Connectivity and components. Trees.

Graph theory

for MSc students in Computer Science

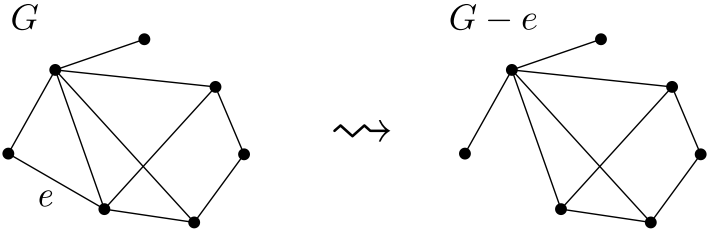
University of Szeged

Szeged, 2024.

Let G be a multigraph, an edge $e \in E(G)$ and a vertex $v \in V(G)$ of it.

$G - e$ denotes the graph obtained by removing the edge e from G (i.e. e is deleted from the edge set and the incidences are inherited from G).

$G - v$ denotes the graph obtained by removing the vertex v from G the multigraph obtained from G (i.e. v is deleted from the vertex set and the edges incident to v are deleted from the edge set, and the incidences are inherited from G).



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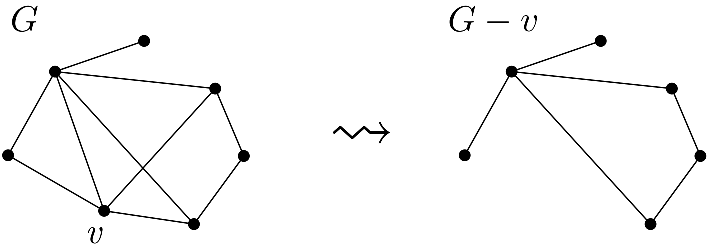
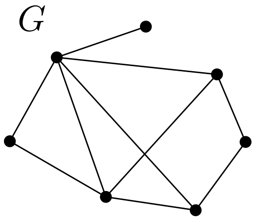


Def. The (multi)graph R is a sub(multi)graph of the (multi)graph G, if R can be obtained from G by removing some (or no) edges and vertices. If R is a submultigraph of G, then we also say that G contains R. Notation: $R \subseteq G$.



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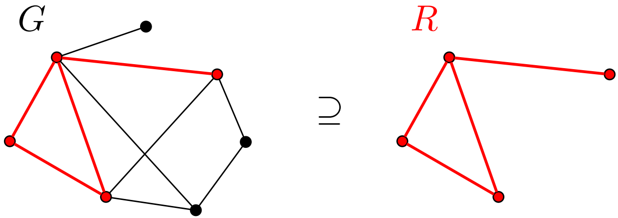

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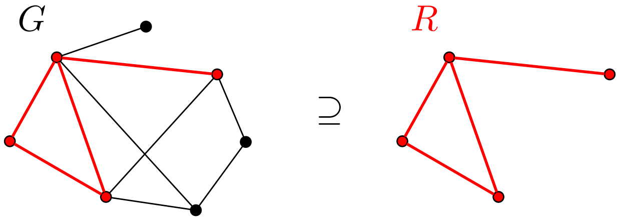

Remark. G is a subgraph of itself.

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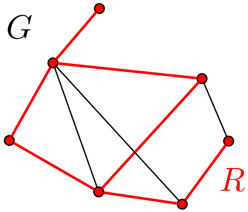
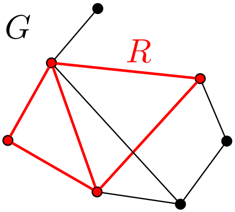


a spanning subgraph



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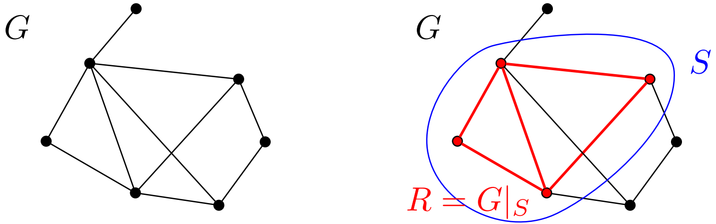
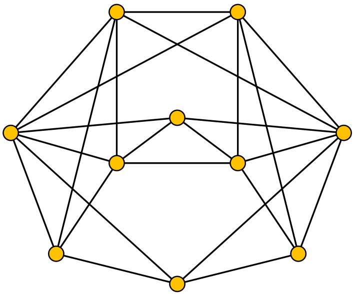


Definition. A walk $\mathcal{W}$ in a multigraph G is a sequence

$$\mathcal{W} : (v_0, e_1, v_1, e_2, v_2, e_3, v_3, \dots, v_{\ell-1}, e_\ell, v_\ell),$$

where $v_0, v_1, \ldots, v_\ell \in V(G)$, $e_1, \ldots, e_\ell \in E(G)$, and the two endvertices of e_i are v_{i-1} and v_i, for every $i = 1, \ldots, \ell$.
We say that ℓ is the length of the walk. ($\ell = 0$ is a possibility: '(v_0)' is a path of length 0.) A walk is closed iff $v_0 = v_\ell$, otherwise we call it non-closed.



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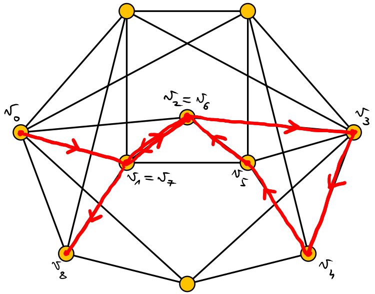
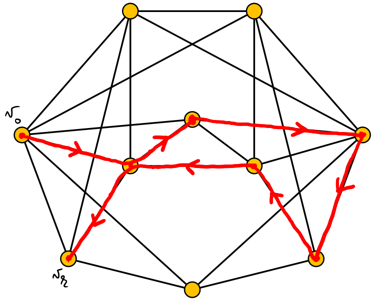


A non-closed walk of length 8

Definition. Let $\mathcal{W}$ be a walk $(v_0, e_1, v_1, e_2, v_2, e_3, v_3, \ldots, v_{\ell-1}, e_\ell, v_\ell)$ in G.

• The walk $\mathcal{W}$ is called trail iff it has no repeated edges (i.e. all e_i's are different).



A trail

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- The walk $\mathcal{W}$ is called trail iff it has no repeated edges (i.e. all e_i's are different).
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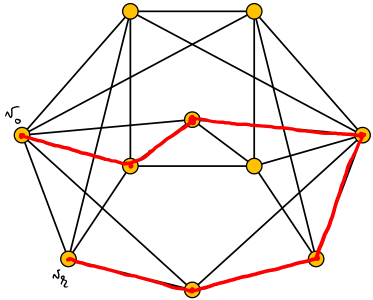

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Proposition. $\mathcal{W}$ is a path. $\Longrightarrow$ $\mathcal{W}$ is a trail. $\Longrightarrow$ $\mathcal{W}$ is a walk.

(However, the reversed implications are false. See the former figures.)

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Convention. On an xy-walk (or xy-path) we mean a walk (or path) with initial vertex x and end vertex y.

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Convention. On an xy-walk (or xy-path) we mean a walk (or path) with initial vertex x and end vertex y.

Proposition. In a multigraph G, for any two vertices x and y, there exists an xy-path in G if and only if there exists an xy-walk in G.

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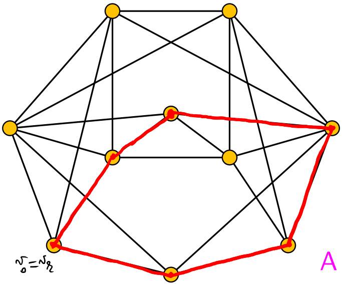
• The walk $\mathcal{W}$ is called trail iff it has no repeated edges (i.e. all e_i's are different).

• The walk $\mathcal{W}$ is called path iff it has no repeated vertices (i.e. all v_i's are different).

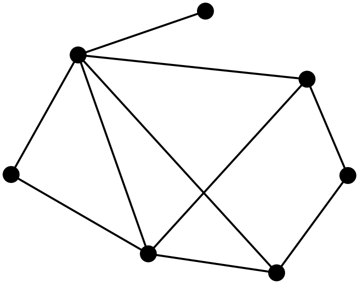
• The walk $\mathcal{W}$ is called cycle iff $\ell > 0$, and $v_0, v_1, \dots, v_{\ell-1}$ are different vertices, but $v_\ell = v_0$, furthermore in the case $\ell = 2$ we have $e_1 \neq e_2$.



A cycle

Definition. A multigraph G is connected, if for any two vertices $x, y \in V(G)$, there exists an xy-walk (or equivalently, an xy-path) in G. A multigraph that is not connected is called disconnected.

A connected graph.



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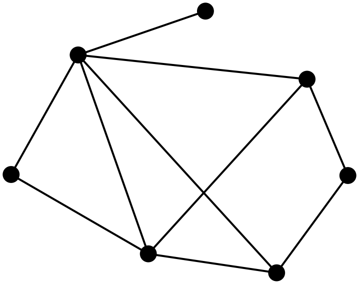

It is enough to check that any vertex can be reached from an arbitrary fixed vertex v by following some sequence of edges. (Why?)

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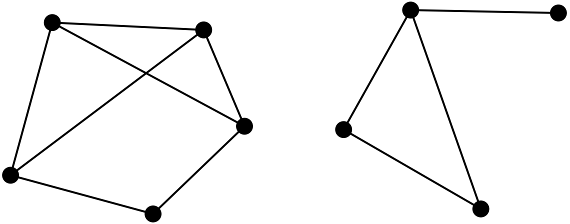

(This graph has 9 vertices.)

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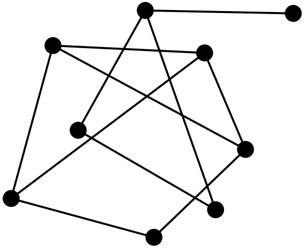
A disconnected graph #2.



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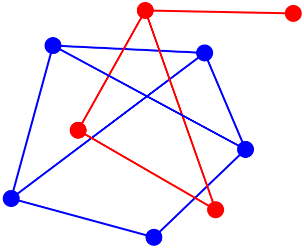

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The following theorem gives the structure of disconnected multigraphs:

Theorem. Every multigraph G is a vertex-disjoint union of connected multigraphs $G_1 \ldots, G_k$; and this decomposition is unique. (That is, $G_1, \ldots, G_k$ are connected induced submultigraphs of G such that there is no edge in G between G_i and G_j for $i \neq j$.)

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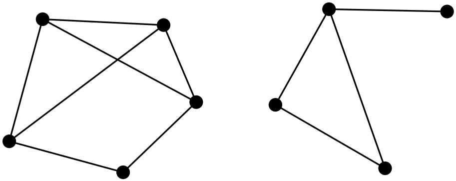

A disconnected graph with 2 components

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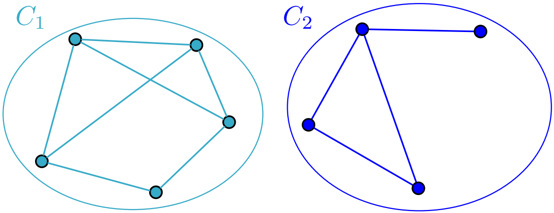

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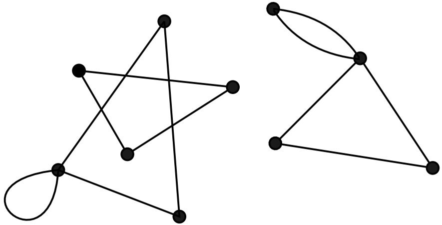

A disconnected multigraph with 3 components

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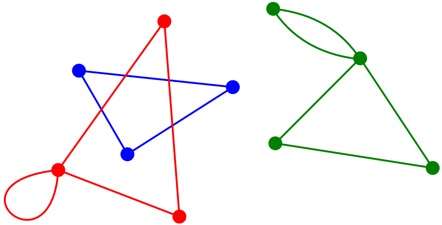

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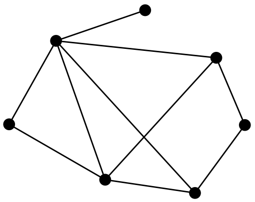

A connected graph (with 1 components)

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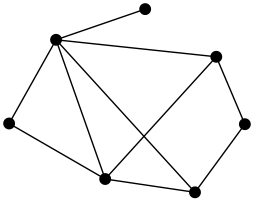
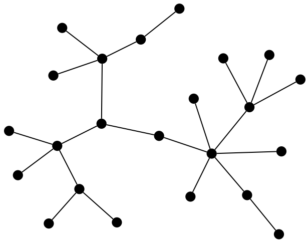


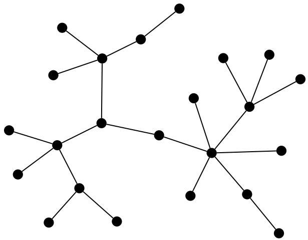
A connected graph (with 1 components)

Remark. We note that a multigraph is disconnected, if and only if it has more than one components.

Def. A multigraph is a tree, if it is connected and it does not contain a cycle.



Def. A multigraph is a tree, if it is connected and it does not contain a cycle.



Remark. Every tree is a simple graph by definion. (A loop would form a cycle of length 1, two parallel edges would form a cycle of length 2.)

Remark. Graphs without cycles are called acyclic graphs.

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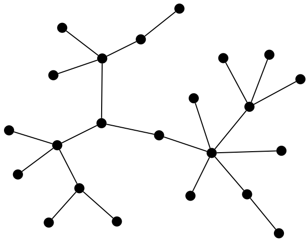
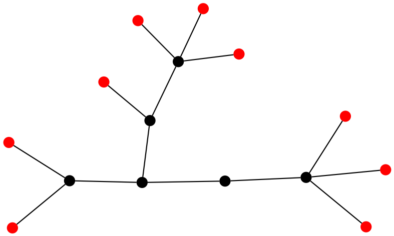


Thm. For any graph G, the following statements are equivalent.

a) G is a tree.
b) G is connected, but the removal of any edge would disconnect it (i.e. $G - e$ is disconnected for all $e \in E(G)$).
c) or any two vertices $x, y \in V(G)$, there exists exactly one xy-path in G.

Def. In a tree T, the vertices with degree 1 are called the leaves of T.



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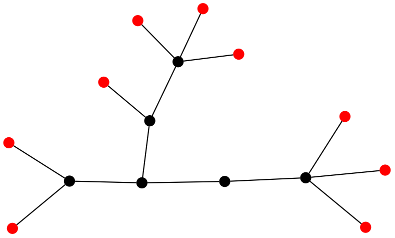

Claim. Every tree with at least two vertices has a leaf. (In fact, every such tree has at least two leaves.)

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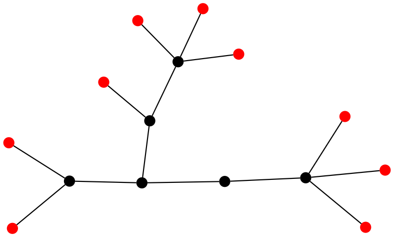

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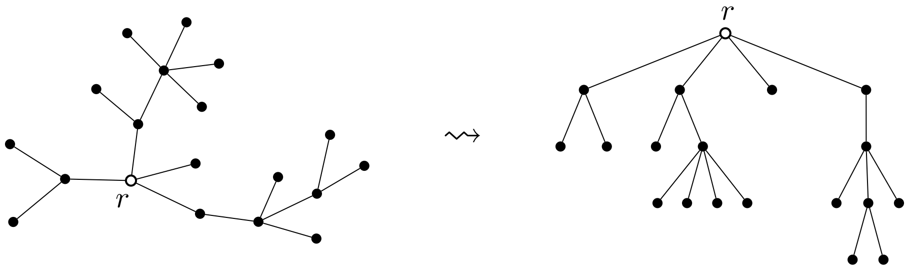
Thm. A tree on n vertices has exactly $n-1$ edges.

Every tree T can be drawn like a family tree, as illustrated in the figure. (An arbitrary vertex $r \in V(T)$ can be designated as the root of the family tree.)



In the drawing, the vertices of T are arranged in levels, such that

(i) there is exactly one vertex on the top level, the (root) vertex r;

(ii) every edge of T connects two vertices on adjacent levels;

(iii) for any non-root vertex u, there is exactly one edge in T that connects u to a vertex on the level just above the level of u.

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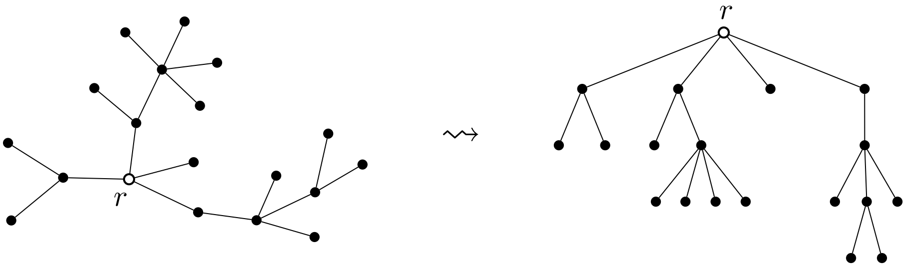

Remark. After designating a root vertex r in a tree T, this drawing of is unique in the following sense: The level of any vertex $v \in V(T)$ is uniquely determined. If the length of the unique rv-path in T is ℓ, then v belongs to the $(\ell+1)^{\text{th}}$ level (from top to bottom). We refer to this drawing as rooted tree T (with root r).

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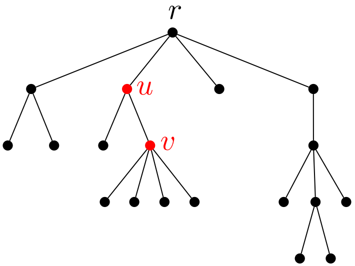
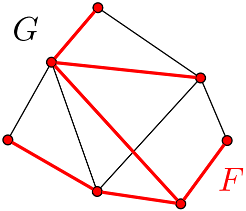


Def. If u and v are adjacent vertices in a rooted tree T, and the level of v is lower than the level of u, then u is called the parent of v, and v is called the child of u. (And other related names are used, like siblings, etc.)

Def. A spanning tree of a multigraph G is spanning subgraph (that contains all vertices of G) which is a tree.



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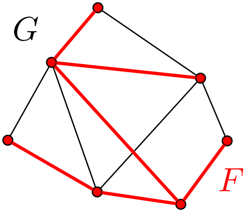

Thm. Every connected (multi)graph has a spanning tree.

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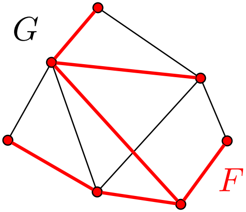

Thm. Every connected (multi)graph has a spanning tree.

Corollary. Every connected graph on n vertices has at least $n-1$ edges.