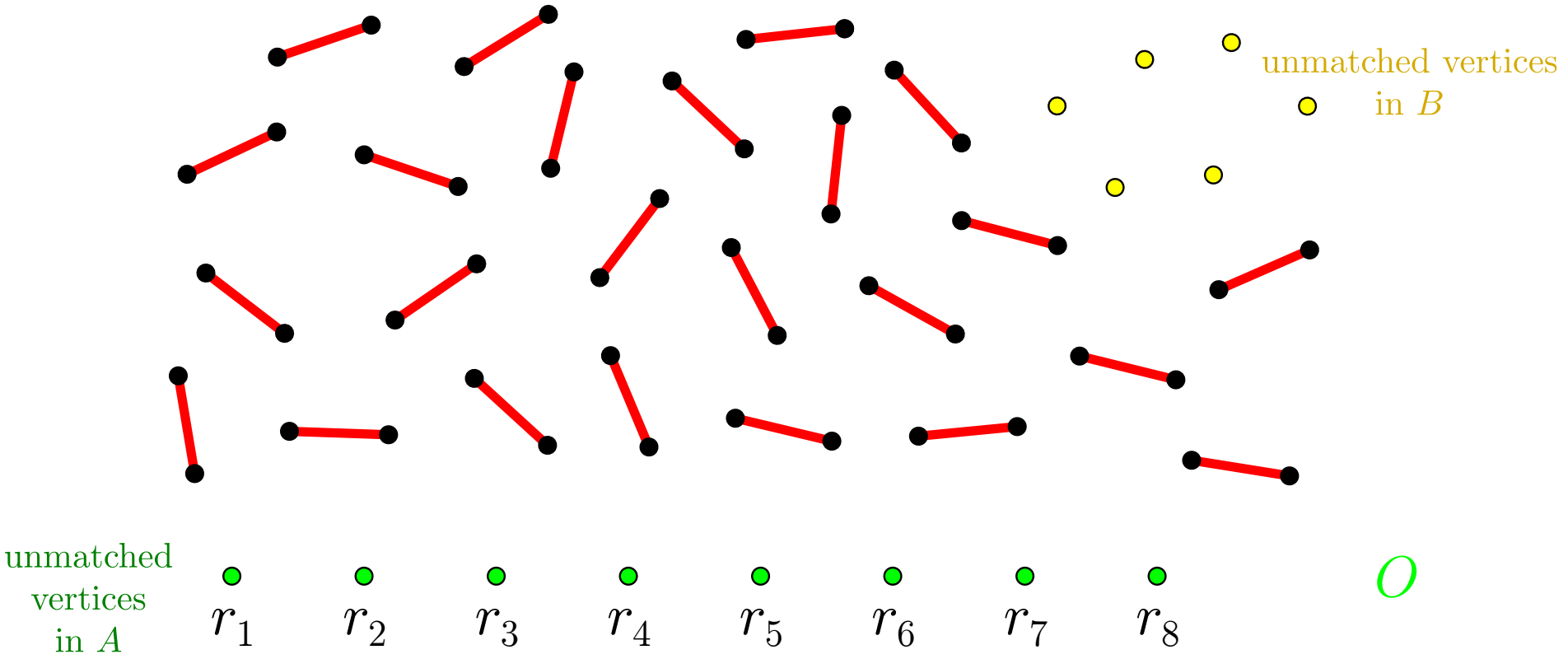
Matching algorithms
(Hungarian method + blossom algorithm)

Graph theory

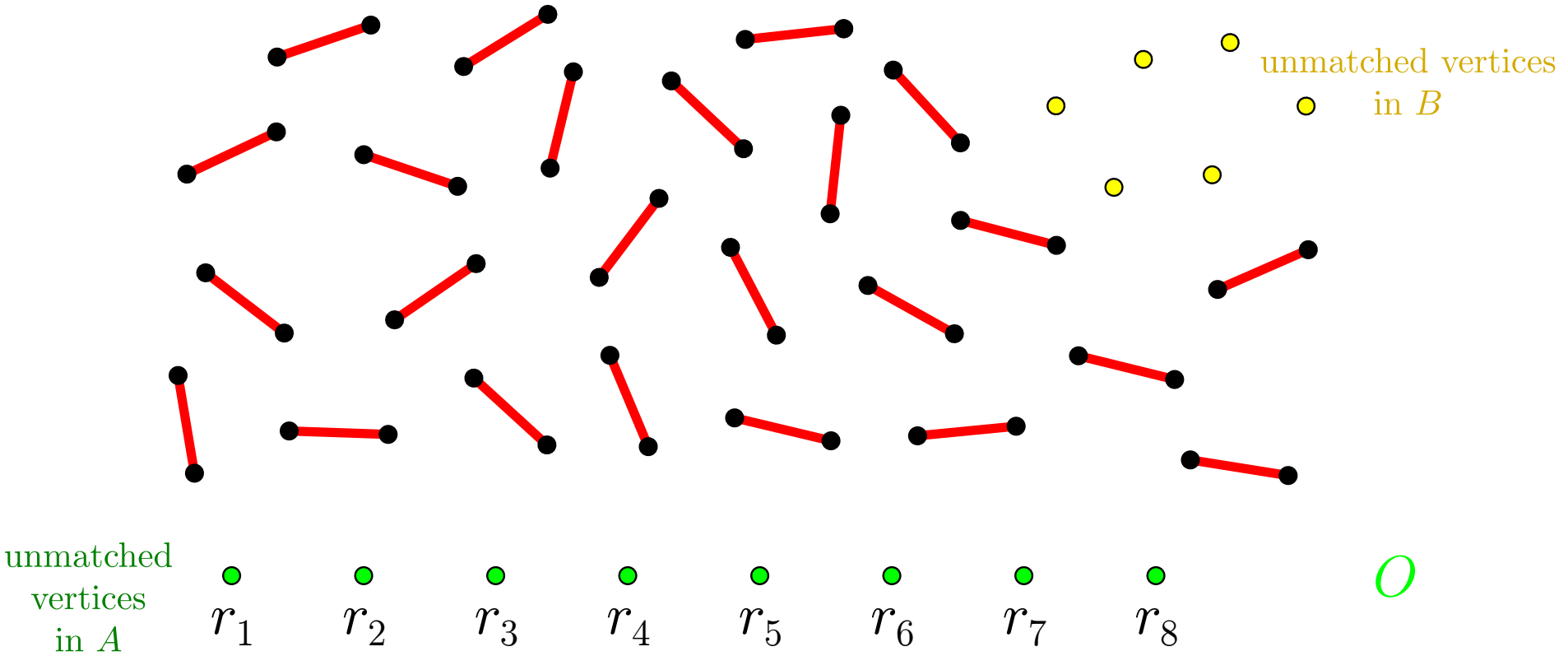
University of Szeged

Szeged, 2018.



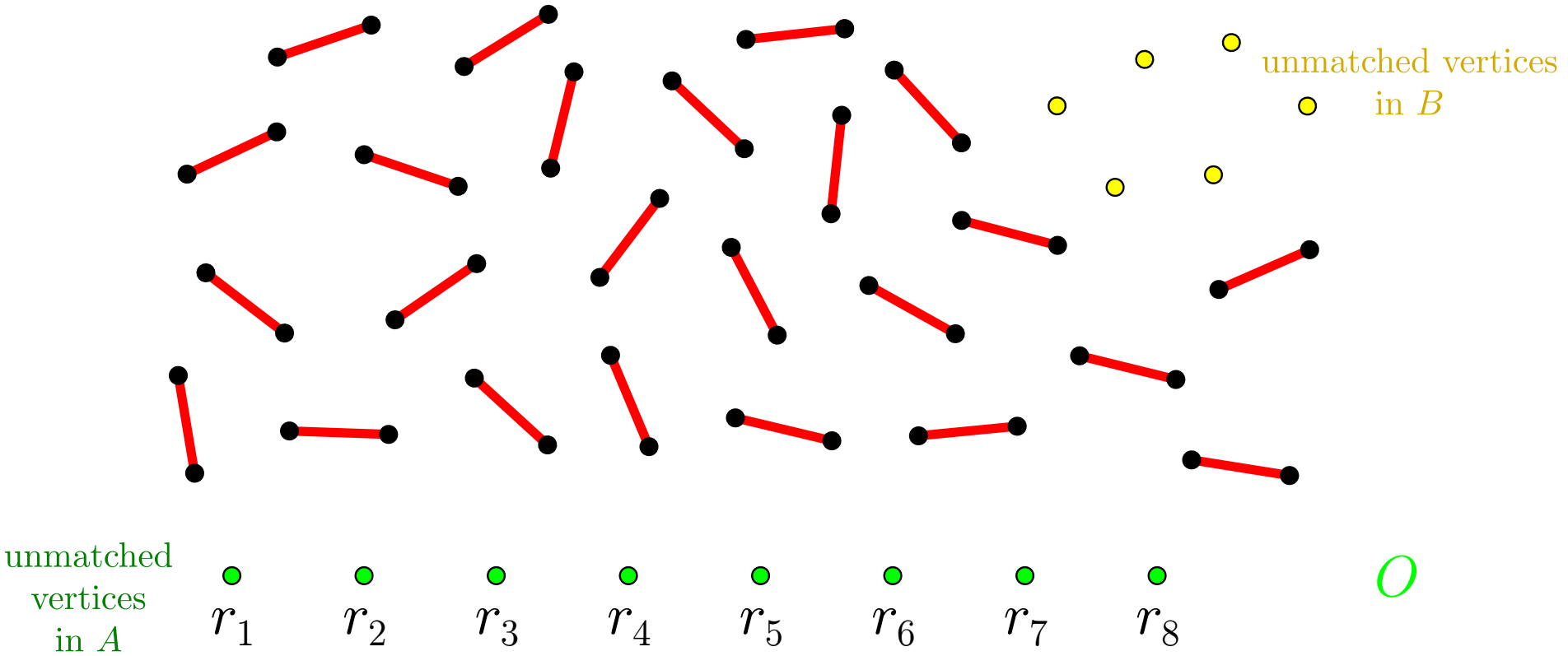
HUNGARIAN METHOD

INPUT: A bipartite graph G and a non-perfect matching M in G.
OUTPUT: An augmenting path for M, or „M is of maximum size".



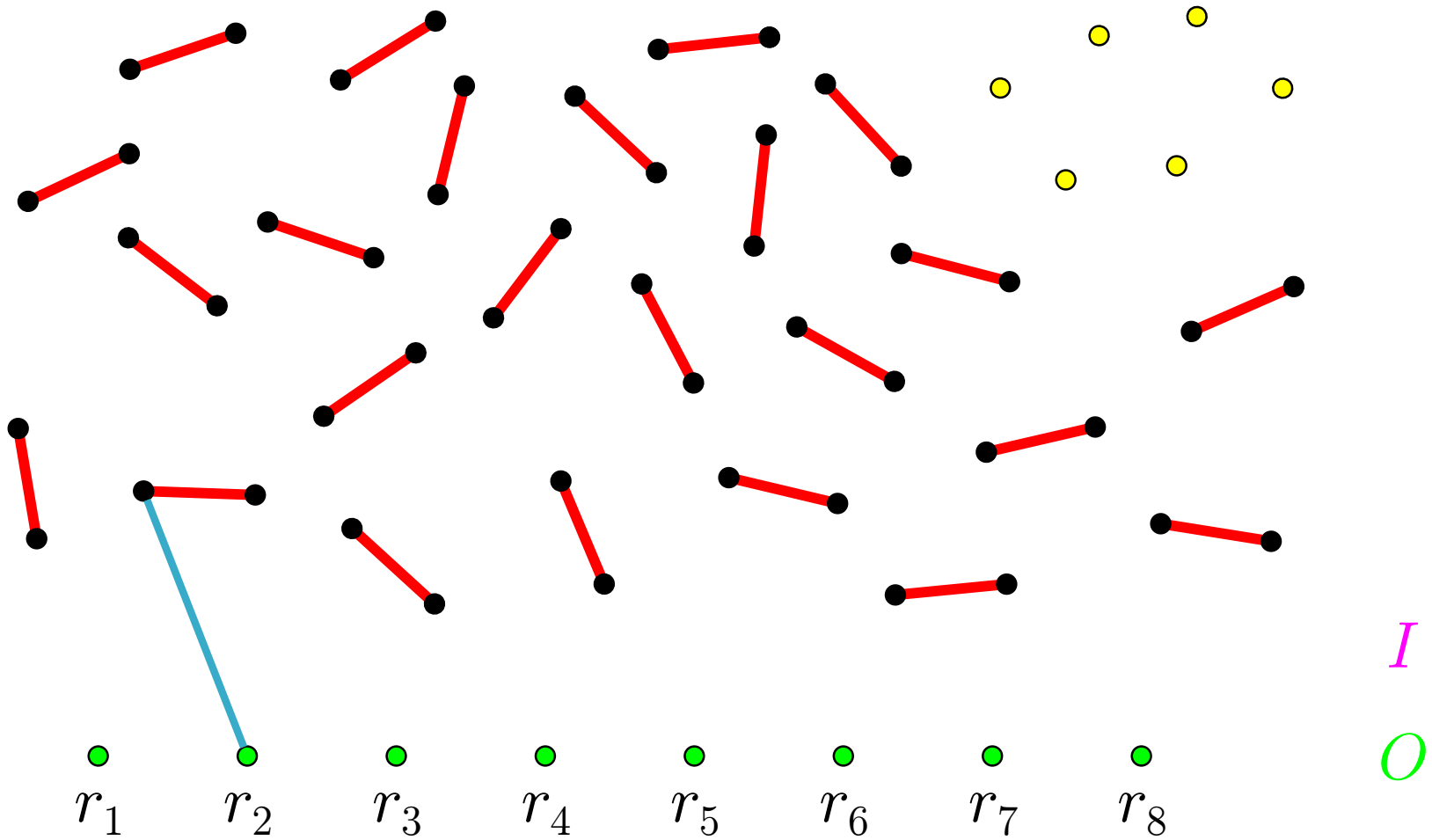
HUNGARIAN METHOD

INPUT: A bipartite graph G and a non-perfect matching M in G. Sketch of the algorithm: Starting from the unmatched points of A (as roots), build a forest from the alternating paths of G. (Color classes of G: A and B.) The forest is constructed by a „greedy method", which will be discussed on the next slide.



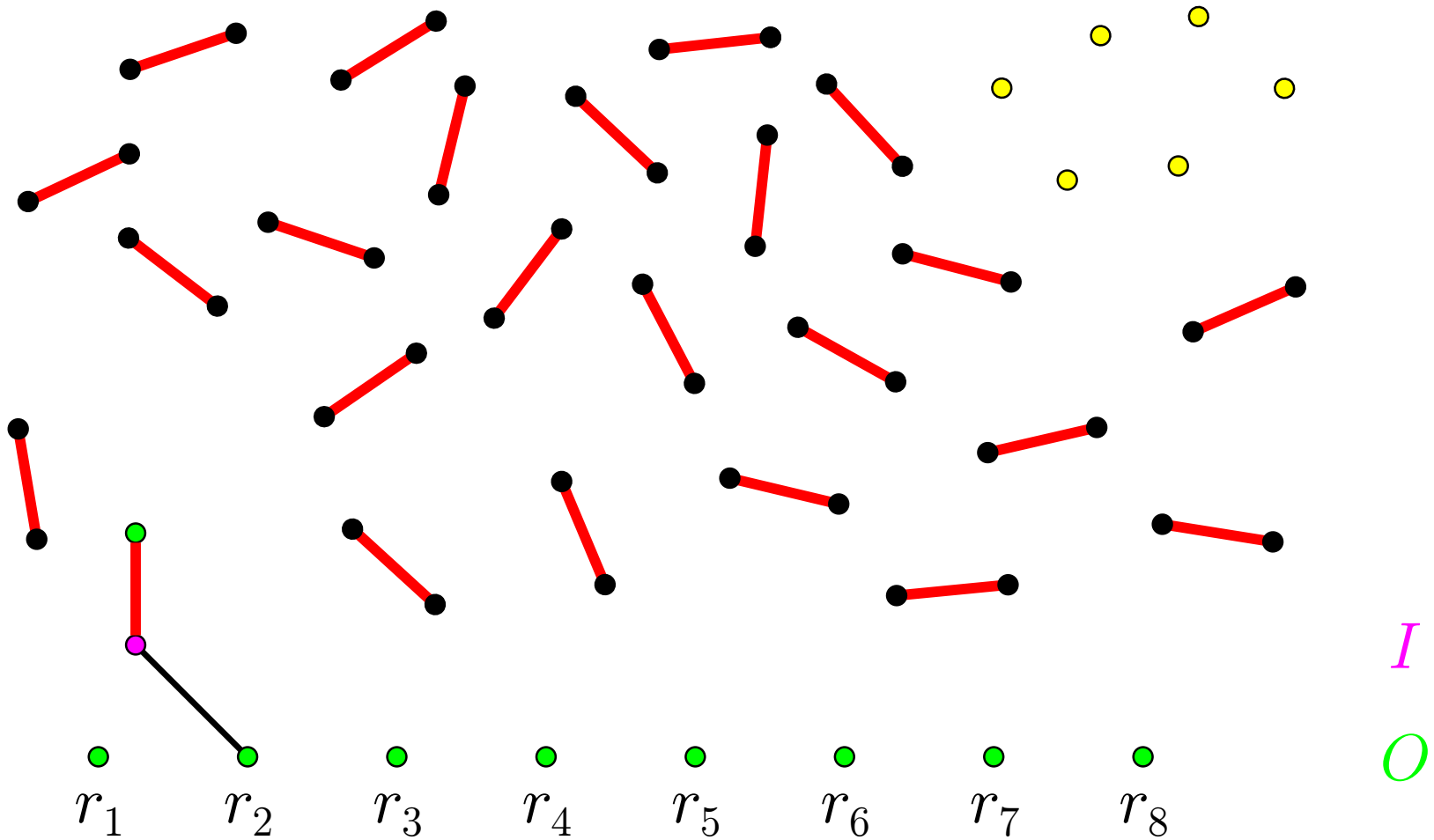
Sketch of the algorithm: Starting from the unmatched points of A (as roots), build a forest from the alternating paths of G. (Color classes of G: A and B.) The forest is constructed by a „greedy method", which will be discussed on the next slide.

We define two sets of vertices, the sets of inner and outer vertices, I and O. Initially, $O := \{r_1, \ldots, r_8\}$ and $I := \emptyset$.



Greedy expansion step: If some outer vertex u is adjacent to some matched vertex v outside the forest, then add v to the forest, together with its pair v' in M (and edges uv and vv').

v is designated as inner vertex, v' is designated as outer vertex.



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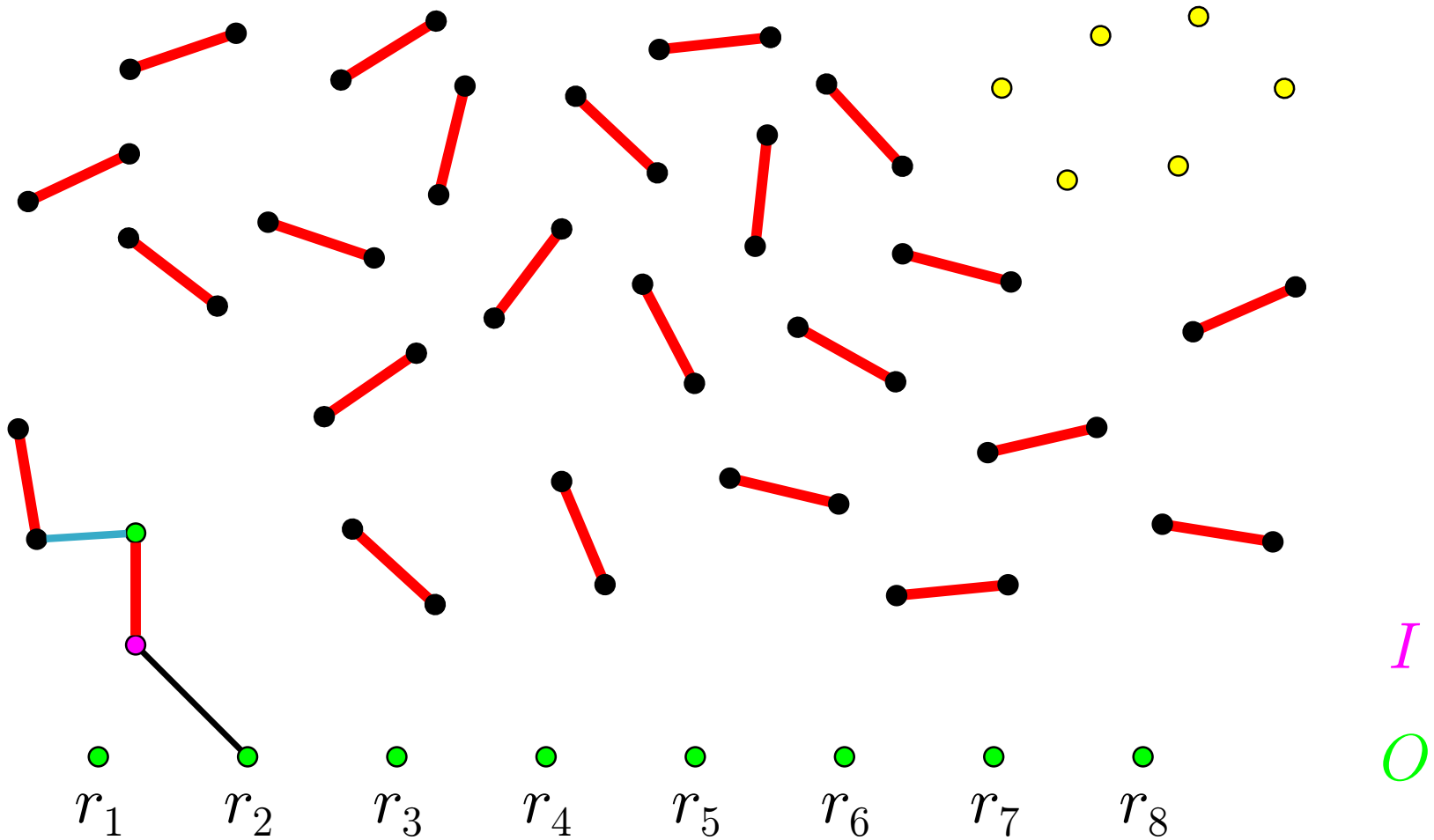

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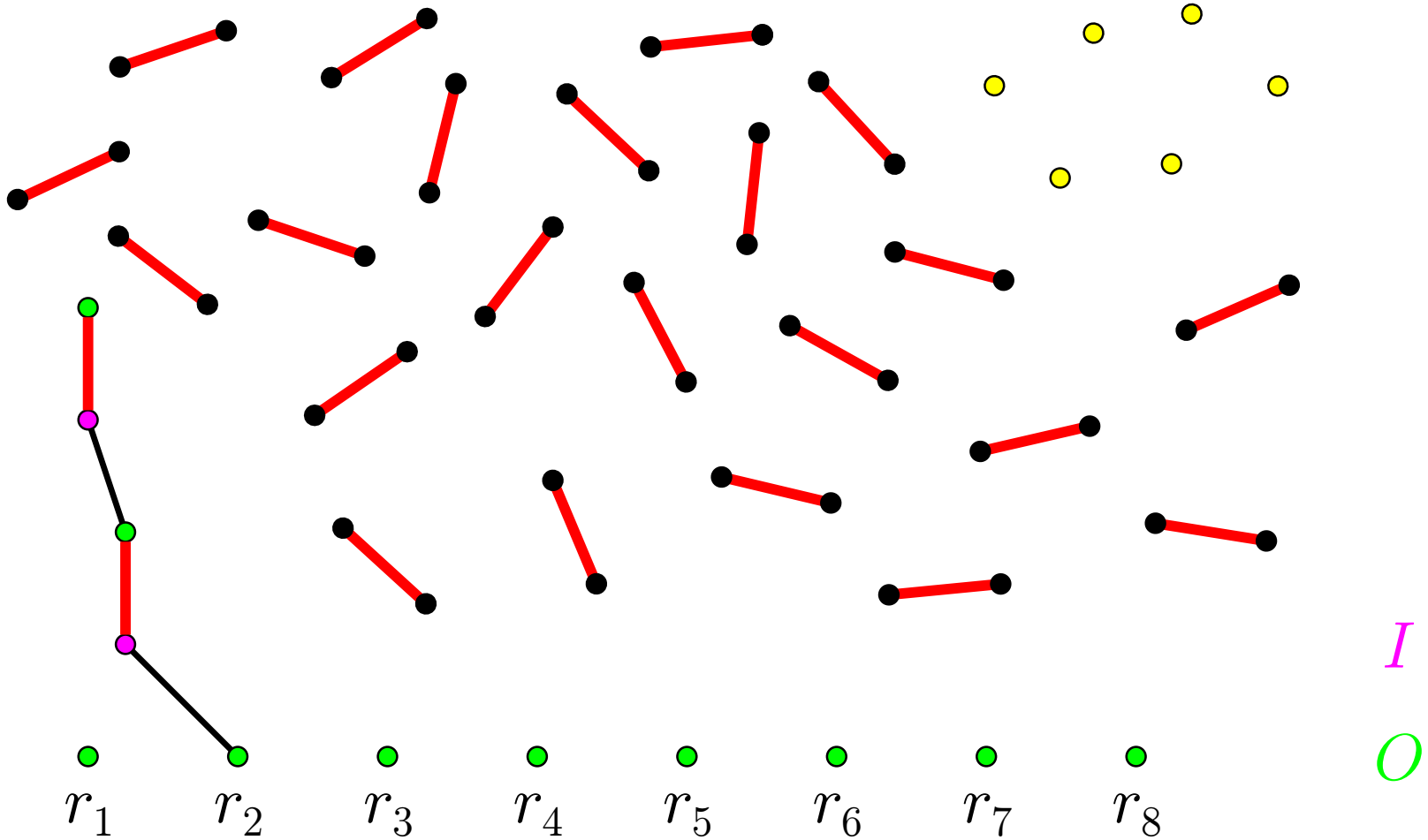

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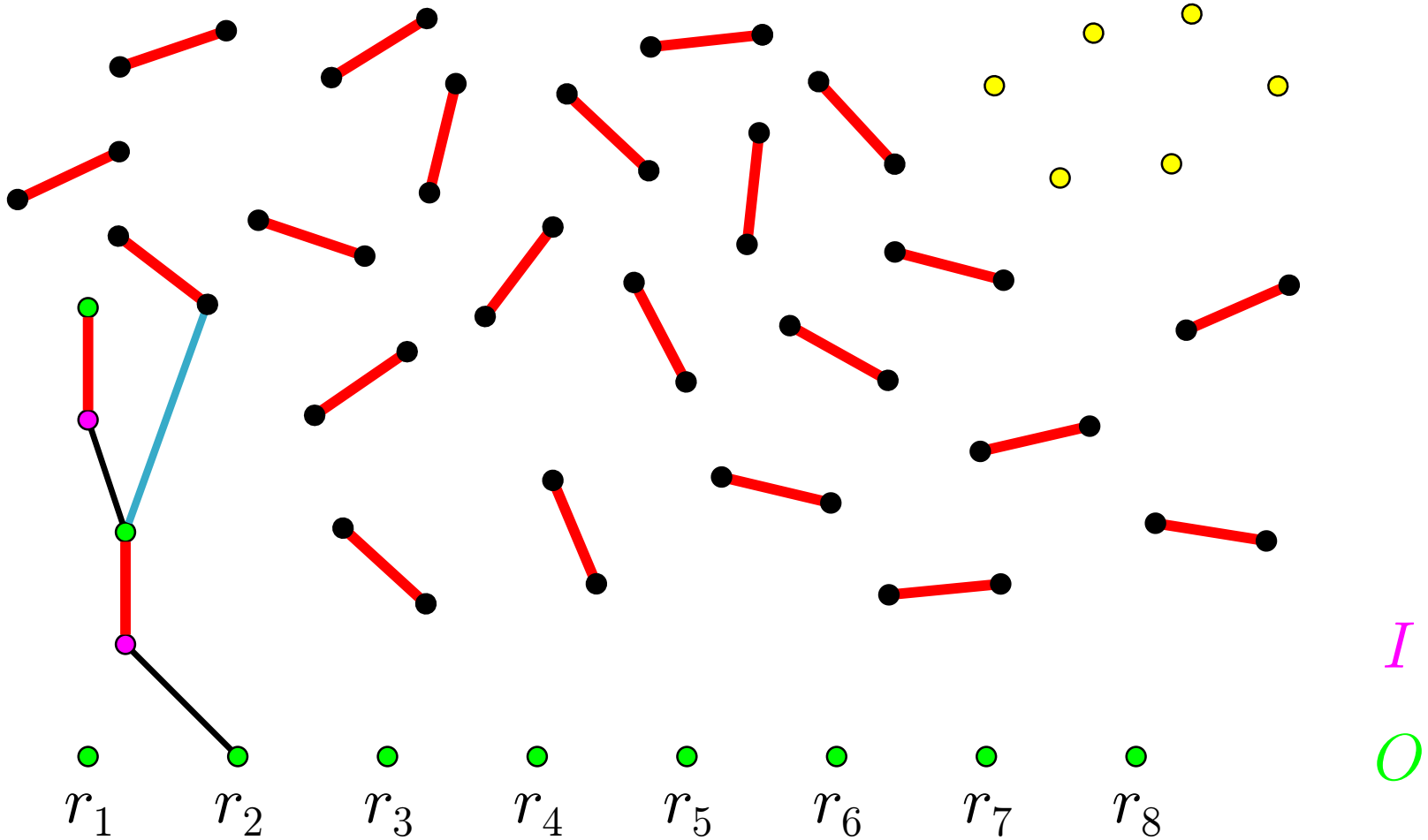

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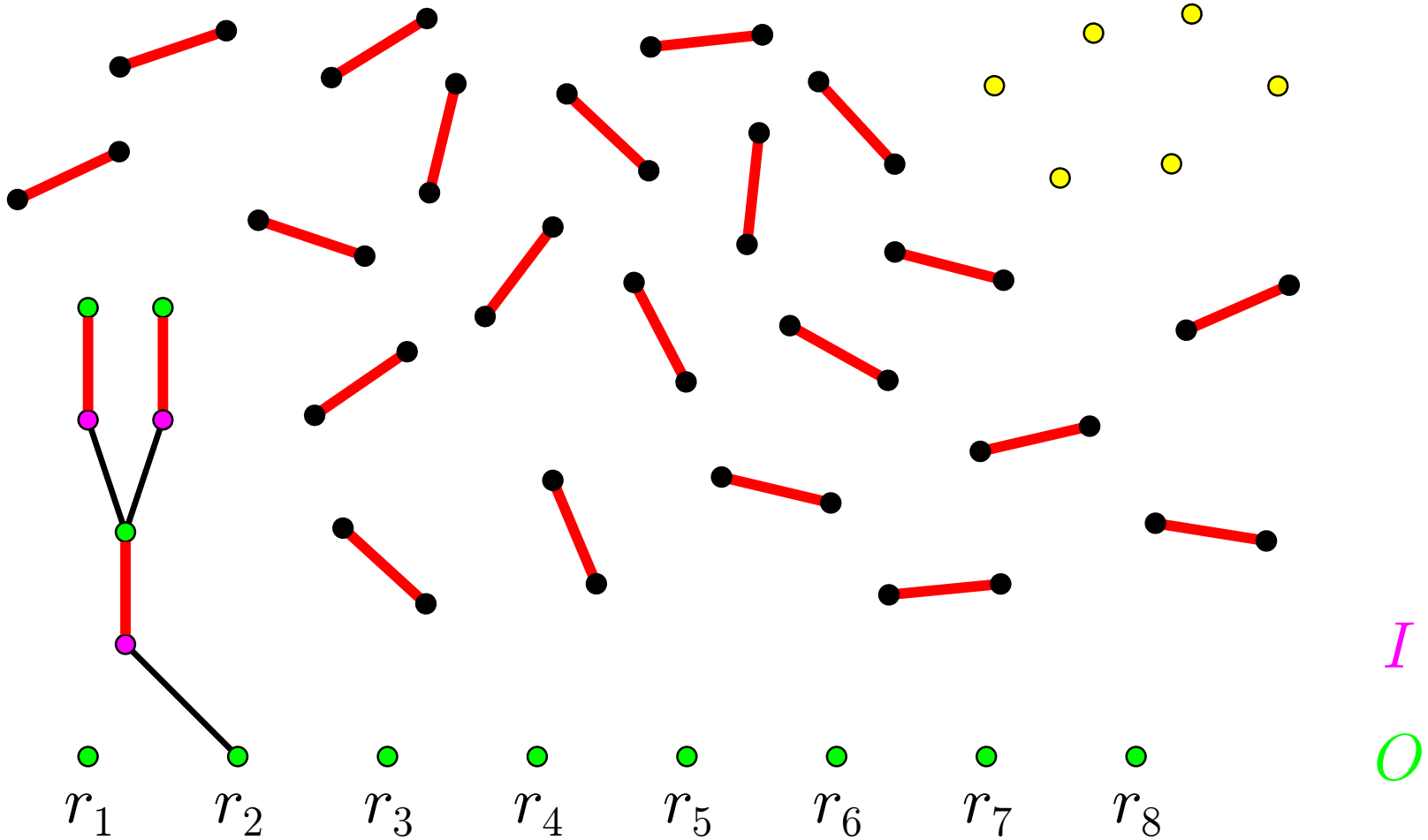

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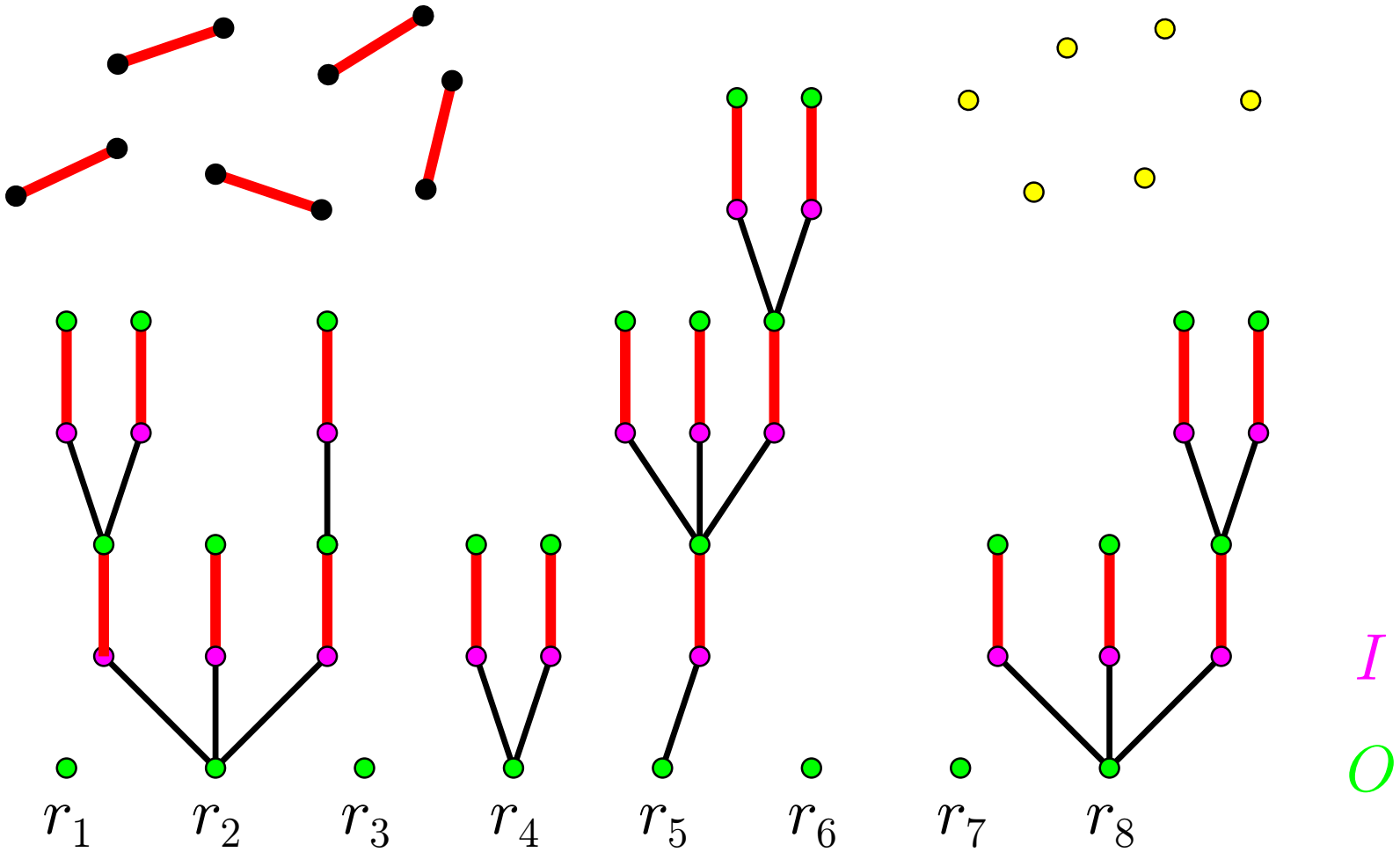
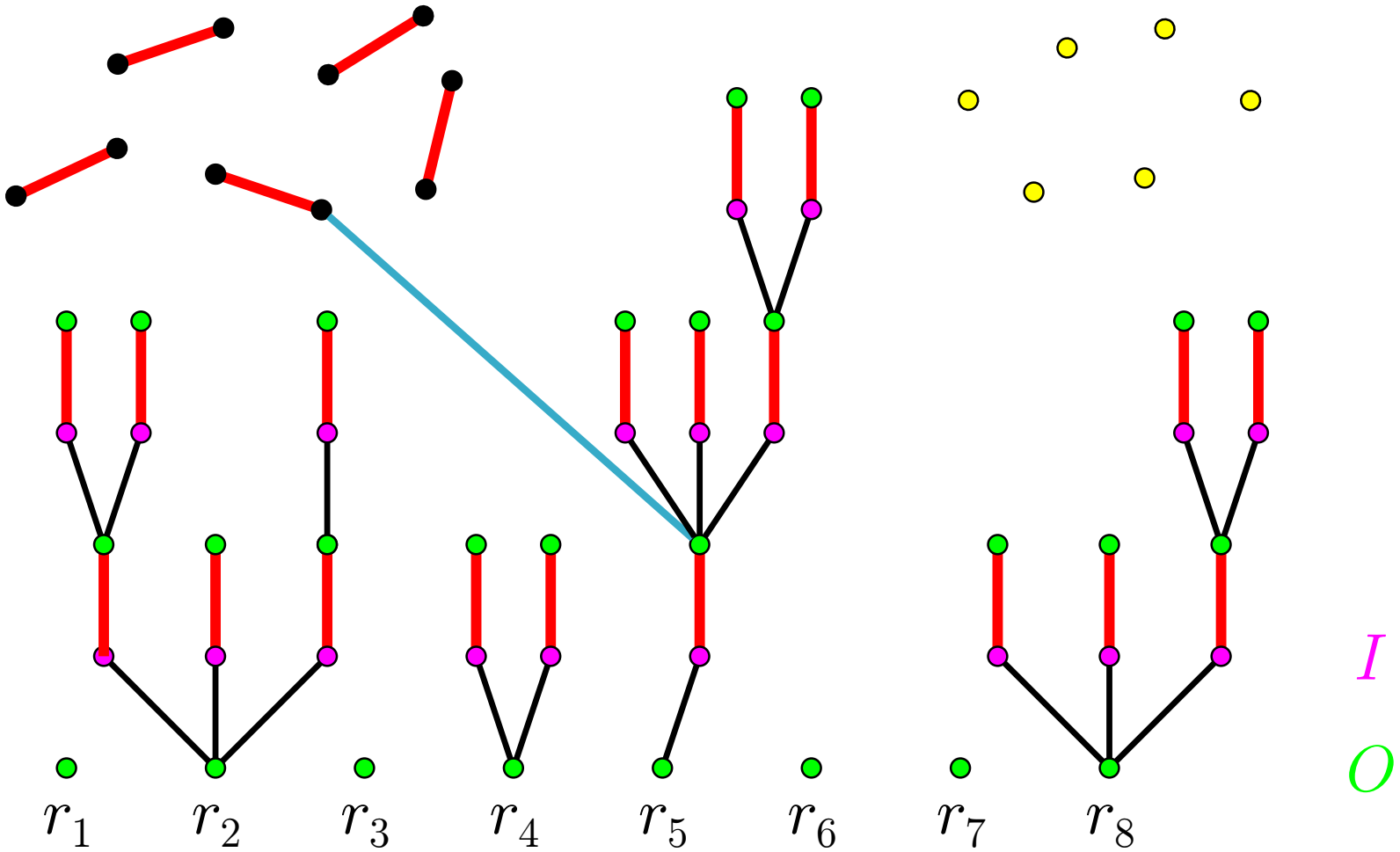


Greedy expansion step: If some outer vertex u is adjacent to some matched vertex v outside the forest, then add v to the forest, together with its pair v' in M (and edges uv and vv').

Remark. It is always true that $O \subseteq A$ and $I \subseteq B$.



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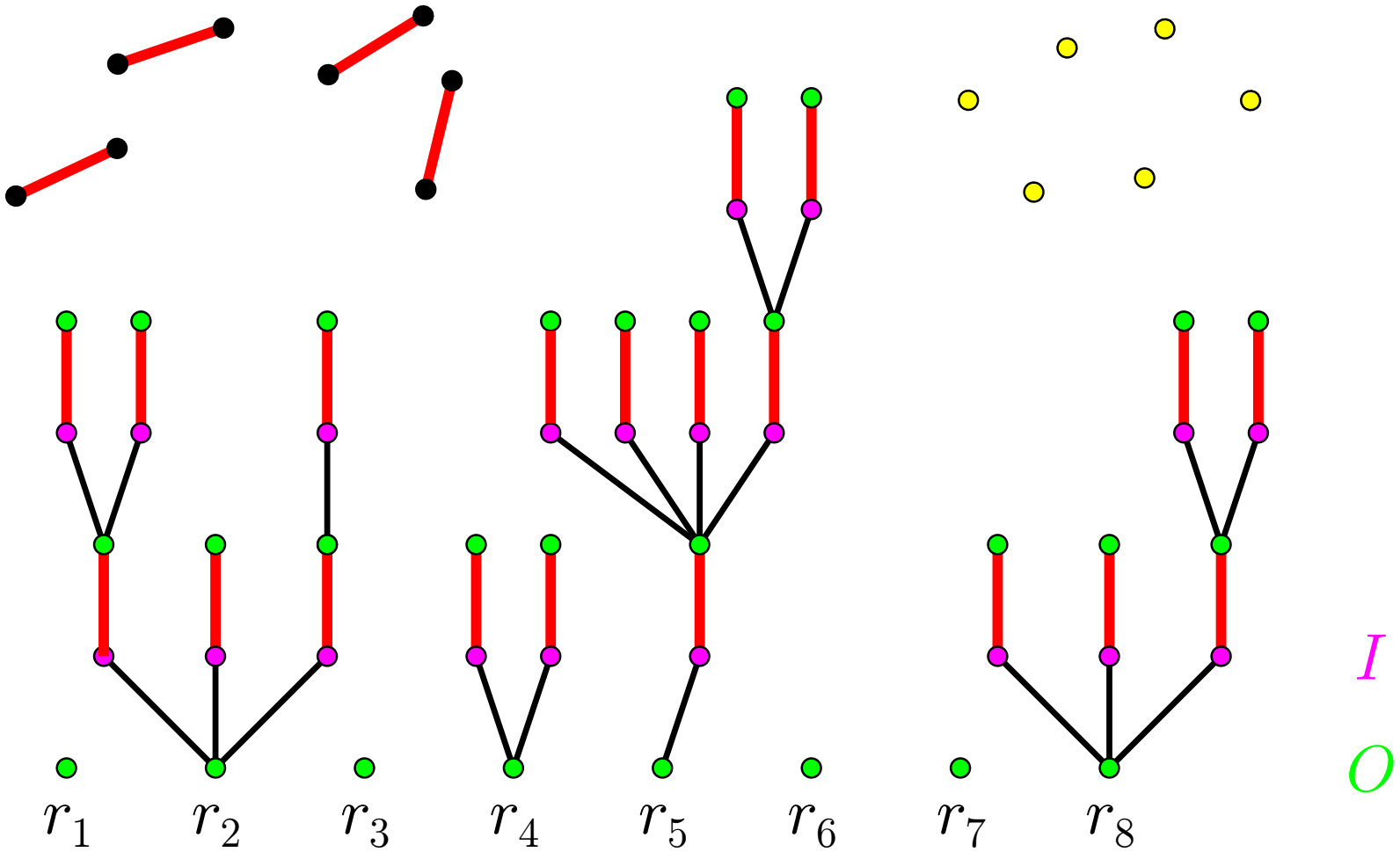

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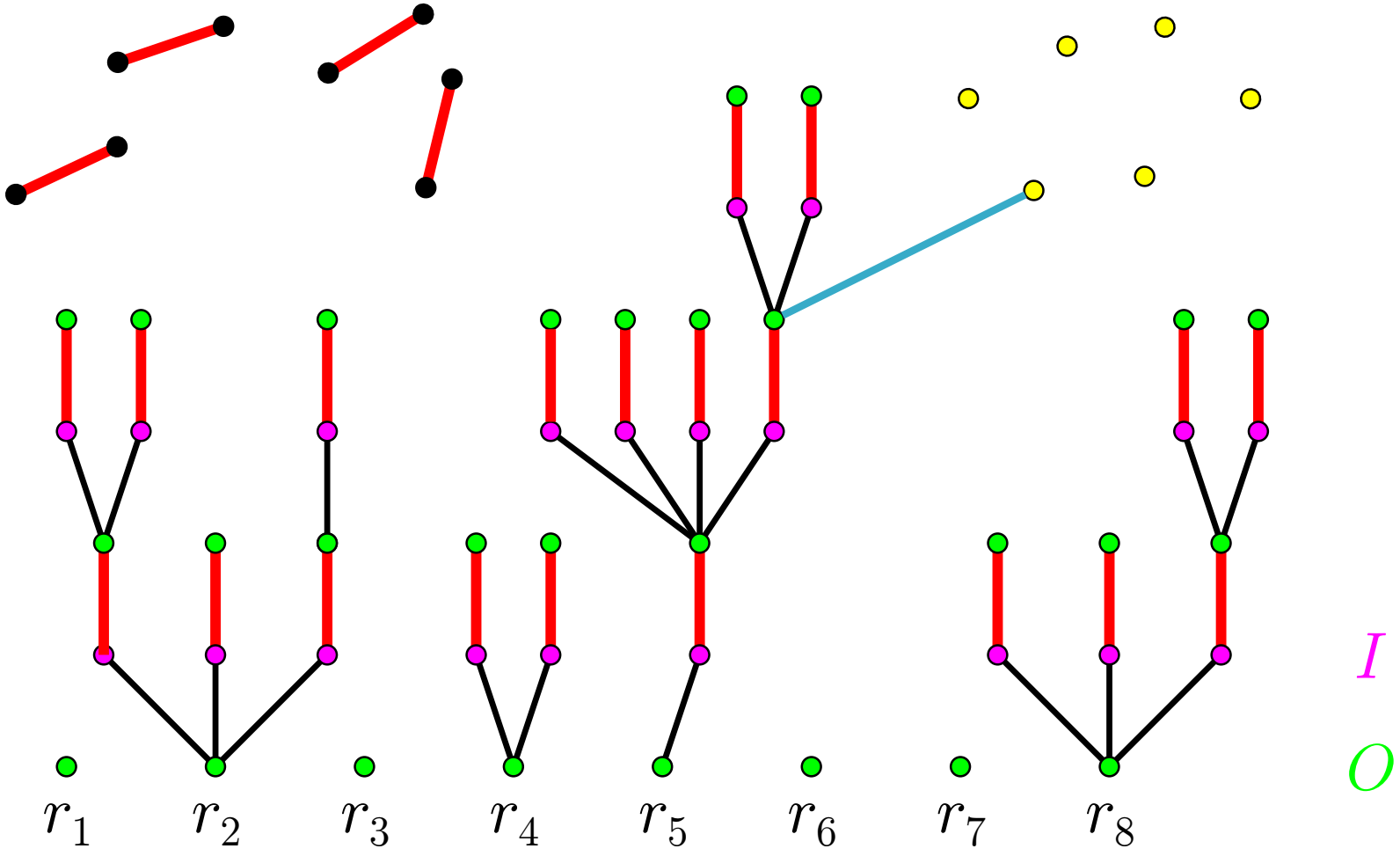
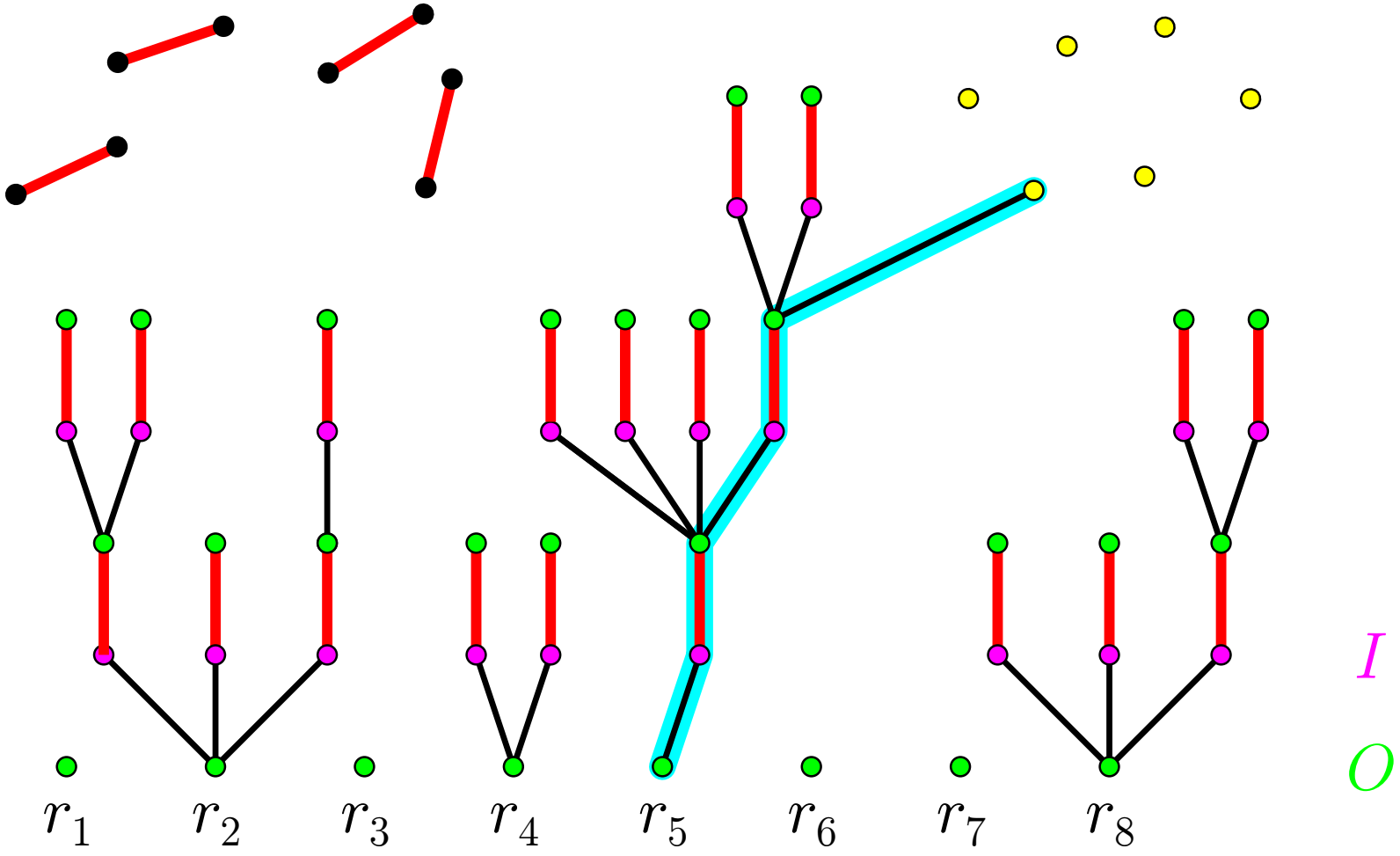


Successful search: If some outer vertex is adjacent to some unmatched vertex, then we found an augmenting path. STOP.



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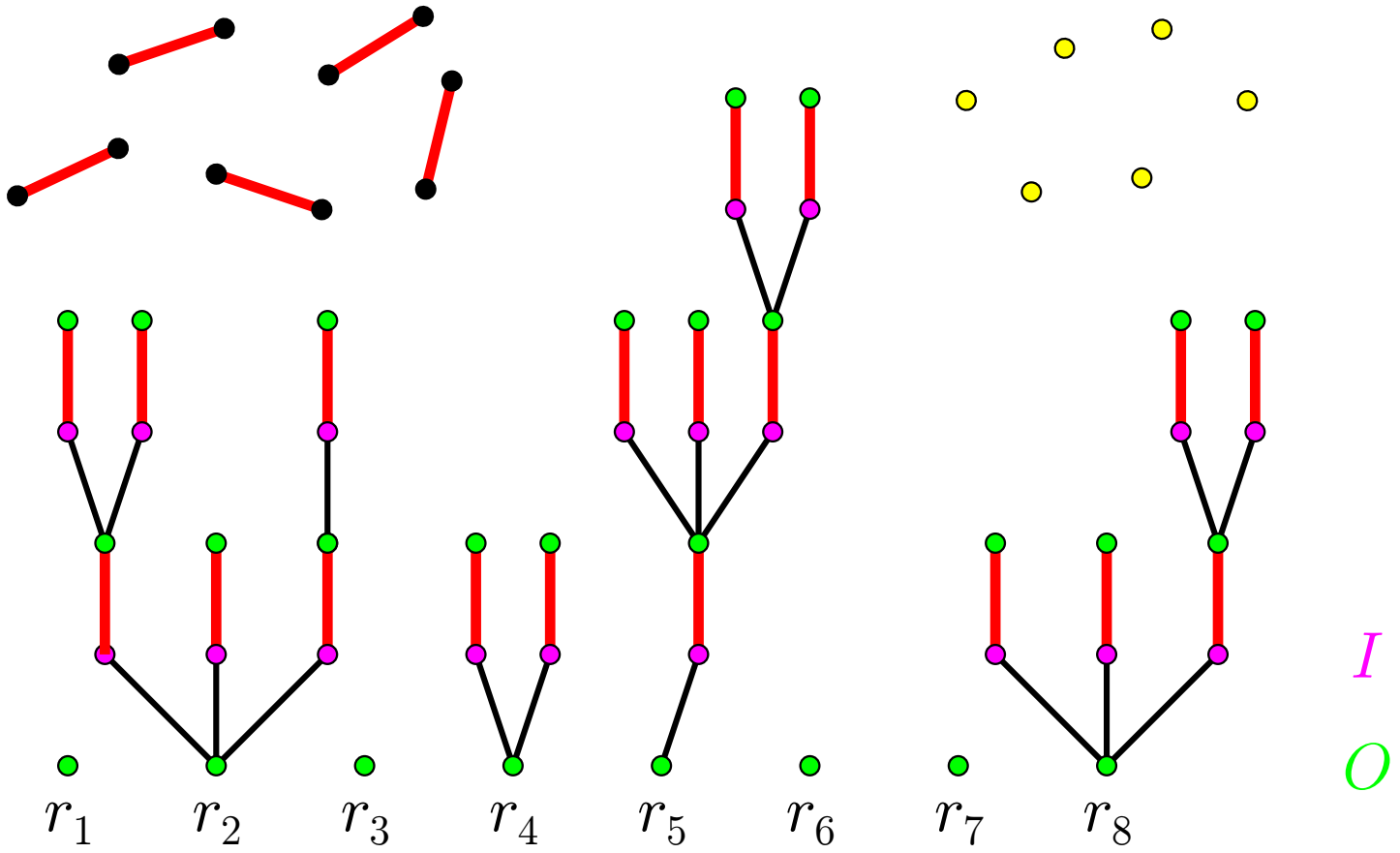
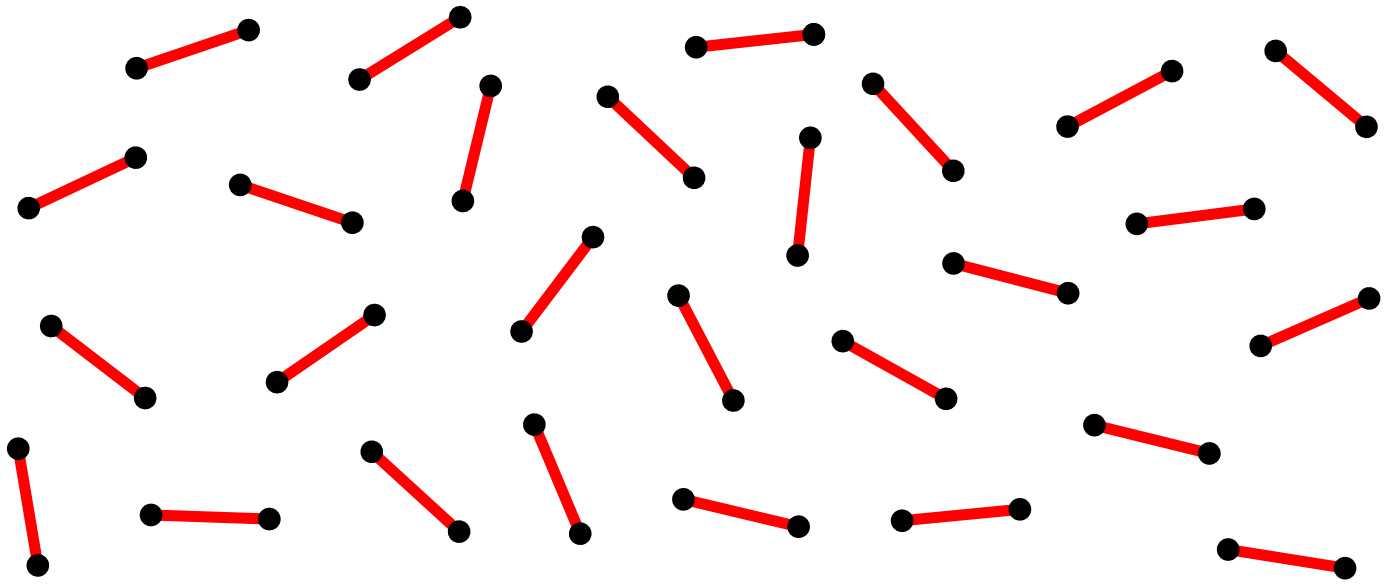
OUTPUT: The obtained augmenting path.



Unsuccessful search: If no augmenting path can be found, and the forest cannot be extended, i.e. if every outer vertex is adjacent to only (inner) vertices of the forest, then STOP.
OUTPUT: „M is of maximum size, i.e. $\nu(G) = |M|$. This is justified by the Kőnig blocking set O."



BLOSSOM ALGORITHM

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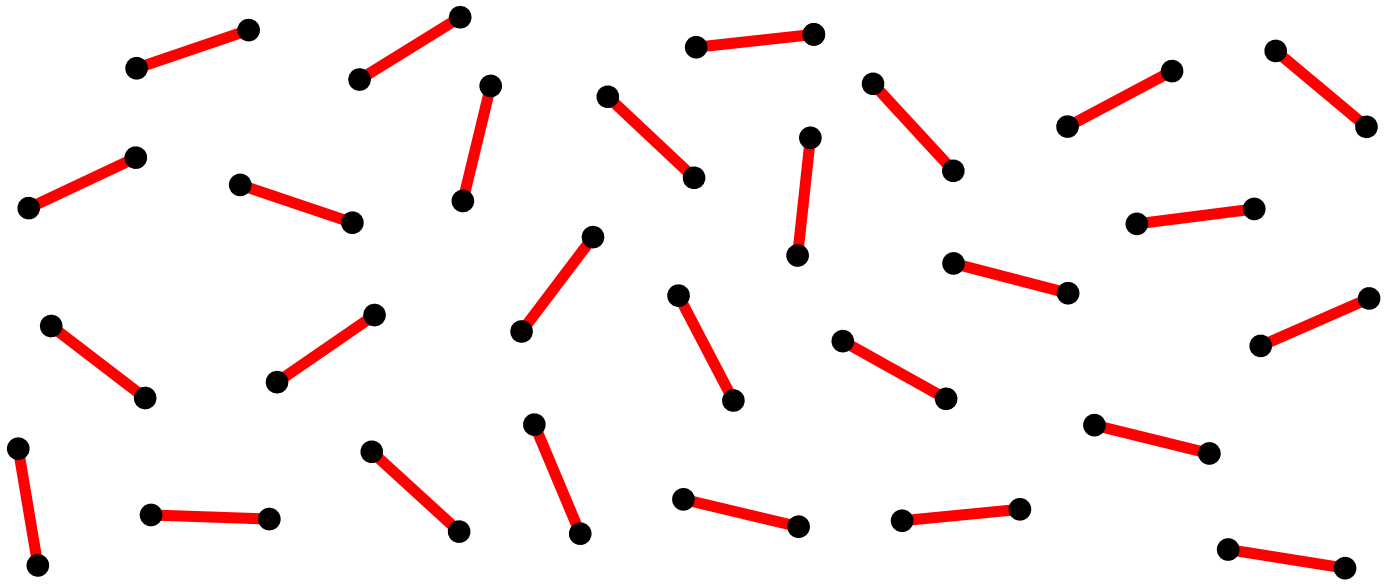
OUTPUT: An augmenting path for M, or „M is of max. size".



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INPUT: An arbitrary graph G + a non-perfect matching M in G.
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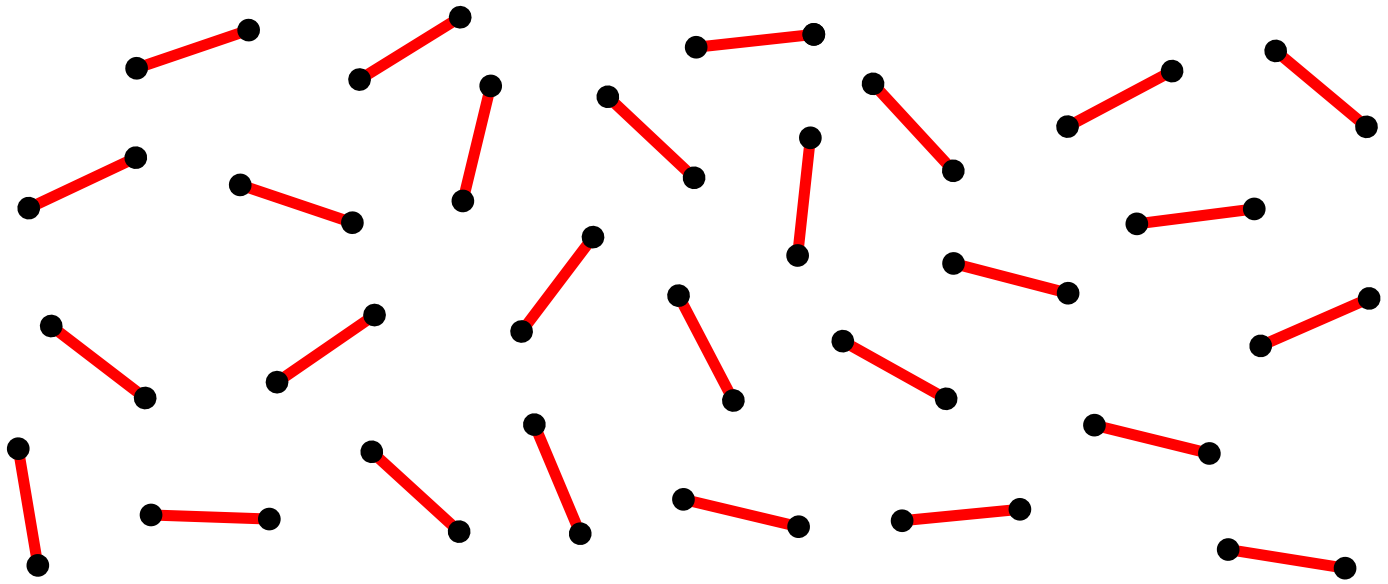
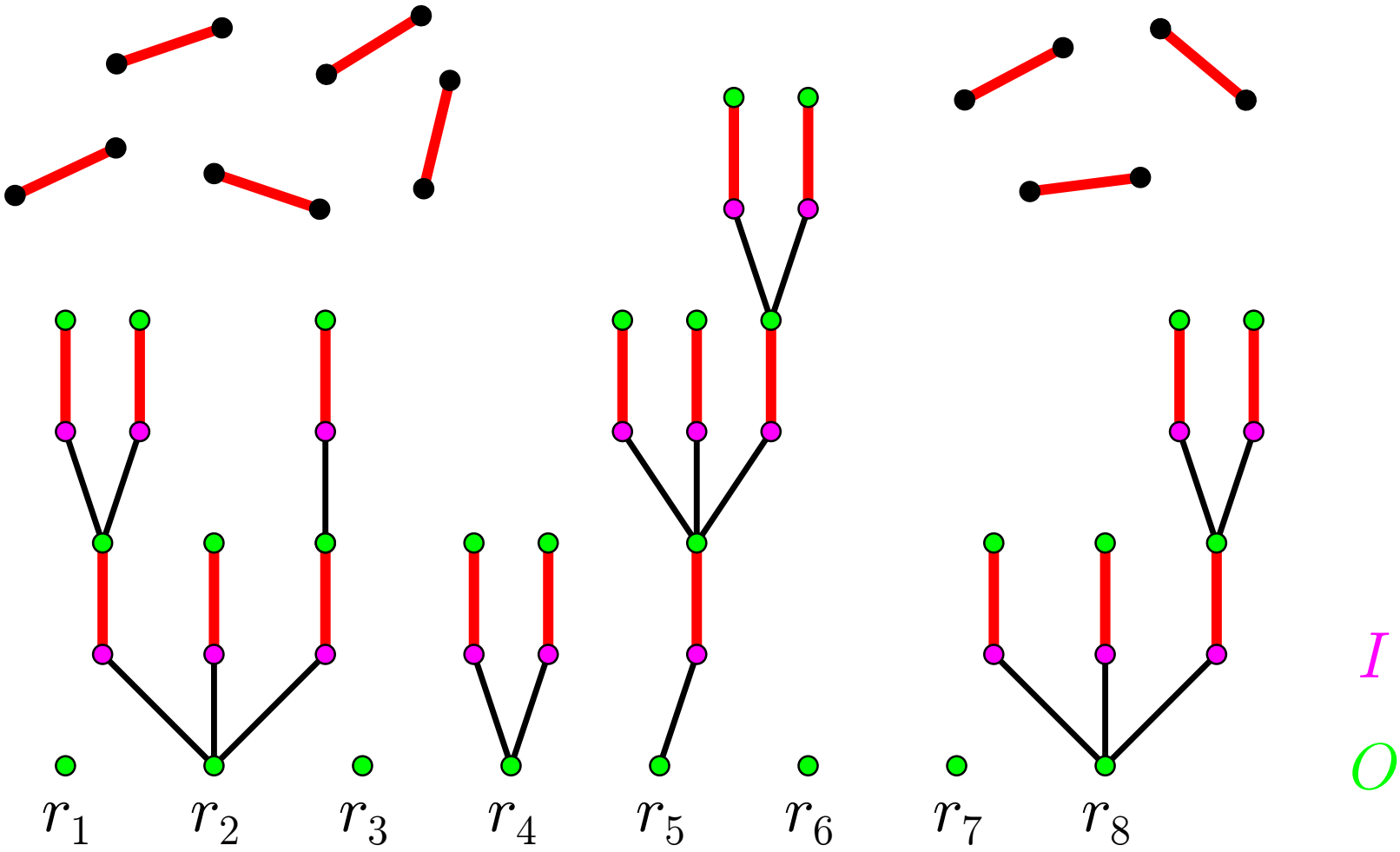
Now the roots of the „forest of alternating paths" are all the unmatched vertices in G.
So there will be no unmatched vertices outside the forest!



On the next slides we discuss the 3 possible types of steps of blossom algorithm. The „default" step is the greedy expansion step seen in Hungarian method (Type 1).



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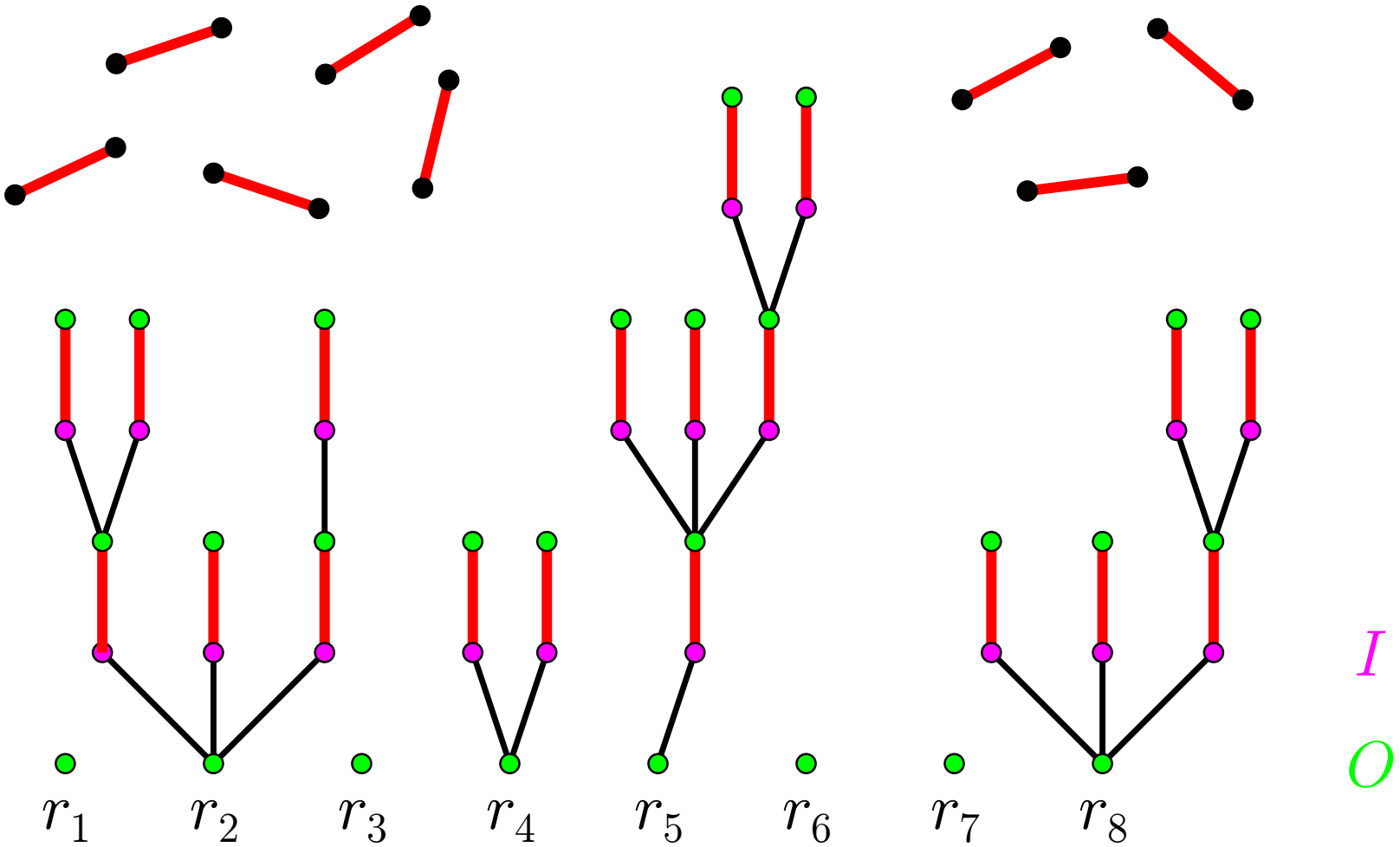
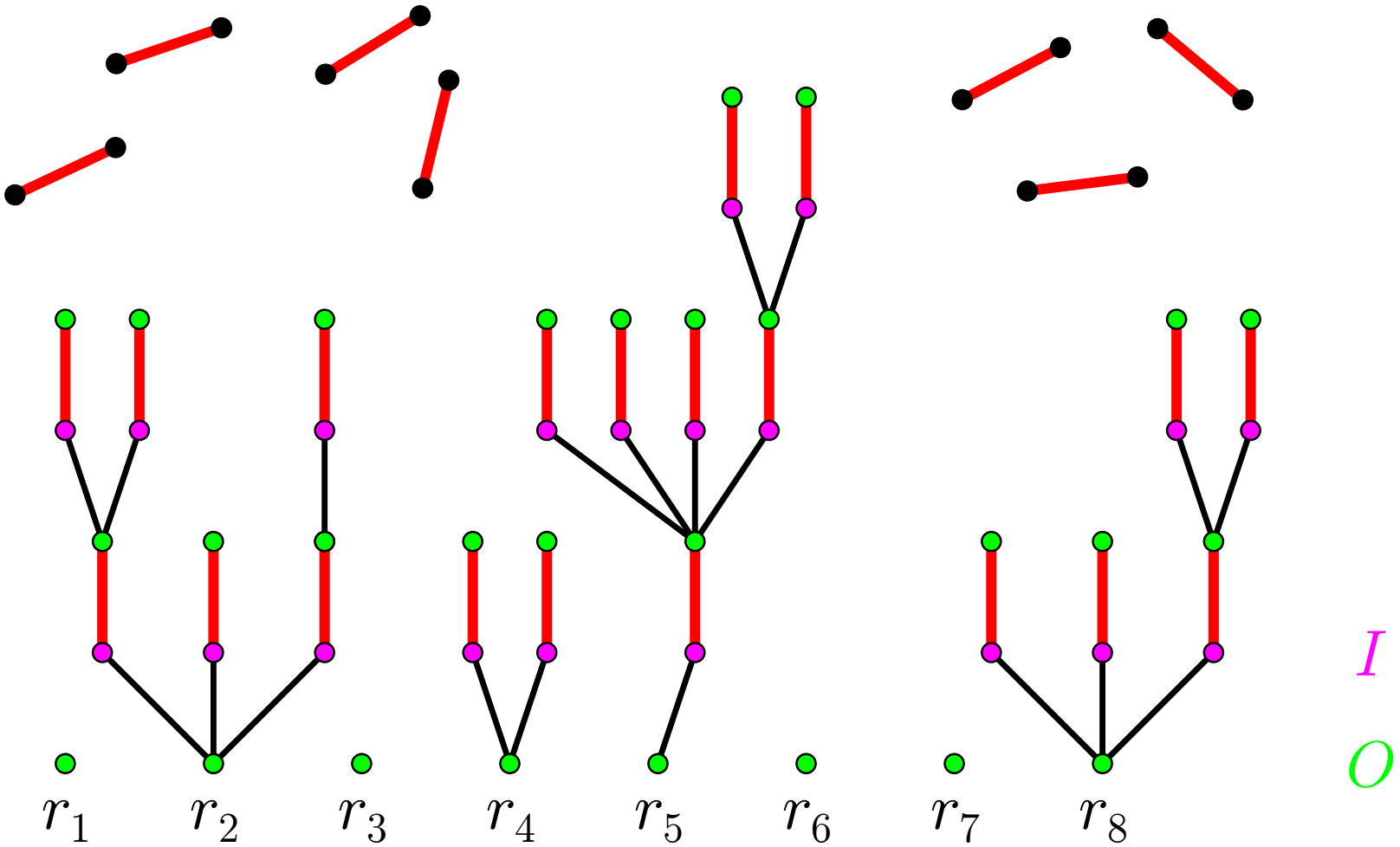


Type 1 (greedy expansion): If some outer vertex u is adjacent to some vertex v outside the forest, then extend the forest with v together with its pair in M, in the same way as in the bipartite case (including the definition of inner and outer vertices).



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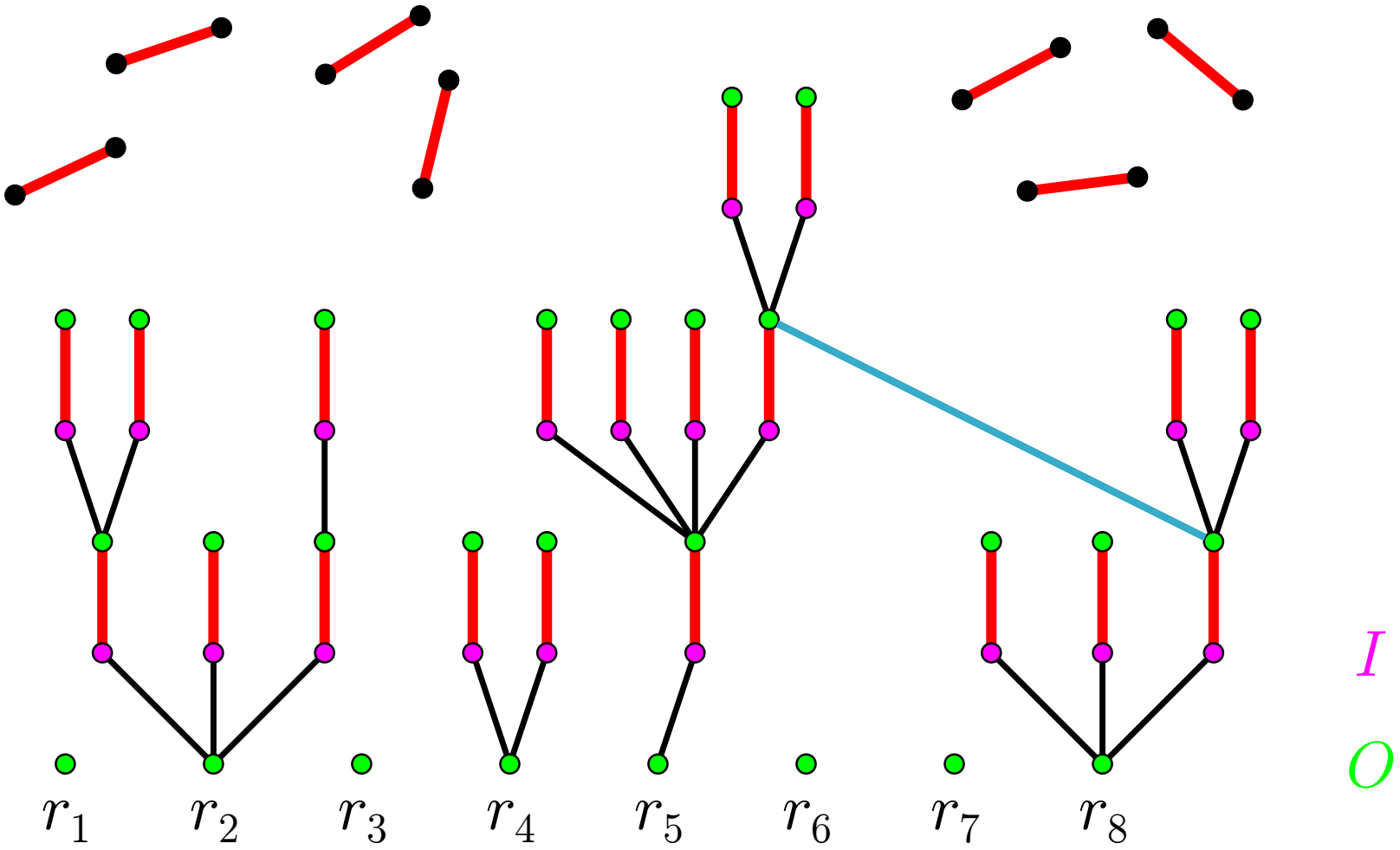
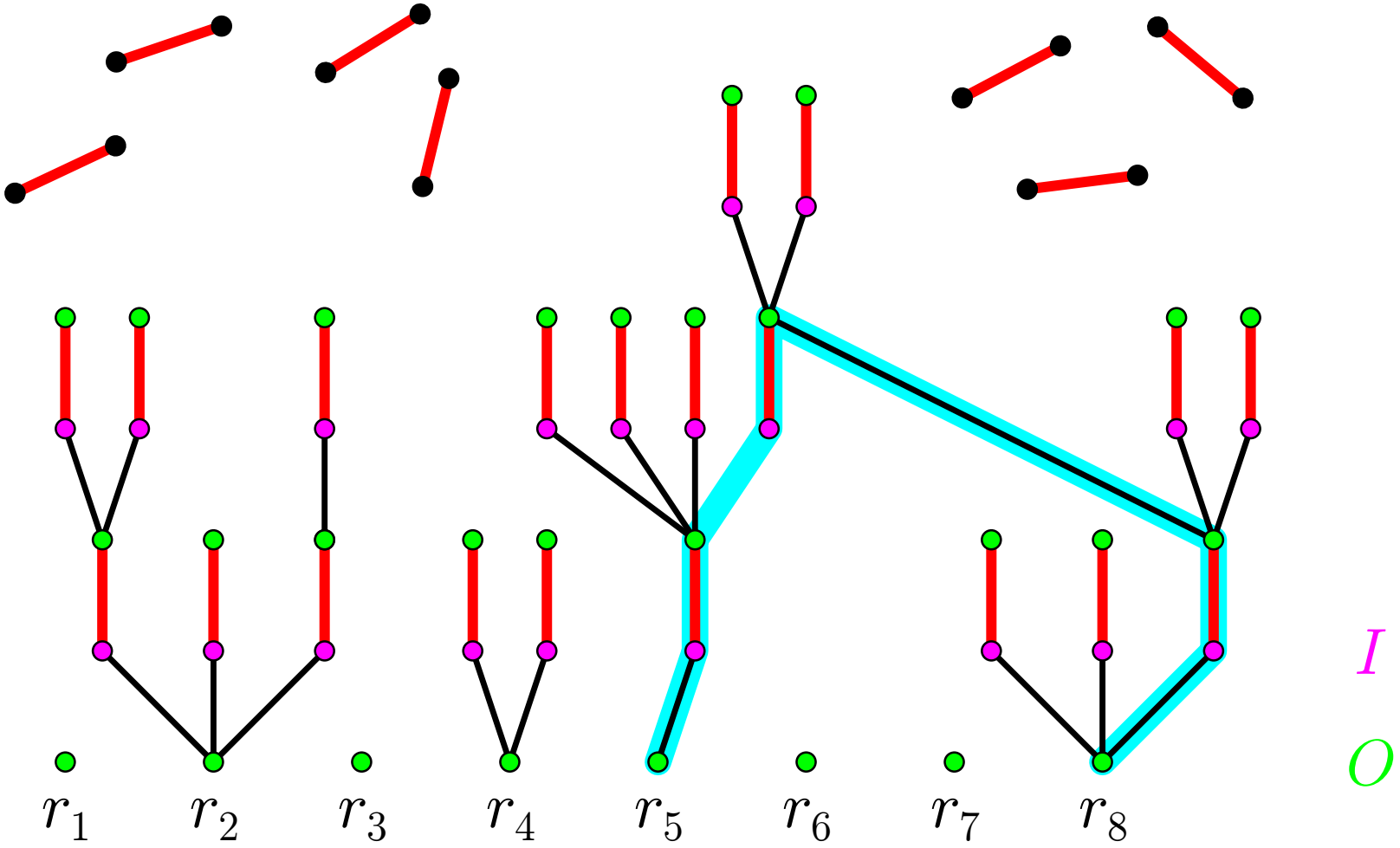


Type 2 (successful search): If some outer vertex is adjacent to some outer vertex of an other component of the forest, then an augmenting path is found (in the actual graph, cf. Type 3). STOP.



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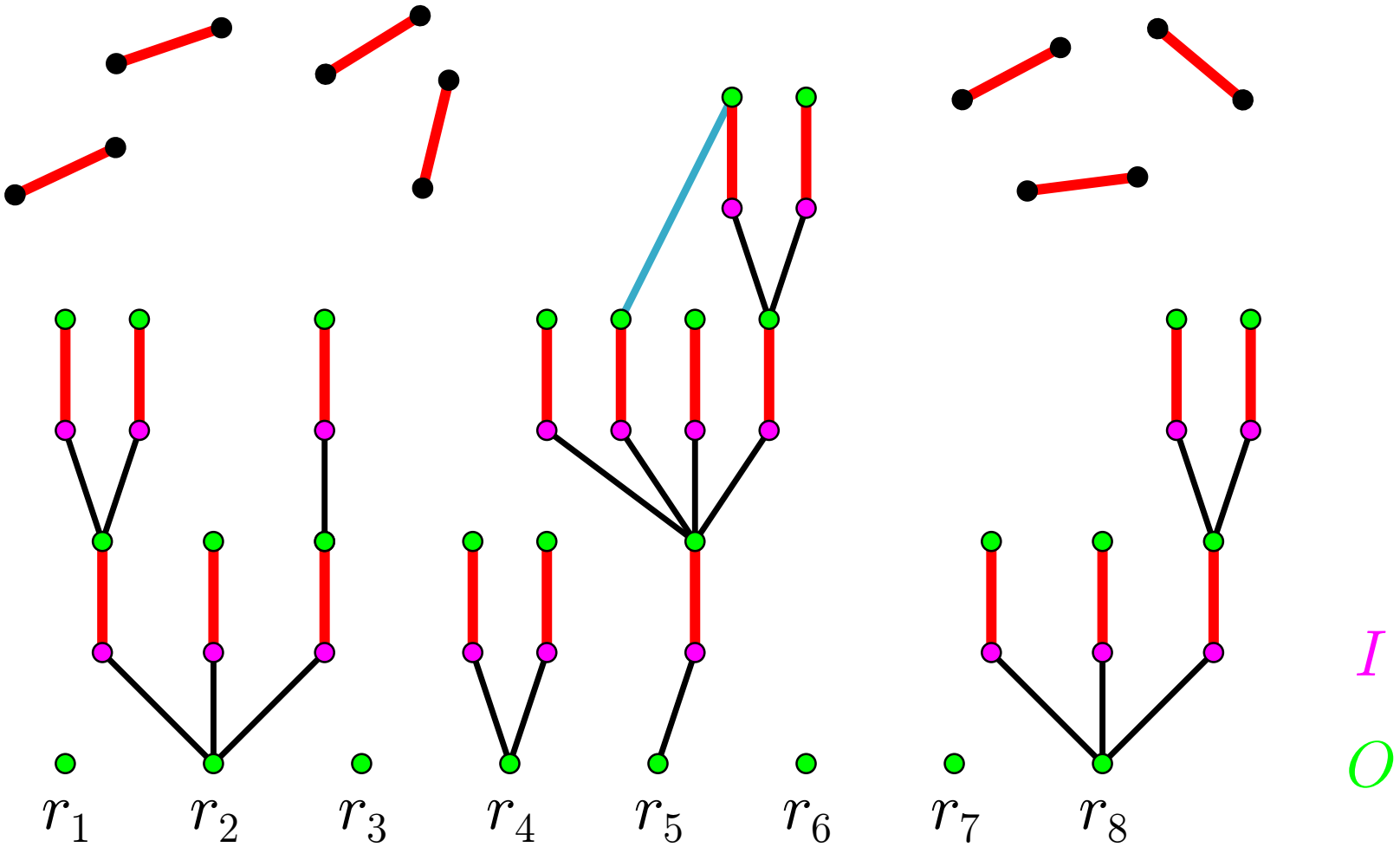
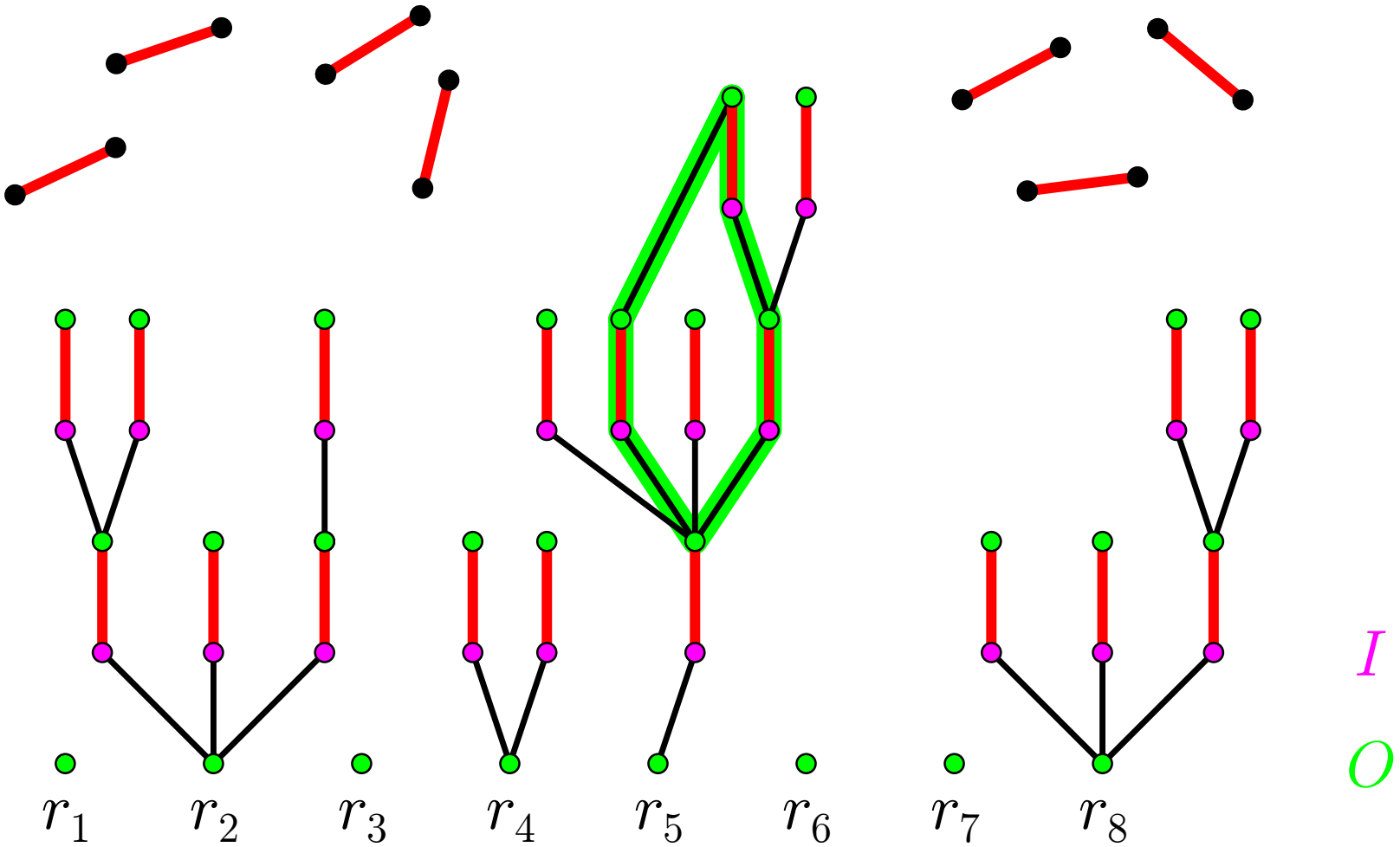


Type 3 (cycle contraction): If there is an edge between two outer vertices in the same component of the forest, then contract the corresponding odd cycle, and continue with the obtained graph. The contracted cycle becomes an outer vertex.



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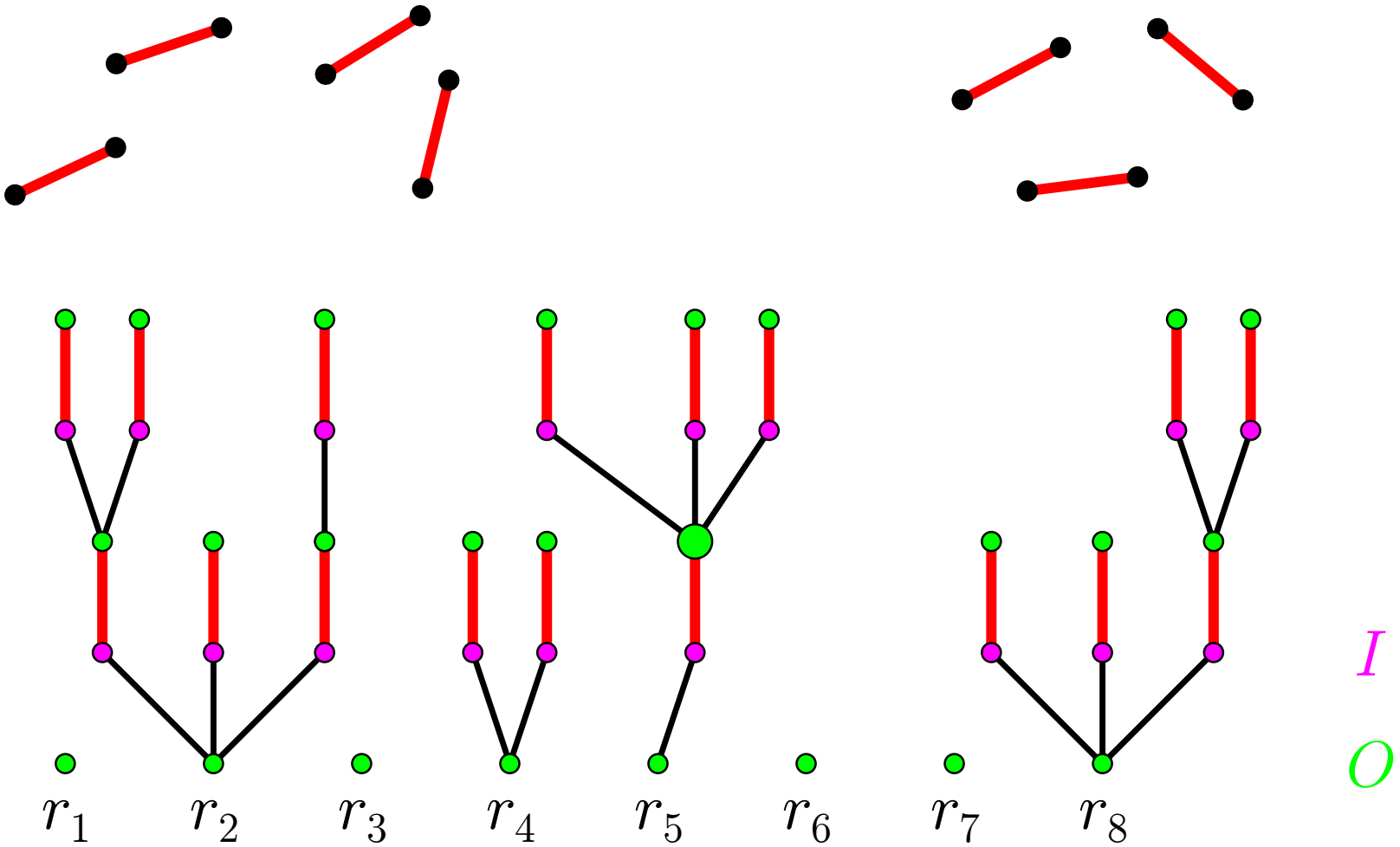

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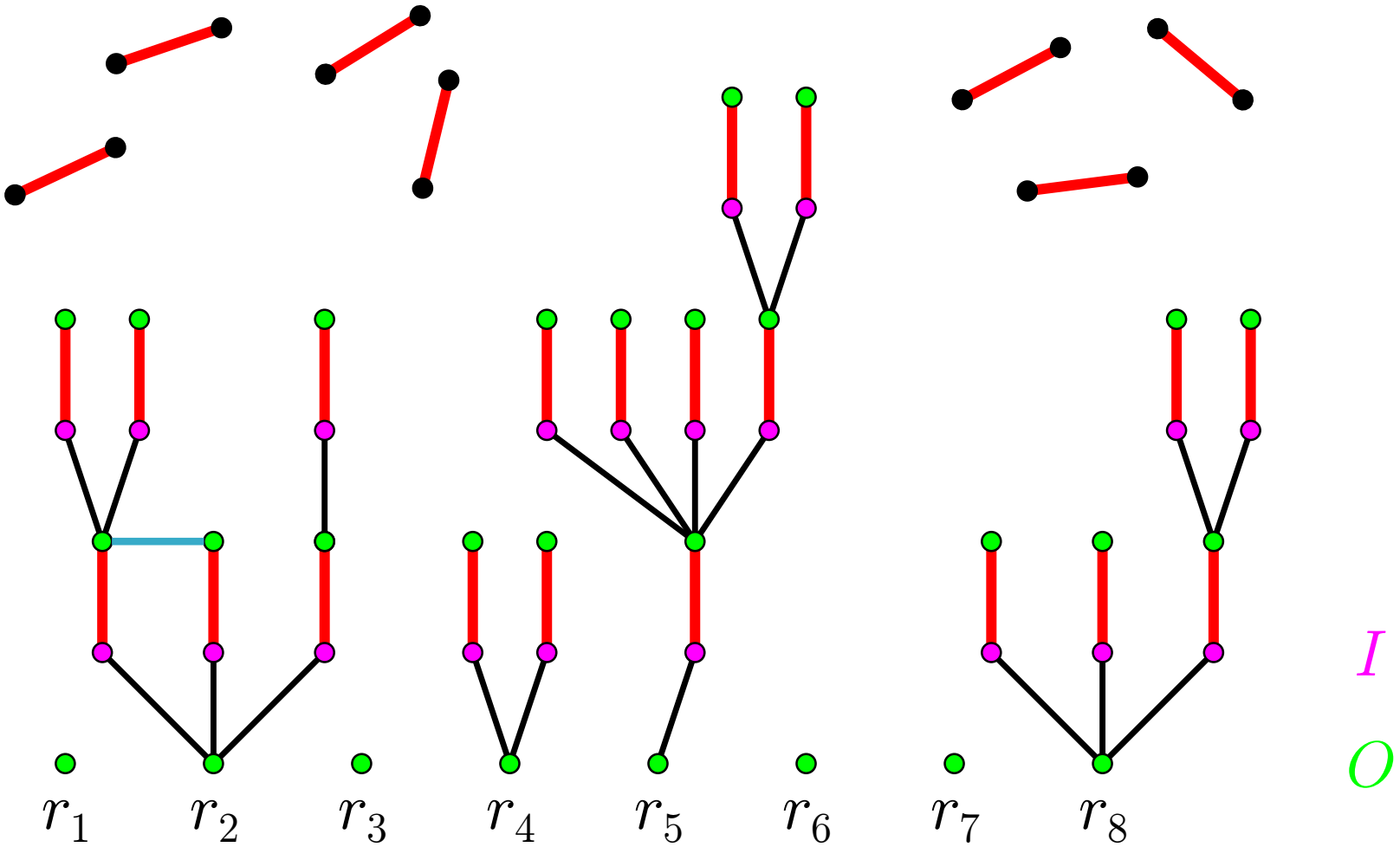

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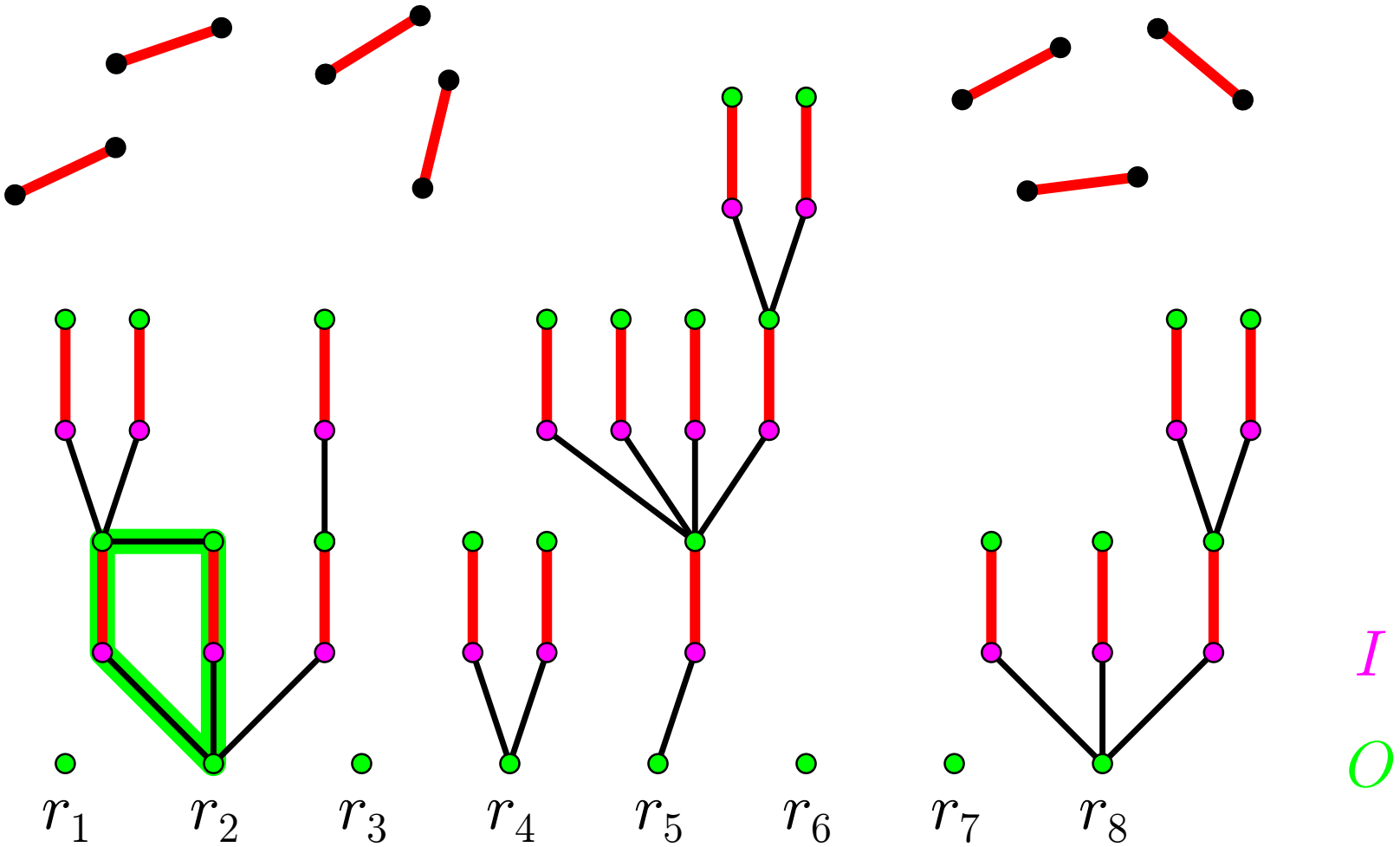

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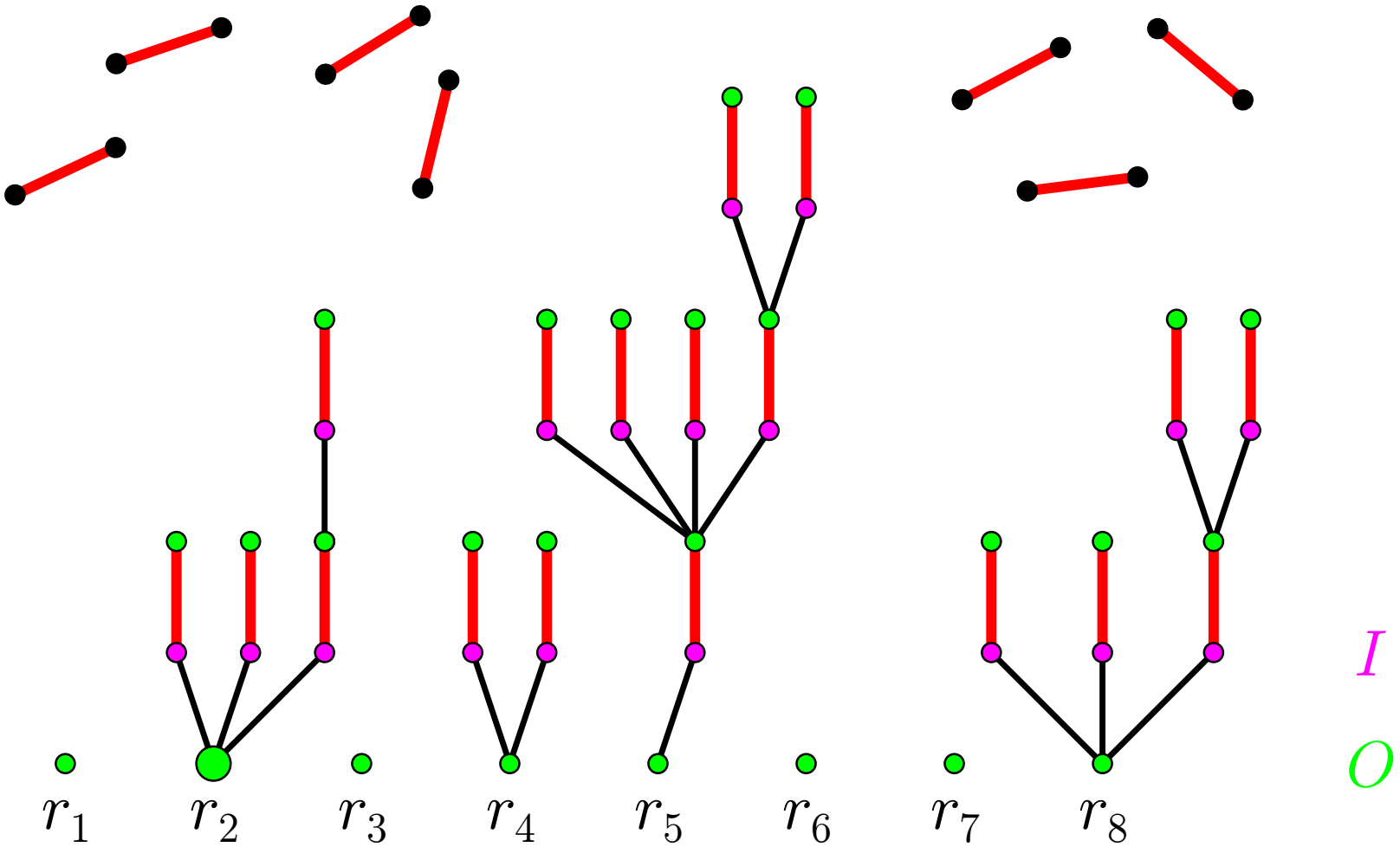

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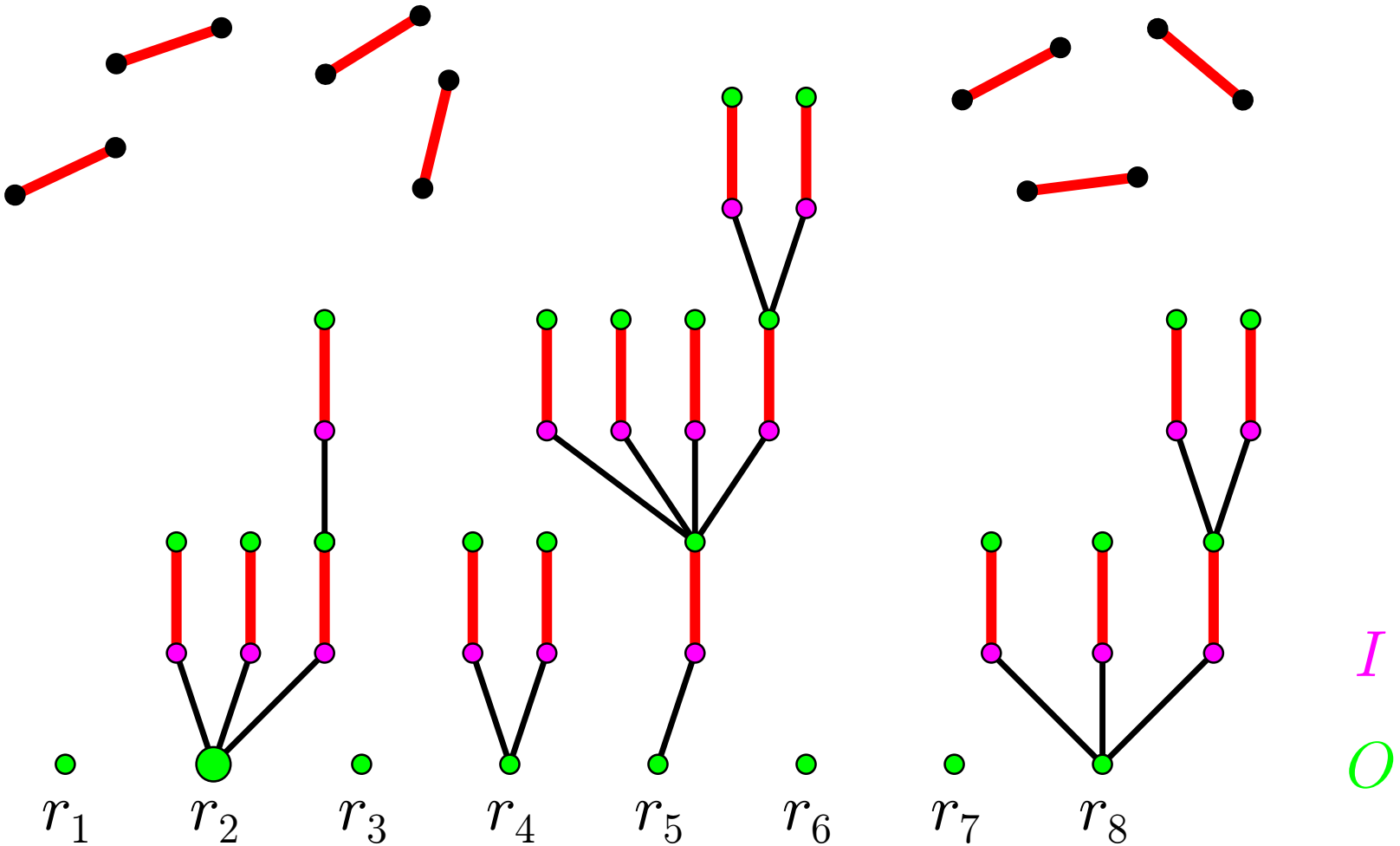

Remark: Steps of types 1-3 can be performed in any order. However, cycle contraction is complicated, so in practice a step of type 3 is recommended to perform only if the other two types of steps cannot be performed (and then try with types 1-2 again).

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The two possible outcomes:

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* In case of unsuccessful search, the "preimage" of I in the original graph G is a Tutte-blocking set, which proves that $\nu(G) \leq |M|$. (Why?) This TBS can be also added to the OUTPUT as proof.

The blossom algorithm is fast (polynomial time), so it can be used in practice. If we start from an arbitrary matching of G (e.g. from one arbitrary edge, considered as trivial matching), then by performing the blossom algorithm repeadly, we always get a larger size matching, until we reach to a perfect matching or a maximal size non-perfect matching (in case of unsuccessful search). So the value of $\nu(G)$ can be determined efficiently.

Moreover, not only the value of $\nu(G)$ is calculated in this way, but the algorithm constructs a maximum size matching with a Tutte blocking set proving the maximality. So the correctness of the algorithm can be verified without knowing/understanding the algorithm itself!

In case of bipartite graphs the same holds for the Hungarian method.