Projective embeddings of 3- and 4-nets in perspective position

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*Joint work with Gábor Nagy (University of Szeged) and Gábor Korchmáros (Università degli Studi della Basilicata)
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▶ Derived $(k - 1)$-net.
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Derived dual $(k - 1)$-net.
Dual 3-net in perspective position

- $(\Lambda_1, \Lambda_2, \Lambda_3)$ classes of points (components)
- $|\Lambda_i| = n$
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**Perspective dual 3-net with a center $C$**

A $(\Lambda_1, \Lambda_2, \Lambda_3)$ dual 3-net is in perspective position with a center $C$, if $C \notin \bigcup \Lambda_i$ and if every line through $C$ meeting a component meets each component in exactly one point.
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A 3-net realizes the group $G$, if for all $a, b, c \in G$ we have

$$a \cdot b = c \iff \alpha(a), \beta(b), \gamma(c) \text{ meet in one point.}$$
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$$a \cdot b = c \iff \alpha(a), \beta(b), \gamma(c) \text{ are collinear points.}$$
$Q = \{1, 2, 3, 4, 5\}$
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Latin square

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$Q = \{1, 2, 3, 4, 5\}$

Latin square $\leftrightarrow (Q, \ast)$ quasigroup.
Theorem (Korchmáros, Nagy, Pace, 2013)
A 4-net in PG(2, \(\mathbb{K}\)) has a constant cross-ratio.
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A 4-net in $\text{PG}(2, \mathbb{K})$ has a constant cross-ratio.
Cross-ratio

Theorem (Korchmáros, Nagy, Pace, 2013)
A dual 4-net in $\text{PG}(2, \mathbb{K})$ has a constant cross ratio, that is, for any line intersecting the components, the cross-ratio of the four intersection points is constant.
Transversal line

The $\ell$ line is a transversal of a $(\lambda_1, \lambda_2, \lambda_3)$ 3-net, if $\ell$ intersect all the lines of the 3-net in the total $n$ points.
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Theorem (B., Korchmáros, Nagy, 2014)

Let \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \) be a 3-net of order \( n \) in \( \text{PG}(2, K) \). Assume that \( \ell \) is a transversal. Then there is a scalar \( \kappa \) such that for all \( P \in \ell \cap \lambda \) \((P = m_1 \cap m_2 \cap m_3, m_1 \in \lambda_1, m_2 \in \lambda_2, m_3 \in \lambda_3)\), the cross-ratio of the lines \( \ell, m_1, m_2, m_3 \) is \( \kappa \).
Transversal line

The line $\ell$ is a transversal of a $(\lambda_1, \lambda_2, \lambda_3)$ 3-net, if $\ell$ intersect all the lines of the 3-net in the total $n$ points.

Theorem (B., Korchmáros, Nagy, 2014)

Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be a 3-net of order $n$ in $\text{PG}(2, \mathbb{K})$. Assume that $\ell$ is a transversal. Then there is a scalar $\kappa$ such that for all $P \in \ell \cap \lambda$ ($P = m_1 \cap m_2 \cap m_3$, $m_1 \in \lambda_1$, $m_2 \in \lambda_2$, $m_3 \in \lambda_3$), the cross-ratio of the lines $\ell, m_1, m_2, m_3$ is $\kappa$.

Theorem (B., Korchmáros, Nagy, 2014)

Let $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$ a dual 3-net of order $n$ in $\text{PG}(2, \mathbb{K})$. Assume that $\Lambda$ is in perspective position with respect to point $T$. Then there is a scalar $\kappa$ such that for all lines $\ell$ through $T$, the cross-ration of the points $T, \ell \cap \Lambda_1, \ell \cap \Lambda_2, \ell \cap \Lambda_3$ is $\kappa$. 
Classification of dual 3-nets

- **Algebraic**: its points are on a cubic curve.
- **Proper algebraic**
- **Conic-line type**
- **Regular**: its components lie on three lines.
- **Triangular**: the three lines form a triangle.
- **Pencil type**: the three lines are concurrent.
- **Tetrahedron type**: its components lie on the sides of a non-degenerate quadrangle (sides and diagonals).
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Theorem (B., Korchmáros, Nagy, 2014)
Any regular dual 3-net in perspective position is of pencil type.
Pencil type dual 3-net doesn’t exist in zero characteristic.

In positive characteristic they only exist when the order of the dual 3-net is divisible by the characteristic.

**Theorem (B., Korchmáros, Nagy, 2014)**

No regular dual 3-net in perspective position exists in zero characteristic. This holds for dual 3-nets in positive characteristic whenever the order of the 3-net is smaller than the characteristic.
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No tetrahedron type dual 3-net in perspective position exists in zero characteristic. This holds for dual 3-nets in positive characteristic whenever the order of the 3-net is smaller than the characteristic.
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Proposition (B., Korchmáros, Nagy, 2014)
Let $\mathbb{K}$ be an algebraically closed field whose characteristic is zero or greater than $n$. Then no dual 4-net of order $n$ embedded in $\text{PG}(2, \mathbb{K})$ has a derived dual 3-net which is either triangular or of tetrahedron type.
Conic-line type dual 3-nets

\[ \Lambda_3 \]

\[ \Lambda_2 \]

\[ \Lambda_1 \]

C

\[ \ell \]

T

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Conic-line type dual 3-nets

Proj. coord. system: $T = (0, 0, 1)$, $\ell: Z = 0$, $C: XY = Z^2$.

It can be shown, the $\Lambda$ has a parametrization with $n^{th}$ root of unity:

$$\Lambda_2 = \{(c, c^{-1}), (c\xi, c^{-1}\xi^{-1}), \ldots, (c\xi^{n-1}, c^{-1}\xi^{-n+1})\},$$

where $c \in \mathbb{K}^*$ and $\xi$ is a $n^{th}$ root of unity in $\mathbb{K}$.

The $u: (x, y) \mapsto (-x, -y)$ perspectivity takes $\Lambda_2$ to $\Lambda_3$:

$$\Lambda_3 = \{(-c, -c^{-1}), (-c\xi, -c^{-1}\xi^{-1}), \ldots, (-c\xi^{n-1}, -c^{-1}\xi^{-n+1})\}.$$ 

If $n$ is even, then $\xi^{n/2} = -1$. If $n$ is odd then

$$\Lambda_1 = \{(c^{-2}), (c^{-2}\xi), \ldots, (c^{-2}\xi^{n-1})\}.$$
Conic-line type dual 3-nets

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If \( n \) is even, then \( \xi^{n/2} = -1 \). If \( n \) is odd then

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\Lambda_1 = \{(c^{-2}), (c^{-2}\xi), \ldots, (c^{-2}\xi^{n-1})\}.
\]

Lemma

For \( n \) odd, the above \( (\Lambda_1, \Lambda_2, \Lambda_3) \) conic-line type dual 3-net is in perspective position with center \( T \).
Theorem (B., Korchmáros, Nagy, 2014)

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero or greater than $n$. Then every conic-line type dual 3-net of order $n$ in $\mathrm{PG}(2, \mathbb{K})$ in perspective position is projectively equivalent to the example given on the previous slide.
**Theorem (B., Korchmáros, Nagy, 2014)**

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero or greater than $n$. Then every conic-line type dual 3-net of order $n$ in $\text{PG}(2, \mathbb{K})$ in perspective position is projectively equivalent to the example given on the previous slide.

**Proposition (B., Korchmáros, Nagy, 2014)**

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero or greater than $n$. Then no dual 4-net of order $n$ in $\text{PG}(2, \mathbb{K})$ has a derived dual 3-net of conic-line type.
\[ \Lambda = (\Lambda_1, \Lambda_2, \Lambda_3) \text{ lies on } \Gamma \text{ irreducible cubic curve.} \]

Suppose: \( \text{char}(\mathbb{K}) \notin \{2, 3\} \).

<table>
<thead>
<tr>
<th>shape of ( \Gamma )</th>
<th># of infl. points</th>
<th>canonical form</th>
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<tbody>
<tr>
<td>nonsingular</td>
<td>9</td>
<td>( Y^2 = X(X - 1)(X - c) )</td>
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<tr>
<td>node</td>
<td>3</td>
<td>( Y^2 = X^3 )</td>
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<tr>
<td>cusp</td>
<td>1</td>
<td>( Y^2 = X^3 + X^2 )</td>
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The $j$-invariant classifies elliptic curves up to isomorphism.

$j$-invariant

If a cubic curve $\Gamma$ can be transformed into the form

$$Y^2 = X(X - 1)(X - c)$$

then the $j$-invariant of the curve is

$$j(\Gamma) = 2^8 \frac{(c^2 - c + 1)^3}{c^2(c - 1)^2}.$$
Theorem (B., Korchmáros, Nagy, 2014)

Let $\mathbb{K}$ be an algebraically closed field of characteristic different from 2 and 3. Let $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$ be a dual 3-net of order $n \geq 7$ in $\text{PG}(2, \mathbb{K})$ which lies on an irreducible cubic curve $\Gamma$. If $\Gamma$ is singular or is nonsingular with $j(\Gamma) \neq 0$ then $\Lambda$ is not in perspective position. If $j(\Gamma) = 0$ then there are at most three point $T_1, T_2, T_3$ such that $\Lambda$ is in perspective position with center $T_i$. 
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Proposition (B., Korchmáros, Nagy, 2014)

Let $\mathbb{K}$ be an algebraically closed field, such that $\text{char}(\mathbb{K}) \notin \{2, 3\}$. Then no dual 4-net of order $n \geq 7$ in $\text{PG}(2, \mathbb{K})$ has a derived dual 3-net lying on a plane cubic.
Theorem (B., Korchmáros, Nagy, 2014)

Let $\Gamma$ be a dual 3-net of order $n$ coordinatized by a group. Assume that $\Lambda$ is embedded in a projective plane $\text{PG}(2, K)$ over an algebraically closed field with $\text{char}(K) = 0$ or $\text{char}(K) > n$. If $\Lambda$ is in perspective position and $n \neq 8$ then one of the following two cases occur:

(i) A component of $\Lambda$ lies on a line while the other two lie on a nonsingular conic.

(ii) $\Lambda$ is contained in a nonsingular cubic curve $C$ with zero $j(C)$-invariant, and $\Lambda$ is in perspective position with at most three center.
Theorem (B., Korchmáros, Nagy, 2014)

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Thank you for your attention!
Addition on cubic curve

Let $\Gamma$ be a cubic curve and let $\Gamma^*$ be the set of its smooth points. Let $O \in \Gamma^*$ be a fixed point. In this case we can define the sum of $A, B \in \Gamma^*$ points:

\[ A + B \]
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Cubic curve

Theorem

Let $\Gamma$ be a cubic curve and let $\Gamma^*$ be the set of its smooth points. Let $O \in \Gamma^*$ be a fixed point. Then $(\Gamma^*, +, O)$ is an abelian group.

Theorem

1. If $\Gamma: Y = X^3$, then $(\Gamma^*, +) \cong (K, +)$.
2. If $\Gamma: Y^2 = X^3$, then $(\Gamma^*, +) \cong (K, +)$.
3. If $\Gamma: Y^2 = X^3 + X^2$, then $(\Gamma^*, +) \cong (K^*, \cdot)$. 
In $\text{PG}(2, \mathbb{K})$ defined over an algebraically closed field $\mathbb{K}$ of characteristic $p \geq 0$, let $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$ be a dual 3-net of order $n \geq 4$ which realizes a group $G$. If either $p = 0$ or $p > n$ then one of the following holds.

(i) $G$ is cyclic or direct product of two cyclic groups and $\Lambda$ is algebraic.

(ii) $G$ is dihedral and $\Lambda$ is of tetrahedron type.

(iii) $G$ is the quaternion group of order 8.

(iv) $G$ has order 12 and is isomorphic to $A_4$.

(v) $G$ has order 24 and is isomorphic to $S_4$.

(vi) $G$ has order 60 and is isomorphic to $A_5$. 