1. Basics. Realization of degree sequences

1. Does there exist a multigraph with degrees
a) $9, 7, 6, 6, 5, 4, 3, 3, 3, 1$
b) $8, 7, 6, 6, 5, 4, 3, 3, 3, 1$?

2. Does there exist a loopless multigraph with degrees
a) $65, 32, 16, 8, 4, 2, 1$
b) $64, 32, 16, 8, 4, 2, 1, 1$?

3. (Havel–Hakimi algorithm.) Does there exist a simple graph with degrees
a) $7, 4, 3, 3, 3, 3, 2, 1, 0$
b) $8, 8, 6, 6, 6, 5, 3, 2, 2$
c) $7, 6, 5, 5, 5, 4, 4, 2$
d) $5, 4, 4, 2, 2, 1$?

4. How many simple graphs have degree sequence $7, 7, 7, 7, 5, 5, 4, 4$ (up to isomorphism)?

5. Does there exist a *connected* graph with degree sequence $4, 4, 3, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1$?

6. After a party, every participant tells us how many people (of opposite sex) he or she danced with. We get the following numbers: $9, 9, 9, 9, 6, 6, 6, 5, 3, 3, 3, 3, 3, 3, 3$. Prove that at least one of them is wrong.

7. Prove that there exists a k-regular graph with n vertices exists if and only if kn is even and $k \leq n - 1$.

8. In a chess competition with 10 participants 11 games have been already played. Prove that there exists a participant who played at least 3 games.

9. In a group of 9 people, every person gives 100$ to exactly 5 other people. Prove that there exist two people whose money changed by the same amount.

10. In a chess competition involving 10 girls and 20 boys, every girl played exactly 6 games. We know that there were exactly 34 games in which a boy and a girl played against each other. What is the number of "girl vs. girl" games?

11. The **Erdős–Gallai theorem** says that the sequence $d_1 \geq d_2 \geq \cdots \geq d_n$ of nonnegative integers can be realized by a simple graph if and only if

$$d_1 + \cdots + d_n \text{ is even, and} \tag{1}$$

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k), \quad \text{for all } k \in \{1, \ldots, n\}. \tag{2}$$

Prove that the conditions are necessary.

12. Let $n \geq 2$. Prove that the sequence $d_1, d_2, \ldots, d_n$ of nonnegative integers can be realized by a *tree* if and only if $\sum_{i=1}^{n} d_i = 2(n-1)$ and $d_i > 0$ for all i.

13.$^+$ Let $n \geq 2$, and assume that the sequence $d_1, \ldots, d_n$ can be realized by a simple graph. Prove that the sequence $d_1, \ldots, d_n$ can be realized by a *connected* simple graph if and only if $\sum_{i=1}^{n} d_i \geq 2(n-1)$, and the numbers $d_1, \ldots, d_n$ are all positive.