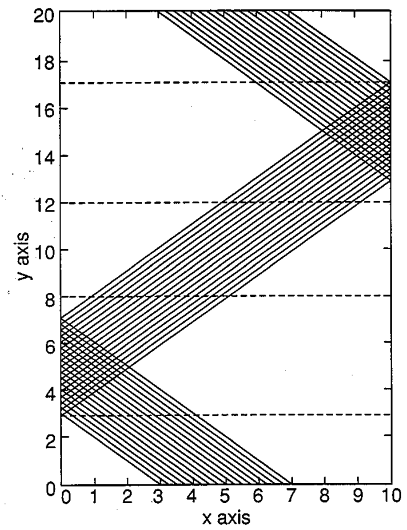

FIGURE 5.6a
Characteristics emanating from the subinterval [3, 7]. *Here* $c = 1$. *Solution* u *is constant along characteristics where they are not crosshatched.*

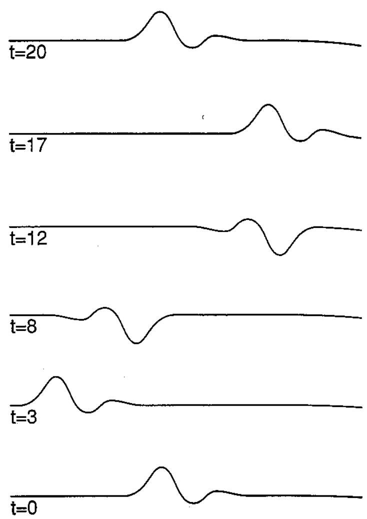

FIGURE 5.6b
Snapshots of the solution at times $t = 0, 3, 8, 12, 17, 20$. *Times correspond to dotted lines on Figure 5.6a.*

Equation (5.50) defines the eigenvalue problem that we studied in Chapter 4 for the heat equation. The eigenvalues are

$$\lambda_n = (\frac{n\pi}{L})^2, \qquad n = 1, 2, \ldots$$

and the eigenfunctions are

$$\varphi_n(x) = \sin(\frac{n\pi x}{L}).$$

Consequently, the solutions of (5.49) are

$$\psi_n(t) = A_n \cos(\omega_n t) + B_n \sin(\omega_n t).$$

$\omega_n = c\sqrt{\lambda_n} = c(n\pi/L)$ is the angular frequency. Consequently we look for a solution of (5.46) - (5.48) in the form of a series (again an eigenfunction expansion)

$$u(x,t) = \sum_1^\infty \psi_n(t)\varphi_n(x) = \sum_1^\infty [A_n \cos(\omega_n t) + B_n \sin(\omega_n t)] \sin(\frac{n\pi x}{L}). \quad (5.51)$$

Now we have two sequences of coefficients A_n and B_n to be determined by the two initial conditions (5.48):

$$f(x) = u(x,0) = \sum_1^\infty A_n \varphi_n(x),$$

and

$$g(x) = u_t(x,0) = \sum_1^\infty \omega_n B_n \varphi_n(x).$$

Hence as in Chapter 4,

$$A_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx \quad (5.52)$$

and

$$\omega_n B_n = \frac{2}{L}\int_0^L g(x)\sin(\frac{n\pi x}{L})dx. \quad (5.53)$$

The frequency in cycles per second $\nu_n = \omega_n/2\pi = c(n/2L)$. In acoustics $\nu_1 = c/2L$ is called the fundamental frequency of the string, and $\sin(\pi x/L)$ is called the fundamental mode. The fundamental period of the motion $T = 2L/c$. $\nu_2 = 2\nu_1 = c/L$ is the frequency of the first overtone. Since $\nu_2 = 2\nu_1$, this tone sounds one octave higher than the fundamental tone. The corresponding mode is $\sin(2\pi x/L)$ which now has a node (point where the string does not move) in the interior of the interval at $L/2$. The next frequency $\nu_3 = 3c/2L$ sounds a fifth above ν_2 and the corresponding mode $\sin(3\pi x/L)$ has two interior nodes at $x = L/3$ and $2L/3$. $\nu_4 = 4c/2L = 2\nu_2 = 4\nu_1$ sounds two octaves above the fundamental frequency ν_1, and the mode has three interior nodes (see Figure 5.7).

How well does the series representation of the solution converge? We notice that in (5.51) we do not have the decaying exponentials $\exp(-\lambda_n kt)$ that made the series for solutions of the heat equation converge so rapidly. The solutions of the wave equation do not become smoother as time increases. If the initial data has a "corner," the solution will continue to have a "corner" at later times. A standard example of this kind of behavior is that of the string plucked at its midpoint. Take the initial data (5.48) to be $g \equiv 0$ and

$$f(x) = \begin{cases} x & \text{for } 0 < x < L/2 \\ L - x & \text{for } L/2 < x < L \end{cases}. \quad (5.54)$$