3. Realization of degree sequences. Trees.

1. Does there exist a multigraph with degrees
a) $99, 7, 6, 6, 5, 4, 3, 3, 3, 1$
b) $88, 7, 6, 6, 5, 4, 3, 3, 3, 1$?

2. Does there exist a loopless multigraph with degrees
a) $65, 32, 16, 8, 4, 2, 1$
a) $64, 32, 16, 8, 4, 2, 1, 1$?

3. **(Havel–Hakimi algorithm.)** Does there exist a simple graph with degrees
a) $7, 4, 3, 3, 3, 3, 2, 1, 0$
b) $8, 8, 6, 6, 6, 5, 3, 2, 2$
c) $7, 6, 5, 5, 5, 4, 4, 2$
d) $5, 4, 4, 2, 2, 1$?

4. Does there exist a *connected* graph with degree sequence $4, 4, 3, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1$?

5. The **Erdős–Gallai theorem** says that the sequence $d_1 \geq d_2 \geq \cdots \geq d_n$ of nonnegative integers can be realized by a simple graph if and only if

$$d_1 + \cdots + d_n \text{ is even, and} \tag{1}$$

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k), \quad \text{for all } k \in \{1, \ldots, n\}. \tag{2}$$

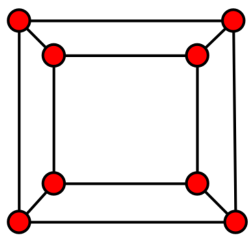
Prove that the conditions are necessary.

6. Let $n \geq 2$. Prove that the sequence $d_1, d_2, \ldots, d_n$ of nonnegative integers can be realized by a *tree* if and only if $\sum_{i=1}^{n} d_i = 2(n-1)$ and $d_i > 0$ for all i.

7.$^+$ Let $n \geq 2$, and assume that the sequence $d_1, \ldots, d_n$ can be realized by a simple graph. Prove that the sequence $d_1, \ldots, d_n$ can be realized by a *connected* simple graph if and only if $\sum_{i=1}^{n} d_i \geq 2(n-1)$, and the numbers $d_1, \ldots, d_n$ are all positive.

8. What happens if we add a new edge to a tree?

9. Is it possible to find two edge-disjoint spanning trees in the cube graph?



10. Show that in a connected graph G with at least 2 vertices, there always exists a vertex whose removal does not disconnect G.