Solution to Problem 11260.

We will use Kummer’s convergence test (see R.T. Prosser, Kummer’s test gives characterizations for convergence divergence of all positive series, Amer. Math. Monthly. 101 (1994), no. 5, 450–452).

Kummer’s Test  Let \( a_n > 0 \) for \( n = 1, 2 \ldots \)

1. If

\[
\lim \inf \left( \alpha_n \frac{a_n}{a_n + 1} - \alpha_{n+1} \right) > 0
\]

with some real numbers \( \alpha_n > 0 \), then the series \( \sum a_n \) are convergent.

2. If \( \sum 1/\alpha_n = \infty \) and

\[
\alpha_n \frac{a_n}{a_{n+1}} - \alpha_{n+1} \leq 0
\]

for \( n > N \) with some integer \( N \) and positive real numbers \( \alpha_n \), then \( \sum a_n = \infty \).

In our case let \( \alpha_n = \alpha + (n + 1) \log(n + 1) \) and

\[
a_n = \prod_{k=1}^{n} \frac{\alpha + k \log k}{\beta + (k + 1) \log (k + 1)},
\]

hence

\[
\alpha_n \frac{a_n}{a_{n+1}} - \alpha_{n+1} = \beta - \alpha,
\]

so if \( \beta > \alpha \) then the series are convergent. Now, following the proof of Kummer’s test, we can easily calculate the sum of the series \( \sum a_n \). Let \( c_n = \alpha_n a_n \). Then by (1) we have \( c_n - c_{n+1} = (\beta - \alpha)a_{n+1} \). Whence we get

\[
\sum_{n=1}^{\infty} (\beta - \alpha)a_{n+1} = \sum_{n=1}^{\infty} c_n - c_{n+1} = c_1.
\]

Thus

\[
\sum_{n=1}^{\infty} a_n = \frac{\alpha_1 a_1}{\beta - \alpha} + a_1 = \frac{1}{\beta - \alpha} \cdot (\alpha + 2 \log 2 + \beta - \alpha) \cdot \frac{\alpha}{\beta + 2 \log 2} = \frac{\alpha}{\beta - \alpha}.
\]

If \( \beta < \alpha \) then to see that the sum is divergent it is enough to show \( \sum 1/\alpha_n = \infty \). There is an \( N \in \mathbb{N} \) such that \( \alpha < k \log k \) for \( k > N \) and hence

\[
\sum_{k=N}^{\infty} \frac{1}{\alpha + k \log k} > \sum_{k=N}^{\infty} \frac{1}{2k \log k} = \frac{1}{2} \sum_{k=N}^{\infty} \frac{1}{k \log k} = \infty
\]

as it is well known. If \( \alpha = \beta \) then the series \( \sum a_n \) reduce to

\[
\alpha \sum_{k=1}^{\infty} \frac{1}{\alpha + (k + 1) \log (k + 1)}
\]

which is divergent as we saw in the previous case.
To sum up, we get that the series

\[ \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{\alpha + k \log k}{\beta + (k + 1) \log(k + 1)} \]

converge if and only if \( \alpha < \beta \), and then its sum is \( \alpha / (\beta - \alpha) \).

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