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Short communication

On an approximate method for the delay logistic equation

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ABSTRACT

This note concerns with the asymptotic properties of solutions of the delay logistic equation. In particular, we point out some false statements in the recent paper Khan et al. [Khan H, Liao SJ, Mohapatra RN, Vajravelu K. An analytical solution for a nonlinear time-delay model in biology. Commun Nonlinear Sci Numer Simulat 2009;14:3141–3148]. Moreover, we show that the author's method is not able to reveal the basic and important features of the dynamics of the delay logistic equation, and gives misleading results.

1. Introduction

In [1], the authors studied the delay differential equation

$$i'(t) = ri(t)\left[1 - \frac{i(t-\tau)}{\kappa}\right], \tag{1}$$

with initial condition

$$i(t) = \alpha, \quad -\tau \leqslant t \leqslant 0,$$

where r, κ are assumed to be positive parameters, and the value α of the constant initial function is also positive. Eq. (1) has two equilibria, κ and 0. The authors of [1] claim that for $\alpha > \kappa$, $i_\infty := \lim_{t\to\infty} i(t)$ exists and takes one of the two possible values $i_\infty = \kappa$ or $i_\infty = 0$, depending on the time delay τ. Their claim was also supported by searching for solutions in a power series form by means of an approximate method, the so called homotopy analysis method (HAM).

In this paper we point out that the above claim is false, since as a matter of fact a positive solution cannot converge to 0 at all, and most solutions converge to a periodic solution if $r\tau > \pi/2$. The method of the authors was not suitable to reveal the important features of the dynamics, and provided misleading results.

2. Dynamics of the delay logistic equation and the approximative method

The delay logistic Eq. (1) was first proposed by Hutchinson [2] in the context of the population dynamics of a single species, with time delay in the per capita growth rate. By the change of variables

$$x(t) = \frac{i(t)}{\kappa} - 1, \quad i(t) = \kappa[1 + x(t)],$$

(1) can be rewritten as

$$x'(t) = -rx(t-\tau)[1 + x(t)], \tag{2}$$

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and for solutions $x(t) > -1$, using

$$y(t) = \ln[1 + x(t)], \quad x(t) = \exp[y(t)] - 1,$$

we obtain

$$y'(t) = -r(\exp[y(t-\tau)] - 1). \tag{3}$$

These latter two forms are usually referred to as Wright's equation. We note that (2) has been arisen in a completely different context, namely, related to the distribution of prime numbers [3]. Eqs. (1)–(3) are essentially equivalent, and they are among the most studied delay differential equations. Note that the above transformations do not change r and τ.

For Eq. (1), the following results hold:

- the equilibrium κ is locally asymptotically stable for $\tau r < \frac{\pi}{2}$,
- varying the delay τ (or analogously r) as a bifurcation parameter, a supercritical Hopf-bifurcation takes place at the equilibrium κ for $\tau = \frac{\pi}{2r}$ and a stable limit cycle appears,
- a periodic solution exists whenever $\tau r > \frac{\pi}{2}$.

This is all very well known for decades and covered in major textbooks [4–8]. Even one book the authors of [1] quote mentions the stable periodic solutions for $r\tau > \pi/2$ in Chapter 3.6 (pp. 117) [9]. The reader is invited to look up the online demonstration project [10] for experimenting with this equation.

Furthermore, note that a positive solution remains positive and cannot converge to 0, as trivially can be seen: if $0 < i(t) \to 0$, then there is a t_0 such that $i(t) < \kappa/2$ for $t > t_0$, and then for $t > t_0 + \tau$ we have $i'(t) > ri(t)/2$, contradicting to $i(t) \to 0$.

Therefore, the authors claim in [1] that solutions converge to 0 for sufficiently large delays is not correct. The misconception in [1] on the limits i_∞ apparently comes from associating (1) with the ordinary differential equation $i'(t) = ri(t)(1 - \alpha/\kappa)$ as the delay τ tends to infinity.

In [1] they search solutions in the form

$$i(t) = \sum_{m=0}^{\infty} a_m e^{-m\beta t}, \tag{4}$$

and use HAM to approximate the coefficients. Though they write in [1] that "it is straightforward that the solution for $\tau > 0$ can be expressed in the form" (4), looking for solutions in this particular form is quite restrictive, and clearly not adequate for finding periodic solutions (for that purpose Fourier series might be used, see [13]). But if the delay τ is greater than $\frac{\pi}{2r}$, then most solutions converge to a periodic orbit. It is also known for delayed monotone negative feedback (such as the delay logistic equation in form (3)) that slowly oscillatory solutions are typical [14].

The approach of asymptotic expansions of solutions can be found in the pioneer work of EM Wright (see [3] from 1955 and references thereof), where he proved several related theorems on the expansion of solutions of (2) that are small at infinity, and he also gave results about specific series solutions. Such expansions of solutions can be found without any homotopy method, by simply plugging expressions like (4) into the equation and finding the coefficients inductively, and showing the convergence of the obtained series.

In his seminal paper [3] in 1955 EM Wright proved his famous 3/2 attractivity theorem, which states that if $\tau r < \frac{3}{2}$, then all solutions of (2) that are greater than -1 converge to zero (equivalently, all positive solutions of (1) converge to κ). He conjectured that the global attractivity holds for $\tau r < \frac{\pi}{2}$, which is the condition for local stability that can be obtained from linearization. Jones proved in 1962 [11] that there exists a periodic solution whenever $\tau r > \frac{\pi}{2}$. Based on one of his remarks in [12], and on the fact that a supercritical Hopf bifurcation of a slowly oscillating periodic solution was demonstrated in [15], the uniqueness of a slowly oscillatory periodic solution has been conjectured for a long time. Despite several partial results, both the Wright and Jones conjectures are still open, and the complete global picture of the dynamics for this simple looking equation is unknown even after several decades, and the related topics are still the subject of active research [13,16,17].

In Fig. 2 of [1], the particular case $\kappa = 0.5$, $r = 2$, $\alpha = 1$ and $\tau = 0.1$ is depicted. The graph indicates convergence to $\kappa = 0.5$, which is correct as we know by applying Wright's theorem that all positive solutions converge to κ, since $r\tau = 0.2 < 3/2$. On the other hand, in Fig. 3 of [1] the case $\kappa = 0.5$, $r = 2$, $\alpha = 1$ and $\tau = 4$ was considered and plotted on the time interval $[-4,6]$. The conclusion of the authors of [1] was that the solution converges to zero. However, we already showed that it is impossible, and indeed, if we plot the numerical solution on a longer time interval, we can observe that after getting close to zero, it starts to increase. For comparison, see our Fig. 1, where in (a) the plot from [1] can be seen, and in (b) the numerical solution plotted on a longer interval. The reason for this is intuitively very clear if we look at (1) and consider the effect of the delay. In fact, this solution (such as most solutions with these parameter values) will converge to a periodic solution with large amplitude, as you can see in Fig. 2.

Furthermore, using 8th order HAM, the authors of [1] gave an approximation of the solution on $[0,6]$. But for this interval we have a straightforward method to calculate the exact solution analytically, by means of simple integration. In general, if the initial function ϕ is given on the interval $[-\tau, 0]$, then the solution for $t \in [0, \tau]$ can be obtained by simple integration as

$$i(t) = i(0)e^{r\int_0^t [1 - i(s-\tau)/\kappa]ds} = i(0)e^{r\int_{-\tau}^{t-\tau} [1-\phi(s)/\kappa]}ds. \tag{5}$$

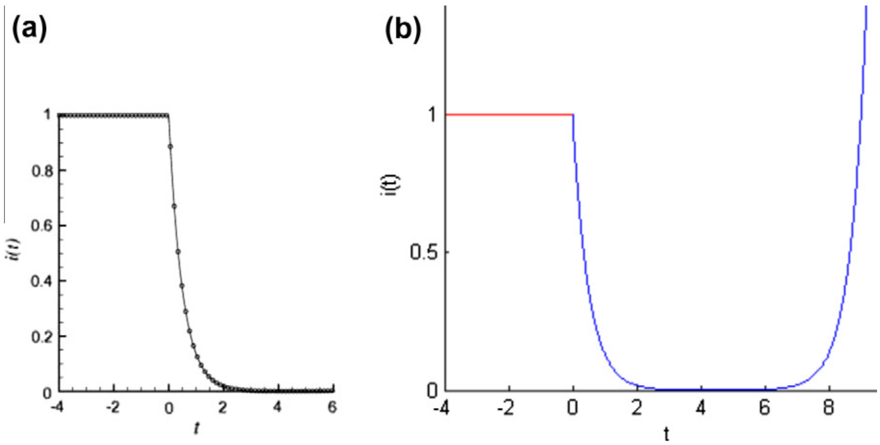

Fig. 1. (a) Plot of the numerical and the approximate solution taken from [1]. Here $\kappa = 0.5$, $r = 2$, $\alpha = 1$, $\tau = 4$. (b) Plot of the same solution on a slightly longer time interval. The first plot is misleading, it suggests that the solution converges to zero, which was the conclusion of the authors of [1]. While in fact the solution starts to increase rapidly soon after the endpoint of first plot.

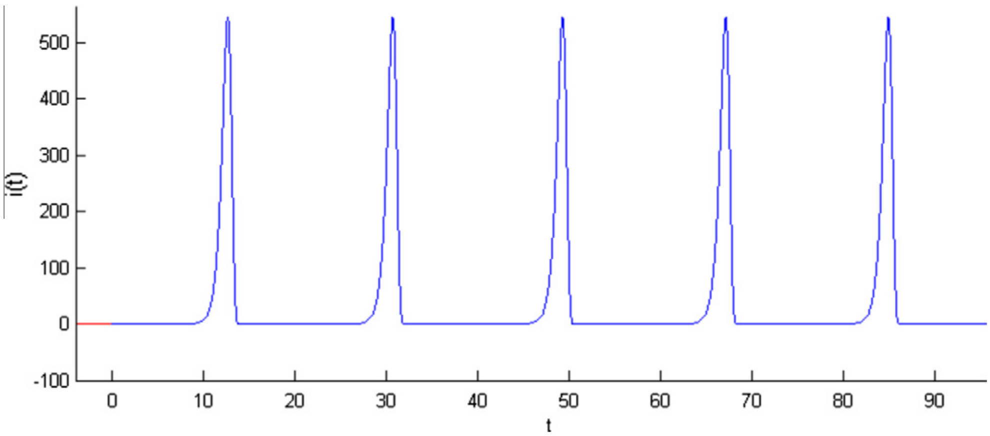

Fig. 2. The same solution as in Fig. 1, plotted on a long time interval. Instead of convergence to 0, we can see periodic oscillation with large amplitude. The long flat parts of the solution make it difficult to analyze it by Fourier series [13].

Then we can repeat this procedure to find the solution for the interval $[\tau, 2\tau]$, $[2\tau, 3\tau]$, etc. In the literature this is called the method of steps.

In this particular case ($\kappa = 0.5$, $r = 2$, $\alpha = 1$, $\tau = 4$) our equation is

$$i'(t) = 2i(t)[1 - 2i(t-4)], \tag{6}$$

and for the interval [0,6], we can find easily the analytic solution without HAM by the method of steps, such as:

- for $t \in [-4, 0]$:

$$i(t) = 1,$$

- for $t \in [0, 4]$:

$$i(t) = e^{-2t},$$

- for $t \in [4, 8]$:

$$i(t) = e^{-8} e^{2\int_4^t [1-2i(s-4)]ds} = e^{-8} e^{2(t-4)+2\int_4^t [-2e^{-2(s-4)}]ds} = e^{-18+2t+2e^{(8-2t)}}.$$

The solution is already increasing on the interval $[t_1, 6]$, where $t_1 = \ln(2)/2 + 4 \approx 4.35$, which is not indicated on the plot of Fig. 1 in [1].

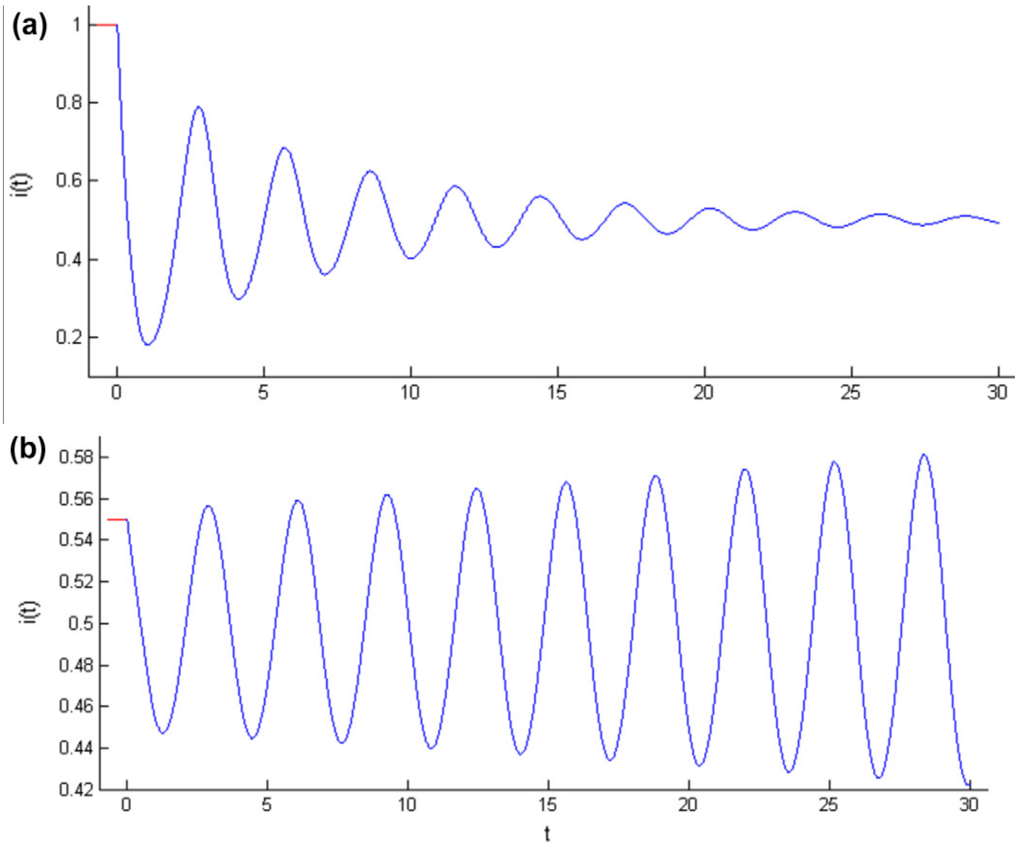

Fig. 3. (a) Convergence to $\kappa = 0.5$, where $r = 2$, $\alpha = 1$ and $\tau = 0.7$, in accordance with Wright's theorem, as $r\tau = 1.4 < 3/2$. (b) Sustained oscillation after increasing the delay to $\tau = 0.8$, when $r\tau = 1.6 > \pi/2$, thus κ is unstable. For purely illustrative purposes we chose $\alpha = 0.55$, but the same phenomena can be observed for other initial values as well.

To study the influence of the time delay, the authors of [1] constructed their Table 3, and they claim that the critical delay, when the transition of i_∞ from κ to 0 happens, is somewhere between 0.85 and 0.87 for (6). In comparison, with those parameters the equilibrium $\kappa = 0.5$ loses its stability at $\tau = \pi/4 \approx 0.785$ which can be found from the standard linearization of (1) at κ and the corresponding characteristic equation. The numerical solution with initial value $\alpha = 1$ tends to a periodic limit cycle if $\tau > \pi/4 \approx 0.785$, showing the inadequacy of the author's approximative method to predict the properties of solutions.

The remark in [1] on "piecewise continuous solutions" of (1) (which appears in the abstract and in the conclusion as well) is also unclear, since (1) has very nice continuous solutions from the considered constant initial functions, being guaranteed by the method of steps as well. In general, for delay differential equations the usual setting is to choose the Banach space $C([-\tau, 0], R)$ of continuous functions on the interval $[-\tau, 0]$ as our phase space equipped with the sup norm.

3. Conclusion

Several inappropriate applications of different approximative methods to various nonlinear problems has been discussed by Fernández in recent papers [18,19].

Restricting the study of the solutions of Eq. (1) to solutions of the form (4) does not capture the basic and important features of the dynamics, since it cannot detect periodic limit cycles, which are paramount for the delay logistic equation. The claim of [1] on i_∞, namely that "Our calculations indicate that there exists a criterion value τ^* dependent on the physical parameters κ, r and α, so that $i_\infty = \kappa$ when $\tau > \tau^*$ but $i_\infty = 0$ when $\tau > \tau^*$" is not correct. The HAM provided only a local approximation, while for short time intervals we can find the analytic solution easily by integration using the method of steps. Globally, for longer time intervals the authors gave completely wrong predictions about the behaviour of the solutions. Moreover, it indicated very inaccurately the value of the critical delay when the positive equilibrium κ loses its stability.

We believe that this note will be useful for people who wish to study nonlinear delay differential equations by means of approximate or numerical methods, to avoid mistakes, to focus on the important open questions and to conduct more careful research in the future.

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