–896.
[16] E. Liz, G. Röst, On the global attractor of delay differential equations with unimodal feedback, Discrete Contin. Dyn. Syst. 24 (2009) 1215–1224.
[17] G. Röst, J. Wu, Domain decomposition method for the global dynamics of delay differential equations with unimodal feedback, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 463 (2007) 2655–2669.
[18] T. Yi, X. Zou, Map dynamics versus dynamics of associated delay reaction–diffusion equations with a Neumann condition, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 466 (2010) 2955–2973.