

# Partial Differential Equations: Theory and numerical methods

Additional exercises to the lecture notes

Mónika Polner

June 15, 2015

# Contents

<b>1</b>	<b>Numerical methods for ODEs</b>	<b>2</b>
1.1	First order ODE's: . . . . .	2
1.1.1	An equivalent integral equation . . . . .	3
1.1.2	Piecewise polynomial approximation in 1D . . . . .	5
1.2	Two-point boundary value problems . . . . .	8
1.2.1	The finite element method . . . . .	8
<b>2</b>	<b>Parabolic problems, the heat equation</b>	<b>12</b>
2.1	FDM for the heat equation . . . . .	12
2.2	FEM for the heat equation . . . . .	12
<b>3</b>	<b>Waves</b>	<b>19</b>
3.1	The wave equation without boundaries . . . . .	19
3.2	Boundary value problems on the half-line . . . . .	21
3.3	Finite difference methods for the wave equation . . . . .	22
<b>4</b>	<b>First order PDEs</b>	<b>24</b>
4.1	Traveling waves, the method of characteristics . . . . .	24
4.2	Nonlinear conservation laws . . . . .	24
4.3	Numerical methods for first order PDEs . . . . .	25
	<b>Bibliography</b>	<b>27</b>

# Chapter 1

## Numerical methods for ODEs

### 1.1 First order ODE's:

Let  $\Omega \subseteq \mathbb{R}^2$  be an open and connected set (i.e., a domain), let  $f : \Omega \rightarrow \mathbb{R}$  be a continuous function and set  $(t_0, u_0) \in \Omega$ . Consider the initial value problem

$$(IVP) \begin{cases} \frac{du}{dt} = f(t, u(t)), \\ u(t_0) = u_0. \end{cases} \quad (1.1.1)$$

The iterative **Euler scheme** is

$$u_{n+1} = u_n + f(t_n, u_n) h.$$

**Exercise 1.** Solve the IVP

$$\frac{du}{dt} = \left(1 - \frac{4}{3}t\right) u, \quad u(0) = 1$$

using Euler's method with step sizes  $h = 0.1, 0.01$ , and plot the error of the approximation.

**Exercise 2.** Consider the IVP

$$\frac{du}{dt} = 3 + e^{-t} - \frac{1}{2}u, \quad u(0) = 1.$$

Use Euler's method with various step sizes to calculate approximate values of the solution for  $0 \leq t \leq 5$ . Compare the calculated results with the corresponding values of the exact solution.

### 1.1.1 An equivalent integral equation

The way to design higher order numerical algorithms is to replace the differential equation by the equivalent integral equation

$$u(t) = u(t_0) + \int_{t_0}^t f(s, u(s))ds.$$

Starting now from  $t_n$  and integrating till  $t = t_{n+1}$ , we obtain

$$u(t_{n+1}) = u(t_n) + \int_{t_n}^{t_{n+1}} f(s, u(s))ds.$$

The *left endpoint rule* for numerical integration. Using the approximation  $f(t_n, u(t_n)) \approx f(t_n, u_n)$ , the integral approximation becomes

$$\int_{t_n}^{t_{n+1}} f(s, u(s))ds \approx hf(t_n, u_n),$$

which results in the Euler method.

### Implicit and predictor-corrector methods

#### *The trapezoidal rule*

Using the integral approximation

$$\int_{t_n}^{t_{n+1}} f(s, u(s))ds \approx \frac{h}{2} [f(t_n, u_n) + f(t_{n+1}, u_{n+1})],$$

we obtain

$$u_{n+1} = u_n + \frac{h}{2} [f(t_n, u_n) + f(t_{n+1}, u_{n+1})], \quad (1.1.2)$$

which results in an *implicit* scheme. Note that this is a nonlinear algebraic equation for the unknown  $u_{n+1}$ , and in case of a system, this is a nonlinear system of equations.

Two ways to go:

- employ a numerical equation solver to compute  $u_{n+1}$  in (1.1.2)
- use a predictor-corrector algorithm

#### *The improved Euler method*

- Use explicit Euler method to *predict* a first approximation  $\tilde{u}_{n+1}$  to the desired solution value  $u_{n+1}$ :

$$\tilde{u}_{n+1} = u_n + hf(t_n, u_n).$$

- Correct this prediction by replacing the required  $u_{n+1}$  on the right hand side of the implicit scheme by  $\tilde{u}_{n+1}$ :

$$u_{n+1} = u_n + \frac{h}{2} [f(t_n, u_n) + f(t_{n+1}, \tilde{u}_{n+1})].$$

- Compute the resulting explicit scheme for the corrected value  $u_{n+1}$ :

$$u_{n+1} = u_n + \frac{h}{2} [f(t_n, u_n) + f(t_n + h, u_n + hf(t_n, u_n))].$$

### The midpoint rule

Use the integral approximation

$$\int_{t_n}^{t_{n+1}} f(s, u(s)) ds \approx hf \left( t_n + \frac{1}{2}h, u \left( t_n + \frac{1}{2}h \right) \right).$$

The steps of the method:

- Use Euler method to *predict* a first approximation:

$$u \left( t_n + \frac{1}{2}h \right) \approx u_n + \frac{1}{2}hf(t_n, u_n).$$

- Solve the resulting explicit scheme for the corrected value  $u_{n+1}$ :

$$u_{n+1} = u_n + hf \left( t_n + \frac{1}{2}h, u_n + \frac{1}{2}hf(t_n, u_n) \right).$$

### Runge-Kutta methods

The most general form is

$$u_{n+1} = u_n + h \sum_{i=1}^m c_i f(t_{i,n}, u_{i,n}),$$

where

- $m$  is the *number of terms*,
- $t_{i,n}$ ,  $i = 1, \dots, m$ , are points in the  $n$ -th interval, i.e.,

$$t_n \leq t_{i,n} \leq t_{n+1},$$

- $c_i \in \mathbb{R}$ ,

- $u_{i,n} \approx u(t_{i,n})$ .

Both the improved Euler and midpoint methods are special cases of two term Runge-Kutta methods.

#### A fourth order Runge-Kutta method

The method has the form

$$u_{n+1} = u_n + \frac{h}{6} [f(t_n, u_n) + 2f(t_{2,n}, u_{2,n}) + 2f(t_{3,n}, u_{3,n}) + f(t_{4,n}, u_{4,n})],$$

where

$$\begin{aligned} t_{2,n} &= t_n + \frac{1}{2}h, & u_{2,n} &= u_n + \frac{1}{2}h f(t_n, u_n) \\ t_{3,n} &= t_n + \frac{1}{2}h, & u_{3,n} &= u_n + \frac{1}{2}h f(t_{2,n}, u_{2,n}) \\ t_{4,n} &= t_n + h, & u_{4,n} &= u_n + h f(t_{3,n}, u_{3,n}). \end{aligned}$$

The improved Euler and midpoint methods are of second order, while the Runge-Kutta method above is a fourth order method.

**Exercise 3.** Compare the Euler, improved Euler, and Runge-Kutta methods for the model initial value problem.

### 1.1.2 Piecewise polynomial approximation in 1D

Piecewise polynomials that can be used to approximate general functions, and which are easy to implement in computer software. For computing piecewise polynomial approximations we present the  $L^2$ -projection technique.

Let  $I = [x_0, x_1]$  be an interval on the real axis and let  $\mathcal{P}_1(I)$  denote the vector space of linear functions on  $I$ , defined by

$$\mathcal{P}_1(I) = \{v : v(x) = c_0 + c_1x, x \in I, c_0, c_1 \in \mathbb{R}\}.$$

We can specify any function in  $\mathcal{P}_1(I)$  by its node values, hence we can introduce a new basis  $\{\psi_0, \psi_1\}$  for  $\mathcal{P}_1(I)$ . This new basis is called a *nodal basis*, and is defined by

$$\psi_j(x_i) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}, \quad i, j = 0, 1$$

Thus, each basis function is a linear function, and they take the following explicit form (see Fig. 1.1)

$$\psi_0(x) = \frac{x_1 - x}{x_1 - x_0}, \quad \psi_1(x) = \frac{x - x_0}{x_1 - x_0}.$$

The basic idea in constructing a piecewise linear function  $v$  is to first subdivide the domain

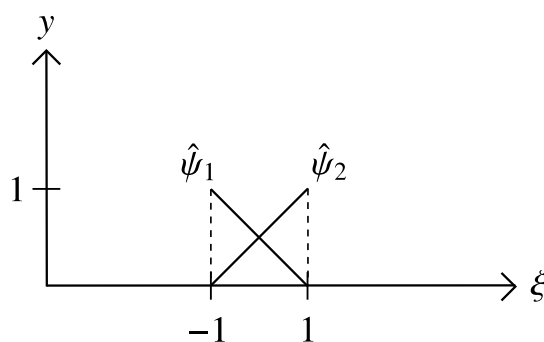


Figure 1.1: Local basis functions

of  $v$  into smaller subintervals. On each subinterval  $v$  is then given by a linear function. Continuity of  $v$  between adjacent subintervals may be enforced.

Let  $I = [a, b]$  be an interval and define a partition

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$$

of this interval into  $n$  subintervals  $I_i = [x_{i-1}, x_i]$ ,  $i = 1, 2, \dots, n$ , of length  $h_i = x_i - x_{i-1}$ .

On the mesh we define the space  $V_h$  of continuous piecewise linear functions by

$$V_h = \{v : v \in C^0(I), v|_{I_i} \in \mathcal{P}_1(I_i)\}$$

where  $C^0(I)$  denotes the space of continuous functions on  $I$ , and  $\mathcal{P}_1(I_i)$  denotes the space of linear functions on  $I_i$ . Thus, by construction the functions in  $V_h$  are linear on each subinterval  $I_i$  and continuous on the whole interval  $I$ .

Any function  $v \in V_h$  is uniquely determined by its nodal values  $\{v(x_i)\}_{i=0}^n$  and conversely. Let the nodal values define our degrees of freedom and introduce a basis  $\{\phi_j\}_{j=1}^n$  for  $V_h$  associated with the nodes and such that

$$\phi_j(x_i) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}, \quad i, j = 0, 1, \dots, n.$$

These are the so-called hat functions, see Figure 1.2. Note that  $\text{supp } \phi_i = I_i \cup I_{i+1}$ .

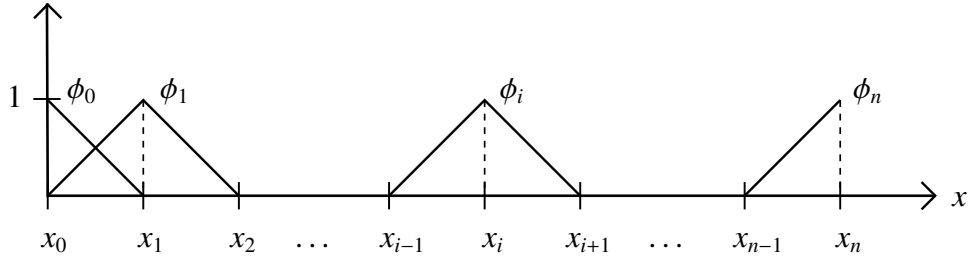


Figure 1.2: Global basis functions

Due to the construction of the hat function basis, any function  $v \in V_h$  can be written as a linear combination of hat functions  $\{\phi_i(x)\}_{i=0}^n$  and corresponding coefficients  $\{\xi_i\}_{i=0}^n$ , with  $\xi_i = v(x_i)$ ,  $i = 0, 1, \dots, n$ , the nodal values of  $v$ . That is,

$$v(x) = \sum_{i=0}^n \xi_i \phi_i(x).$$

The explicit expressions for the hat functions are given by

$$\phi_i(x) = \begin{cases} \frac{x-x_{i-1}}{h_i}, & \text{if } x \in I_i \\ \frac{x_{i+1}-x}{h_{i+1}}, & \text{if } x \in I_{i+1} \\ 0, & \text{otherwise.} \end{cases}$$

### $L^2$ -projection

Orthogonal-, or  $L^2$ -projection is a technique for approximating functions. Given a function  $f \in L^2(I)$  the  $L^2$ -projection  $P_h f \in V_h$  of  $f$  is defined by

$$\int_I (f - P_h f) v \, dx = 0, \quad \forall v \in V_h.$$

In analogy with projection onto subspaces of  $\mathbb{R}^n$ , it defines a projection of  $f$  onto  $V_h$ , since the difference  $f - P_h f$  is required to be orthogonal to all functions  $v \in V_h$ .

To compute the  $L^2$ -projection  $P_h f$  we first note that the definition is equivalent to

$$\int_I (f - P_h f) \phi_i \, dx = 0, \quad i = 0, 1, \dots, n,$$

where  $\phi_i$ ,  $i = 0, 1, \dots, n$  are the hat basis functions. Since  $P_h f \in V_h$  it can be written as the linear combination

$$P_h f(x) = \sum_{j=0}^n \xi_j \phi_j(x)$$



with  $n + 1$  unknown coefficients  $\xi_j$ ,  $j = 0, 1, \dots, n$  to be determined. Inserting this into the definition of the projection, we get

$$\int_I f \phi_i dx = \int_I \left( \sum_{j=0}^n \xi_j \phi_j \right) \phi_i dx = \sum_{j=0}^n \xi_j \int_I \phi_j \phi_i dx, \quad i = 0, 1, 2, \dots, n.$$

Introducing the notations

$$M_{ij} = \int_I \phi_j \phi_i dx, \quad i, j = 0, 1, 2, \dots, n, \quad (1.1.3)$$

$$b_i = \int_I f \phi_i dx, \quad i = 0, 1, 2, \dots, n, \quad (1.1.4)$$

we obtain

$$b_i = \sum_{j=0}^n M_{ij} \xi_j, \quad i = 0, 1, 2, \dots, n,$$

which leads to the linear system in matrix form

$$M\xi = b.$$

The matrix  $M$  is called *mass matrix* and  $b$  is the *load vector*.

Since the hats  $\phi_i$  and  $\phi_j$  lack common support for  $|i - j| > 1$ , only  $M_{ii}$ ,  $M_{i,i+1}$ , and  $M_{i+1,i}$  need to be calculated. All other matrix entries are zero by default. As a consequence, the mass matrix  $M$  is tridiagonal.

**Exercise.** Calculate the entries of the mass matrix. Implement the  $L^2$ -projection  $P_h f$  of your favorite function  $f$ .

## 1.2 Two-point boundary value problems

### 1.2.1 The finite element method

Consider the following two-point boundary value problem: find  $u$  such that

$$-u''(x) = f(x), \quad x \in I = (0, 1) \quad (1.2.1)$$

$$u(0) = u(1) = 0, \quad (1.2.2)$$

where  $f$  is a given function. For a general  $f$  it may be difficult or even impossible to find  $u$  with analytical techniques, hence numerical methods are needed. In the subsequent sections we summarize the most important steps of finite element method (FEM).

### Variational formulation

The derivation of a finite element method always starts by rewriting the differential equation under consideration as variational equation. This so-called variational formulation is obtained by multiplying the differential equation by a test function  $v$ , which is assumed to vanish at the end-points of the interval  $I$ , and integrate by parts. For our model problem this yields:

$$\int_0^1 f v dx = - \int_0^1 u'' v dx = \int_0^1 u' v' dx - u'(1)v(1) + u'(0)v(0) = \int_0^1 u' v' dx,$$

where we used the assumption  $v(0) = v(1) = 0$ . For this example we need to make additional assumptions of the test function, so that the integrals exist. The largest space of such test functions is

$$V_0 = \{v : \|v\| < \infty, \|v'\| < \infty, v(0) = v(1) = 0\}.$$

This leads to the following variational formulation: find  $u \in V_0$  such that

$$\int_0^1 f v dx = \int_0^1 u' v' dx, \quad \forall v \in V_0.$$

The solution  $u$  is called trial function.

### Finite element approximation

We next try to approximate  $u$  by a continuous piecewise linear function. To this end we introduce a mesh on the interval  $I$  consisting of  $n$  subintervals, and the corresponding space  $V_h$  of all continuous piece-wise linears. Since our functions are vanishing at the end-points of  $I$ , we also introduce the following subspace  $V_{h,0}$  of  $V_h$  that satisfies the boundary conditions

$$V_{h,0} = \{v \in V_h : v(0) = v(1) = 0\}.$$

This means that a basis for  $V_{h,0}$  is obtained by deleting the half hats  $\phi_0$  and  $\phi_n$  from the usual set  $\{\phi_j\}_{j=0}^n$  of hat functions spanning  $V_h$ . Replacing the large space  $V_0$  with the much smaller subspace  $V_{h,0} \subset V_0$  of piece-wise linears in the variational formulation, we obtain the following finite element method: find  $u_h \in V_{h,0}$  such that

$$\int_0^1 u_h' v' dx = \int_0^1 f v dx, \quad \forall v \in V_{h,0}. \quad (1.2.3)$$

To compute the finite element approximation  $u_h$  we first note that (1.2.3) is equivalent to

$$\int_0^1 u_h' \phi_i' dx = \int_0^1 f \phi_i dx, \quad i = 1, 2, \dots, n-1.$$

Since  $u_h \in V_{h,0}$ ,

$$u_h = \sum_{j=1}^{n-1} \xi_j \phi_j, \quad \xi_j \in \mathbb{R}.$$

Inserting this into the finite element method (1.2.3), we get

$$\int_0^1 f \phi_i dx = \int_0^1 \left( \sum_{j=1}^{n-1} \xi_j \phi_j' \right) \phi_i' dx = \sum_{j=1}^{n-1} \xi_j \int_0^1 \phi_j' \phi_i' dx, \quad i = 1, 2, \dots, n-1.$$

Introducing the notations

$$A_{ij} = \int_0^1 \phi_j' \phi_i' dx, \quad i, j = 1, 2, \dots, n-1, \quad (1.2.4)$$

$$b_i = \int_0^1 f \phi_i dx, \quad i = 1, 2, \dots, n-1, \quad (1.2.5)$$

we obtain

$$b_i = \sum_{j=1}^{n-1} A_{ij} \xi_j, \quad i = 1, 2, \dots, n-1,$$

which leads to the linear system in matrix form

$$A\xi = b.$$

The matrix  $A$  is called *stiffness matrix* and  $b$  is the *load vector*.

### A model problem.: the stationary heat equation

Many physical phenomena are described by the same partial differential equations, and therefore the methods and mathematical theory can often be developed for certain model problems and still be applied to a wide range of different applications.

Consider a thin metal rod of length  $L$  and cross section area  $A[m^2]$  occupying the interval  $[0, L]$ . The rod is heated by a heat source (e.g., a small electrical current) of intensity  $f[J/(sm)]$ , which has been acting for a long time so that the heat transfer process is at a steady state, and all physical quantities are independent of time. Let  $q[J/(sm^2)]$  be the heat flux. We want to find the distribution of temperature  $T[K]$  within the rod. Using the lecture notes, we arrive to the equation

$$(AkT')' = f,$$

which is the stationary Heat equation.

*Boundary conditions.* There are three types of boundary conditions: Dirichlet, Neumann, and Robin boundary conditions. Robin boundary conditions is a mixture of Dirichlet and Neumann boundary conditions, typically of the form  $AkT'(0) = T(0) - T_\infty$ . In real-world applications this is perhaps the most realistic boundary condition, since it means that the heat flux is proportional to the difference between the temperature of the rod and the ambient media  $T_\infty$ . Consider the general Robin boundary condition

$$AkT'(0) = \kappa(T(0) - T_\infty) + q_\infty$$

where  $\kappa \geq 0$ ,  $T_\infty$ , and  $q_\infty$  are parameters to be chosen. Choosing  $\kappa = 0$  we obtain the Neumann boundary condition  $AkT'(0) = q_\infty$ . Choosing on the other hand  $\kappa$  large means that whenever  $T(0) \neq T_\infty$  there will be a heat flux between the rod and the ambient media, which will counteract this difference. As a consequence, the Robin condition will approximate the Dirichlet condition  $T(0) = T_\infty$  as  $\kappa \rightarrow \infty$ .

Summarizing, we want to solve the BVP:

$$-(au')' = f, \quad x \in I = (0, 1) \tag{1.2.6}$$

$$au'(0) = \kappa_0(u(0) - g_0) \tag{1.2.7}$$

$$-au'(1) = \kappa_1(u(1) - g_1) \tag{1.2.8}$$

$$\tag{1.2.9}$$

where  $a > 0$  and  $f$  are given functions, and  $\kappa_0 \geq 0, \kappa_1 \geq 0, g_0, g_1$  are given parameters. The positiveness assumption on  $a, \kappa_0$ , and  $\kappa_1$  is necessary to assert existence and uniqueness of the solution  $u$ .

Write a finite element solver for this BVP. Let us use it to compute the temperature  $T$  in a rod of length  $L = 6m$ , cross section  $A = 0.1 \text{ m}^2$ , thermal conductivity  $k = 5 - 0.6x \text{ J/(Ksm)}$ , internal heat source  $f = 0.03(x - 6)^4 \text{ J/sm}$ , held at constant temperature  $T = -1 \text{ K}$  at  $x = 2$ , and thermally insulated at  $x = 8$ . Thus, we want to solve

$$-(0.5 + 0.7x)T'' = 0.3x^2, \quad 2 < x < 8, \quad T(2) = -1, \quad T'(8) = 0.$$

To approximate the Dirichlet condition  $T(2) = -1$  we use the Robin condition with parameters  $\kappa_0 = 10^6$  and  $g_0 = -1$ . Similarly, to impose the Neumann condition  $T'(8) = 0$  we let  $\kappa_1 = 0$ . The value of  $g_1$  does not matter.

# Chapter 2

## Parabolic problems, the heat equation

### 2.1 FDM for the heat equation

We discussed during the lectures an explicit, an implicit and the Crank-Nicolson methods. In the next section we add a new discretization method: the finite element method. The exercises at the end of the next section will refer to all numerical methods.

### 2.2 FEM for the heat equation

We consider the model problem

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left( a \frac{\partial u}{\partial x} \right) = q, \quad x \in (0, L), t \in (0, T), \quad (2.2.1a)$$

$$au'(0, t) = \kappa_0 (u(0, t) - g_{D,0}) + g_{N,0}, \quad (2.2.1b)$$

$$-au'(L, t) = \kappa_L (u(L, t) - g_{D,L}) - g_{N,L}, \quad (2.2.1c)$$

$$u(x, 0) = f(x) \quad (2.2.1d)$$

where  $u = u(x, t)$  is the solution,  $a = a(x) \geq a_0 > 0$  is a given positive function,  $q = q(x, t) \in C(\bar{\Omega})$  is a given source function where  $\Omega = (0, L)$ ,  $\kappa_0, \kappa_L, g_{D,0}, g_{D,L}, g_{N,0}, g_{N,L} \in L^2(\partial\Omega)$  are given functions,  $\kappa_0, \kappa_L \geq 0$  and  $f(x)$  is a given initial condition.  $\bar{\Omega}$  denotes the closure of  $\Omega$ :  $\bar{\Omega} \stackrel{\text{def}}{=} \Omega \cup \partial\Omega$ .

The boundary conditions (2.2.1b) and (2.2.1c) describe how the solution  $u$  has to act like on the boundary of the domain. Usually three types of boundary conditions can occur. The *Dirichlet type* as known as *first* or *essential boundary condition* specifies the value of the solution on the boundary:

$$u|_{\partial\Omega} = g_D$$

where  $g_D : \partial\Omega \rightarrow \mathbb{R}$  is a continuous function.

The *Neumann-, second or natural boundary condition* models the exterior effect acting on the boundary:

$$au' |_{\partial\Omega} = g_N.$$

The *Robin- or third type boundary condition* is a mixture of Dirichlet and Neumann:

$$Bu |_{\partial\Omega} = \kappa(u - g_D).$$

The generalized Robin condition can be used to approximate Dirichlet and Neumann. Choosing  $\kappa = 0$  we obtain Neumann condition, and  $\kappa \rightarrow \infty$  gives Dirichlet type.

We introduce some notations. Consider the Sobolev space  $W_2^1(\Omega)$ ,

$$H^1(\Omega) = W_2^1(\Omega),$$

where the inner product is

$$(u, v)_{H^1(\Omega)} = \int_{\Omega} uv \, dx + \int_{\Omega} \nabla u \nabla v \, dx,$$

and the norm is

$$\|u\|_{H^1(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2.$$

Let  $V=H^1(0, L)$ . Multiplying (2.2.1a) by a test function  $v \in V$  and integrating by parts over  $(0, L)$  gives

$$\begin{aligned} \int_0^L qv \, dx &= \int_0^L \frac{\partial u}{\partial t} v \, dx - \int_0^L \frac{\partial}{\partial x} \left( a \frac{\partial u}{\partial x} \right) v \, dx \\ &= \int_0^L \frac{\partial u}{\partial t} v \, dx - a \frac{\partial}{\partial x} u(L, t) v(L, t) + a \frac{\partial}{\partial x} u(0, t) v(0, t) + \int_0^L a \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \, dx \\ &= \int_0^L \frac{\partial u}{\partial t} v \, dx + (\kappa_L (u(L, t) - g_{D,L}) - g_{N,L}) v(L, t) + (\kappa_0 (u(0, t) - g_{D,0}) + g_{N,0}) v(0, t) \\ &\quad + \int_0^L a \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \, dx. \end{aligned} \tag{2.2.2}$$

The variational formulation: find  $u \in V$  such that

$$\begin{aligned} & \int_0^L \frac{\partial u}{\partial t} v \, dx + \kappa_L u(L, t)v(L, t) + \kappa_0 u(0, t)v(0, t) + \int_0^L a \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \, dx \\ &= \int_0^L qv \, dx + g_{N,L}v(L, t) - g_{N,0}v(0, t) + \kappa_L g_{D,L}v(L, t) + \kappa_0 g_{D,0}v(0, t), \quad \forall v \in V, 0 < t < T. \end{aligned} \quad (2.2.3)$$

Since  $u$  depends on  $x$  and  $t$  we first discretize with respect to  $x$ . Let  $0 = x_0 < x_1 < \dots < x_n = L$  be a mesh on interval  $\Omega = [0, L]$ , and let  $V_h \subset V$  be the corresponding subspace of continuous piecewise linear functions. Space discrete counterpart: find  $u_h$  such that, for every fixed  $t$ ,  $u_h \in V_h$  and

$$\begin{aligned} & \int_0^L \frac{\partial u_h}{\partial t} v \, dx + \kappa_L u_h(L, t)v(L, t) + \kappa_0 u_h(0, t)v(0, t) + \int_0^L a \frac{\partial u_h}{\partial x} \frac{\partial v}{\partial x} \, dx \\ &= \int_0^L qv \, dx + g_{N,L}v(L, t) - g_{N,0}v(0, t) + \kappa_L g_{D,L}v(L, t) + \kappa_0 g_{D,0}v(0, t), \quad \forall v \in V_h, 0 < t < T. \end{aligned} \quad (2.2.4)$$

This is equivalent to the following: find  $u_h$  such that, for every fixed  $t$ ,  $u_h \in V_h$  and

$$\begin{aligned} & \int_0^L \frac{\partial u_h}{\partial t} \phi_i \, dx + \kappa_L u_h(L, t)\phi_i(L) + \kappa_0 u_h(0, t)\phi_i(0) + \int_0^L a \frac{\partial u_h}{\partial x} \frac{\partial \phi_i}{\partial x} \, dx \\ &= \int_0^L q\phi_i \, dx + g_{N,L}\phi_i(L) - g_{N,0}\phi_i(0) + \kappa_L g_{D,L}\phi_i(L) + \kappa_0 g_{D,0}\phi_i(0), \\ & \quad i = 0, 1, \dots, n, 0 < t < T \end{aligned} \quad (2.2.5)$$

where  $\phi_i$ ,  $i = 0, 1, \dots, n$  are the hat basis functions for  $V_h$ . We seek the solution  $u_h$  as a linear combination

$$u_h(x, t) = \sum_{j=0}^n \xi_j(t) \phi_j(x),$$

$$\xi(t) = \begin{pmatrix} \xi_0(t) \\ \xi_1(t) \\ \vdots \\ \xi_n(t) \end{pmatrix} = \begin{pmatrix} u_h(x_0, t) \\ u_h(x_1, t) \\ \vdots \\ u_h(x_n, t) \end{pmatrix},$$

and substitute:

$$\begin{aligned}
 & \sum_{j=0}^n \frac{\partial \xi_j}{\partial t} \int_0^L \phi_j \phi_i \, dx + \kappa_L \xi_n(t) \phi_j(L) \phi_i(L) + \kappa_0 \xi_0(t) \phi_j(0) \phi_i(0) + \sum_{j=0}^n \xi_j \int_0^L a \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial x} \, dx \\
 &= \int_0^L q \phi_i \, dx + g_{N,L} \phi_i(L) - g_{N,0} \phi_i(0) + \kappa_L g_{D,L} \phi_i(L) + \kappa_0 g_{D,0} \phi_i(0), \\
 & \qquad \qquad \qquad i = 0, 1, \dots, n, \quad 0 < t < T. \tag{2.2.6}
 \end{aligned}$$

The elements of the mass matrix, stiffness matrix and load vector are

$$\begin{aligned}
 M_{ij} &= \int_0^L \phi_j \phi_i \, dx, \quad i, j = 0, 1, 2, \dots, n, \\
 A_{ij} &= \int_0^L a \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial x} \, dx + \kappa_L \phi_j(L) \phi_i(L) + \kappa_0 \phi_j(0) \phi_i(0), \quad i, j = 0, 1, 2, \dots, n, \\
 b_i(t) &= \int_0^L q(t) \phi_i \, dx + g_{N,L} \phi_i(L) - g_{N,0} \phi_i(0) + \kappa_L g_{D,L} \phi_i(L) + \kappa_0 g_{D,0} \phi_i(0), \\
 & \qquad \qquad \qquad i = 0, 1, \dots, n, \quad 0 < t < T.
 \end{aligned}$$

The equation (2.2.6) in matrix form is

$$\sum_{j=0}^n M_{ij} \frac{\partial \xi_j(t)}{\partial t} + \sum_{j=0}^n A_{ij} \xi_j(t) = b_i(t), \quad i = 0, 1, \dots, n, \quad 0 < t < T, \tag{2.2.7}$$

$$M \frac{\partial \xi(t)}{\partial t} + A \xi(t) = b(t), \quad 0 < t < T. \tag{2.2.8}$$

This is called the spatial semi-discretization of the heat equation. After that, we discretize this system of ordinary differential equations with respect to  $t$ .

Let  $0 = t_0 < t_1 < \dots < t_m = T$  be a time grid on  $[0, T]$ , with time steps  $k_l = t_l - t_{l-1}$ ,  $l = 1, 2, \dots, m$  and let denote  $\xi_l$  an approximation to  $\xi(t_l)$ . We apply the backward Euler method to approximate the solution  $\xi(t)$  of the system (2.2.8). We substitute  $\frac{\partial \xi(t_l)}{\partial t}$  with the quotient  $\frac{\xi(t_l) - \xi(t_{l-1})}{k_l}$ , the error of discretization is  $O(k_l)$ .

Consider the problem

$$M \frac{\partial \xi(t)}{\partial t} + A \xi(t) = b(t), \quad 0 < t < T, \tag{2.2.9a}$$

$$\xi(0) = \xi_0. \tag{2.2.9b}$$



Integrating the system (2.2.9a) over  $(t_{l-1}, t_l)$  we have

$$\begin{aligned} \int_{t_{l-1}}^{t_l} b(t) dt &= \int_{t_{l-1}}^{t_l} M \frac{\partial \xi(t)}{\partial t} dt + \int_{t_{l-1}}^{t_l} A \xi(t) dt \\ &= M (\xi(t_l) - \xi(t_{l-1})) + A \int_{t_{l-1}}^{t_l} \xi(t) dt. \end{aligned} \quad (2.2.10)$$

Using the right-end point quadrature to approximate the integrals, where the error is  $O(k_l)$ , we obtain

$$k_l b_l = M (\xi_l - \xi_{l-1}) + k_l A \xi_l,$$

or equivalently

$$(M + k_l A) \xi_l = M \xi_{l-1} + k_l b_l.$$

Since  $\xi_0$  is given, we can solve the originated system. Then, we get the solution  $\xi$ , the numerical solution of differential equation (2.2.1) in the points  $(x_j, t_l)$ ,  $j = 0, 1, \dots, n$ ,  $l = 0, 1, \dots, m$ .

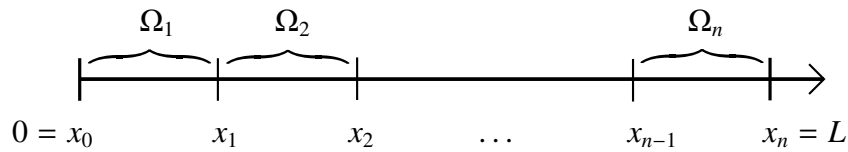


Figure 2.1: Mesh on interval  $\Omega$

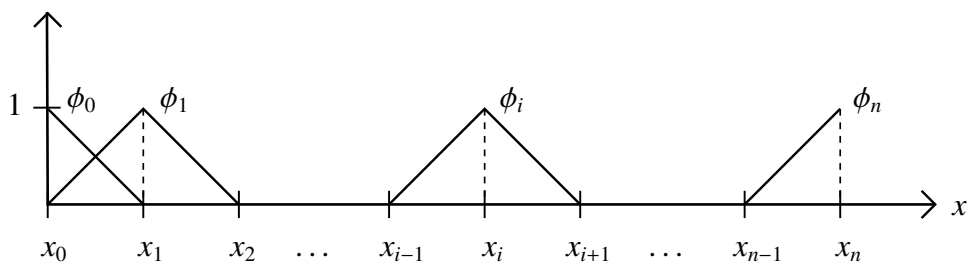


Figure 2.2: Global basis functions

**Exercise 4.** Consider the IBVP

$$\begin{cases} u_t = k u_{xx}, & 0 < x < L, t > 0, \\ u(x, 0) = f(x), & 0 < x < L \\ + \text{boundary conditions,} \end{cases} \quad (2.2.11)$$

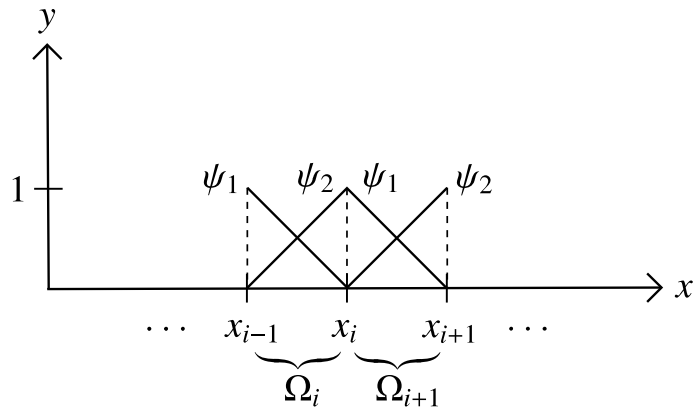


Figure 2.3: Element basis functions

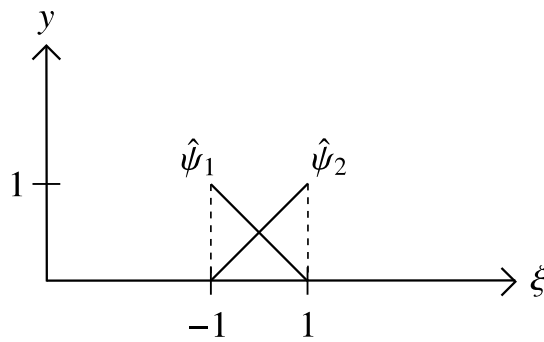


Figure 2.4: Local basis functions

with homogeneous Dirichlet boundary conditions,  $k = 1$ ,  $L = 1$  and initial function

$$f(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2} \\ 2 - 2x, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Construct the numerical solution using the explicit, the implicit, the Crank-Nicolson and the finite element method.

**Exercise 5.** Consider the heat equation with inhomogeneous Dirichlet boundary conditions  $u(0, t) = -2$ ,  $u(L, t) = 3$  and initial function  $f(x) = \frac{x}{2} - 2 + \sin(\frac{4\pi}{10}x)$ .

- (a) Determine the steady-state solution  $U(x)$  towards which the time depending solution is tending.

- (b) Plot the solution on the interval  $[0, 10]$  for several values of time and  $k = 1$ .
- (c) How large should  $t$  be so that  $\max |u(x, t) - U(x)| \leq 1/10$ ? Plot  $\max |u(x, t) - U(x)|$  for this  $t$  on the interval  $[0, 10]$ .

Hint: We have solved this exercise analytically in the lecture, hence you can verify your numerical scheme.

**Exercise 6.** Consider the IBVP with homogeneous Neumann boundary conditions:

$$\begin{cases} u_t = ku_{xx}, & 0 < x < L, t > 0, \\ u(x, 0) = f(x), & 0 < x < L, \\ u_x(0, t) = u_x(L, t) = 0, & t \geq 0. \end{cases}$$

- (a) Using your Matlab code developed (both the Crank-Nicholson method and FEM), plot the solution for several time values when  $L = 10, k = 1$  and  $f(x) = e^{-(x-5)^2}$ .
- (b) Compute the integral

$$\frac{1}{10} \int_0^{10} f(x) dx$$

using the error function. Does this value agree with the asymptotic value of the solution seen in your plots when  $t \rightarrow \infty$ ?

Hint: We know from the lecture that  $u(x, t) - \bar{f}$  tends to zero exponentially as  $t \rightarrow \infty$ , where

$$\bar{f} = \frac{1}{L} \int_0^L f(x) dx.$$

**Exercise 7.** Consider the heat equation with  $L = 1, f(x) = x(1 - x)$ , source term

$$q(x, t) = 2(t + 1) + x(1 - x)$$

and homogeneous Dirichlet boundary conditions.

- (a) Show that

$$u(x, t) = (t + 1)x(1 - x)$$

solves the IBVP and violates the maximum principle.

- (b) Apply the FEM to this problem and compare the numerical results with the solution of part (a).

# Chapter 3

## Waves

### 3.1 The wave equation without boundaries

The main purpose of this lecture is to understand how waves propagate before they interact with the boundaries. We consider the IVP

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R}, t \in \mathbb{R} \\ u(x, 0) = f(x), & x \in \mathbb{R} \quad (\text{initial position}) \\ u_t(x, 0) = g(x), & x \in \mathbb{R} \quad (\text{initial velocity}) \end{cases} \quad (3.1.1)$$

**Theorem 3.1.1** Suppose  $f \in C^2(\mathbb{R})$  and  $g \in C^1(\mathbb{R})$ . Then (3.1.1) has a unique  $C^2$  solution given by d'Alembert's formula:

$$u(x, t) = \frac{1}{2} [f(x + ct) + f(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y) dy \quad (3.1.2)$$

for all  $x \in \mathbb{R}$  and  $t \in \mathbb{R}$ .

Summing up, we see that

$$u(x, t) = F(x - ct) + G(x + ct),$$

i.e., the solution is the sum of two waves:

- term  $F(x - ct)$  gives a wave propagating to the right with speed  $c$ ,
- while  $G(x + ct)$  gives a wave moving to the left with speed  $c$ .

**Exercise 8.** Consider the IVP (3.1.1). Let  $c = 1$ ,  $f(x) = e^{-x^2}$  and suppose there is no initial velocity, so  $g \equiv 0$ . Implement this wave propagation and observe that the initial

displacement splits into two waves, one moving to the right and the other moving to the left, each of constant speed  $c$ , each of exactly the same shape as  $f$ , but only half as tall.

**Exercise 9.** Take now two initially separated pulses, centered at  $x = 0$  and  $x = 10$ , for example

$$f(x) = 2e^{-x^2} + e^{-(x-10)^2} \text{ and } g \equiv 0.$$

The solution will consist of four pulses, two moving to the right and two to the left. Observe that, when the right-moving pulse collides with the left-moving pulse, they emerge from collision unchanged.

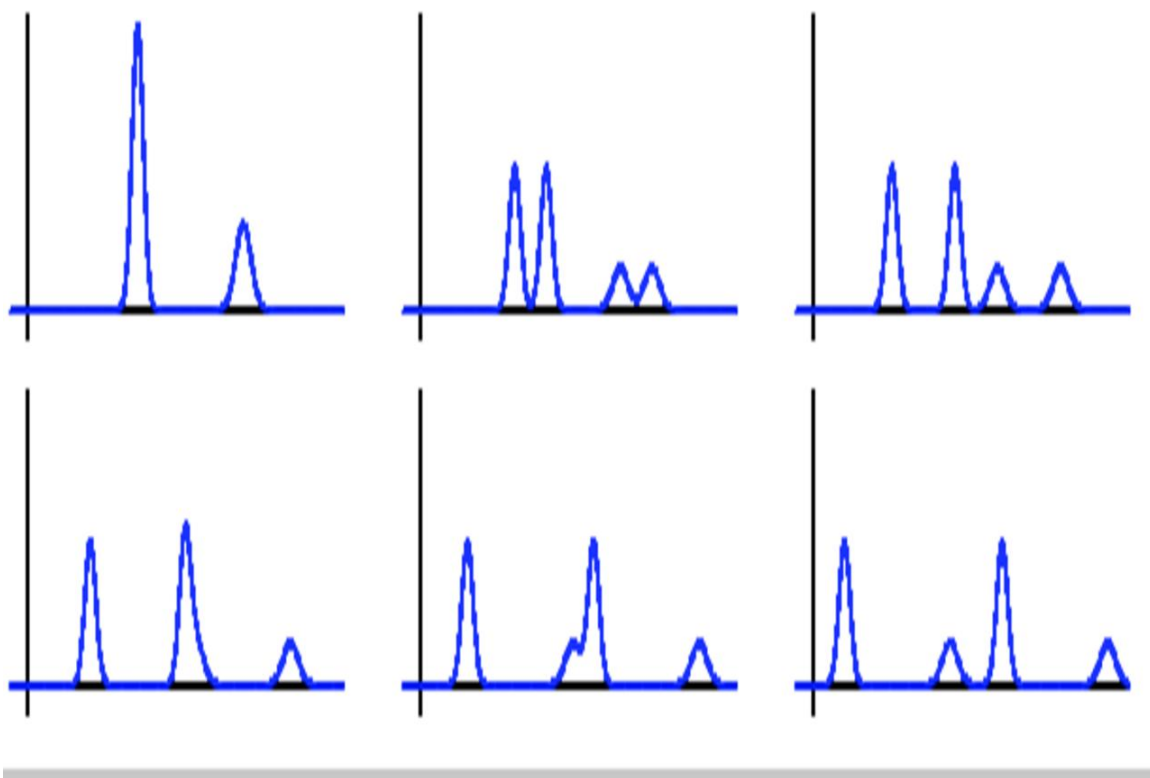


Figure 3.1: Interaction of waves

**Exercise 10.** Suppose now that there is no initial displacement, so  $f(x) \equiv 0$  and the motion is purely the result of the initial velocity

$$g(x) = e^{-x^2}.$$

Physically, this models a violin string at rest being hit by a hammer at the initial time. Show that the d'Alembert solution of this problem can be written in terms of the error function:

$$u(x, t) = \frac{\sqrt{\pi}}{4c} (\operatorname{erf}(x + ct) - \operatorname{erf}(x - ct)).$$

The effect of the initial hammer blow is felt gradually further and further away along the string. A nonzero initial velocity leaves the string permanently deformed.

## 3.2 Boundary value problems on the half-line

The initial value problem on the half-line with Dirichlet condition at  $x = 0$  is:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x > 0, t \in \mathbb{R} \\ u(x, 0) = f(x), & x \geq 0 \\ u_t(x, 0) = g(x), & x \geq 0 \\ u(0, t) = 0, & t \in \mathbb{R} \end{cases} \quad (3.2.1)$$

**Exercise 10.** Consider the IBVP for the half-line with  $c = 1$ . Let the initial displacement be

$$f(x) = \begin{cases} 0, & 0 \leq x < 2 \\ 1, & 2 \leq x \leq 4 \\ 0 & x > 4 \end{cases}$$

and initial velocity  $g = 0$ . Simulate the solution using the d'Alembert formula for the half line.

**Exercise 11.** Consider the IBVP for the half-line with  $c = 1$ . Let the incident wave form be

$$G(x) = (x - 5)(x - 6)e^{-(x-5)^2}.$$

(a) Show that the solution of the IBVP with Dirichlet condition  $u(0, t) = 0$  has the form

$$u(x, t) = G(x + t) - G(t - x)$$

and plot the solution for  $t = 2, 4, 6, 8$ . Describe how the shape of the incident wave is changed after reflection.

(b) Answer the same questions for the solution

$$u(x, t) = G(x + t) + G(t - x)$$

of the IBVP with Neumann condition  $u_x(0, t) = 0$ .

(c) Make movies of the reflection for both boundary conditions.

### 3.3 Finite difference methods for the wave equation

**Exercise 12.** Solve the IVP

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < 10, t > 0 \\ u(x, 0) = f(x) = e^{-4(x-3)^2}, & 0 < x < 10 \\ u_t(x, 0) = g(x) \equiv 0, & 0 < x < 10 \\ u(0, t) = u(L, t) = 0, & t \geq 0 \end{cases} \quad (3.3.1)$$

with the finite difference scheme presented in the lecture.

**Exercise 13.** Consider the IVP, called *hammer blow* in the literature,

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < 10, t > 0 \\ u(x, 0) = f(x) \equiv 0, & 0 < x < 10 \\ u_t(x, 0) = g(x) = \begin{cases} 0, & 0 \leq x < 2.5 \\ -1, & 2.5 \leq x < 7.5 \\ 0, & 7.5 \leq x < 10 \end{cases} \\ u(0, t) = u(L, t) = 0, & t \geq 0. \end{cases}$$

Our Matlab program uses the explicit finite difference scheme to numerically solve the wave equation. For this example the (time) period of motion is  $T = 20$ . Observe in the plot the motion of the string at several times during one period.

**Exercise 14.** Consider the IVP, called *plucked string* in the literature,

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < 10, t > 0 \\ u(x, 0) = f(x) = \begin{cases} x/5, & 0 \leq x \leq 5 \\ -1 - (x - 5)/5, & 5 \leq x \leq 10 \end{cases} \\ u_t(x, 0) = g(x) \equiv 0, & 0 < x < 10 \\ u(0, t) = u(L, t) = 0, & t \geq 0. \end{cases}$$

### 3.3. FINITE DIFFERENCE METHODS FOR THE WAVE EQUATION

Our Matlab program uses the explicit finite difference scheme to numerically solve the wave equation. For this example the (time) period of motion is  $T = 20$ . Observe in the plot the motion of the string at several times during one period.



# Chapter 4

## First order PDEs

The material of this chapter can be found in the lecture notes. Only the list of exercises is listed here.

### 4.1 Traveling waves, the method of characteristics

**Exercise 1.** (traveling waves) Consider the transport equation with characteristic speed  $c = 2$  and initial function  $u(x, 0) = f(x) = e^{-2(x-1)^2}$ . Observe the characteristic lines, the solution surface and two dimensional contours of the solution.

**Exercise 2.** (Non-uniform transport equation) Consider the non-uniform transport equation with characteristic speed  $c(x) = x$  and initial function  $u(x, 0) = f(x) = e^{-2(x-2)^2}$ . Observe the characteristic lines, the solution at some instances of time, the solution surface and two dimensional contours of the solution.

**Exercise 3.** Consider the non-uniform transport equation with characteristic speed  $c(x) = -x$  and initial function  $u(x, 0) = f(x) = \sin x$ . Observe the characteristic lines, the solution at some instances of time, the solution surface and two dimensional contours of the solution.

### 4.2 Nonlinear conservation laws

$$u_t + (F(u))_x = 0, \text{ or } \boxed{u_t + c(u)u_x = 0}, \text{ where } c(u) = F'(u). \quad (4.2.1)$$

We know that since the solution is constant along a characteristic curve, the derivative

$dx/dt$  is constant, and hence the characteristic curve must be a straight line

$$x(t) = c(f(x_0))t + x_0, \quad \text{where } x_0 = x(0).$$

Equivalently, the characteristic line through  $(x_0, t)$  is

$$x(t) = x_0 + c(f(x_0))t, \quad \text{where } u(x, 0) = f(x). \quad (4.2.2)$$

**Project: Traffic flow.** In our application,  $u(x, t)$  is the traffic density on a highway. Then the conservation law tells that the rate of change in the number of vehicles on the road between  $a$  and  $b$  (with no exits or entrances) equals the number of vehicles entering at point  $a$  minus the number of leaving at point  $b$ . Consider the conservation law (4.2.1), with characteristic speed  $c(u) = \beta - 2u$  and initial function

$$f(x) = u(x, 0) = \begin{cases} 0, & \text{if } x \leq 0 \\ \beta x^2(3 - 2x), & \text{if } 0 \leq x \leq 1 \\ \beta, & \text{if } x \geq 1. \end{cases} \quad (4.2.3)$$

- Plot some characteristics emanating from  $x_0 \in [-1, 2]$ , for  $t \in [0, 1]$ .
- Plot the solution  $u(x, t)$  on the interval  $x \in [-1, 2]$  for some time values  $t > 0$ . Observe what happens with the initial function. Plot the solution  $u(x, t)$  starting from  $x_0 = 0.5$  for  $(x, t) \in [-1, 2] \times [0, 1]$ .
- Do the characteristic lines intersect for  $t > 0$ ? If so, what is the first time  $t^* > 0$ ? Observe the behavior of the solution for times close to  $t^*$ .
- Repeat the same exercise for the initial function

$$g(x) = u(x, 0) = \begin{cases} \beta, & \text{if } x \leq 0 \\ \beta(1 - x^2(3 - 2x)), & \text{if } 0 \leq x \leq 1 \\ 0, & \text{if } x \geq 1. \end{cases} \quad (4.2.4)$$

## 4.3 Numerical methods for first order PDEs

**Exercise 1.** Consider the transport equation  $u_t + cu_x = 0$ , with  $c = -1$  and initial function

$$f(x) = u(x, 0) = \begin{cases} 0, & x \leq -1 \\ x + 1, & -1 \leq x \leq 0 \\ 1, & x \geq 0. \end{cases} \quad (4.3.1)$$

Plot the solution calculated with the method of characteristics, the forward and the upwind methods.

**Exercise 2.** Consider the transport equation with  $c = 1$  and the same initial function. Plot the solution calculated with the method of characteristics, the forward and the upwind methods.

**Project: Burgers' equation.** Consider the Burgers' equation with the initial function

$$f(x) = u(x, 0) = \begin{cases} 1, & \text{if } x \leq 0 \\ \frac{1}{4}(x+1)(x-2)^2, & \text{if } 0 \leq x \leq 2 \\ 0, & \text{if } x \geq 2. \end{cases}$$

For this problem, the characteristics first intersect at  $t_*$ , and the solution begins to develop a vertical tangent in its profile. From here on, we continue as a weak solution with a step discontinuity. We can calculate that at  $t_{**}$ , the weak solution becomes a step wave. Our numerical method can capture all these steps. The CFL condition:

$$\frac{c_{\max}}{\rho} \leq 1,$$

where

$$c_{\max} = \max |u(x, t)| = \max |f(x)| = 1.$$

Calculate  $t_*$ ,  $t_{**}$  and plot all the steps described above.

# Bibliography

- [1] Susanne C. Brenner, L. Ridgway Scott, *The Mathematical Theory of Finite Element Methods*, Springer, 2008.
- [2] Jeffery Cooper, *Introduction to Partial Differential Equations with MATLAB*, Birkhauser, 1998.
- [3] Lawrence C. Evans, *Partial Differential Equations*, American Math Society, 1998.
- [4] Claes Johnson, *Numerical Solution of Partial Differential Equations by the Finite Element Method*, Cambridge University Press, 1987.
- [5] Mats G. Larson, Fredrik Bengzon, *The Finite Element Method: Theory, Implementation and Practice*, Springer, 2010.
- [6] Ben Q. Li, *Discontinuous Finite Elements in Fluid Dynamics and Heat Transfer*, Springer-Verlag, 2006.
- [7] Endre Süli, *Lecture Notes on Finite Element Methods for Partial Differential Equations*, Mathematical Institute, University of Oxford, 2012.