Projective embeddings of 3- and 4-nets in perspective position

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*Joint work with Gábor Nagy (University of Szeged) and Gábor Korchmáros (Università degli Studi della Basilicata)

Definitions

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A finite k-net of order n is an incidence structure consisting of $k \geq 3$ pairwise disjoint classes of lines, each of size n, such that every point incident with two lines from distinct classes is incident with exactly one line from each of the k classes.

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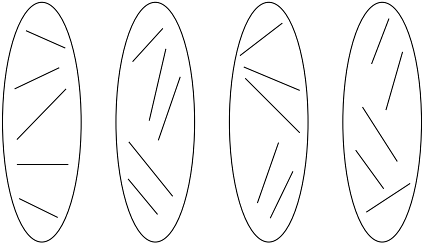

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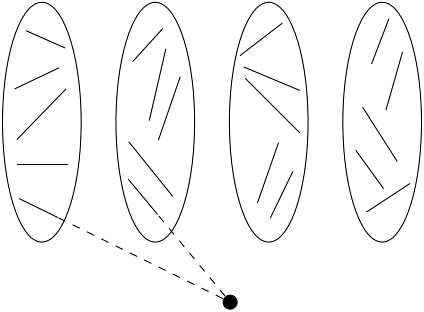

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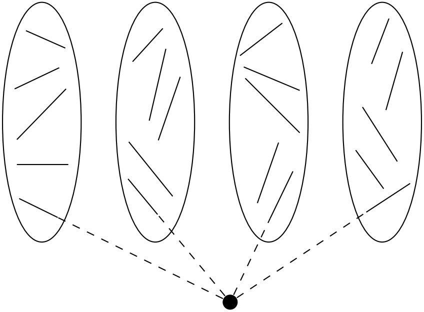

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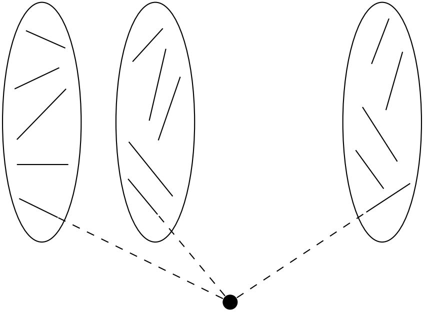

- Derived $(k-1)$-net.

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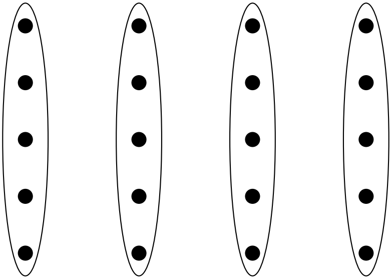

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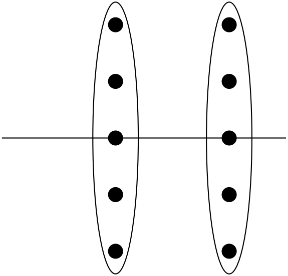

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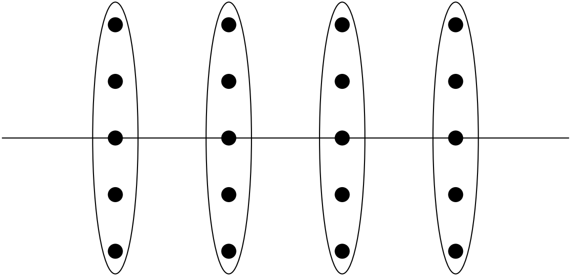

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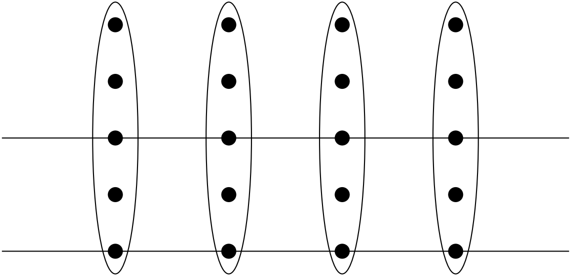

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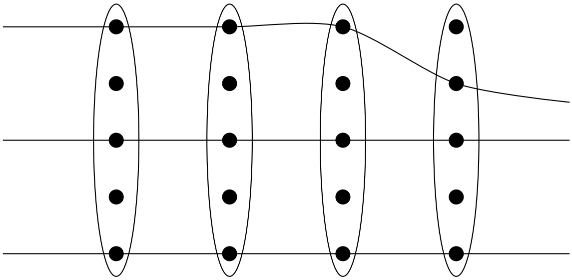

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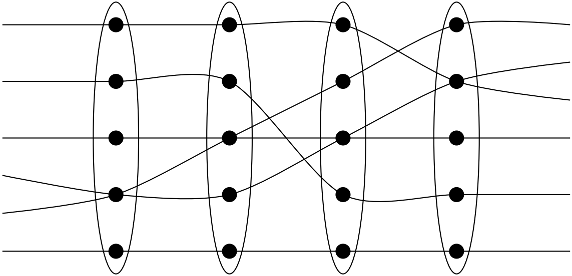

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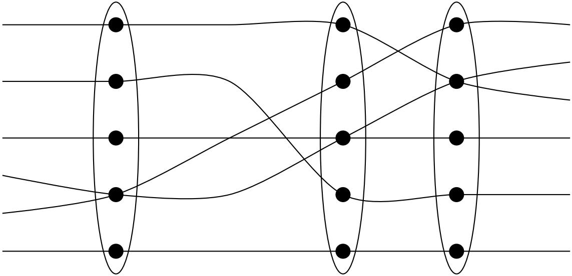

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Dual 3-net in perspective position

- $(\Lambda_1, \Lambda_2, \Lambda_3)$ classes of points (components)
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A $(\Lambda_1, \Lambda_2, \Lambda_3)$ dual 3-net is in perspective position with a center C, if $C \notin \cup\Lambda_i$ and if every line through C meeting a component meets each component in exactly one point.

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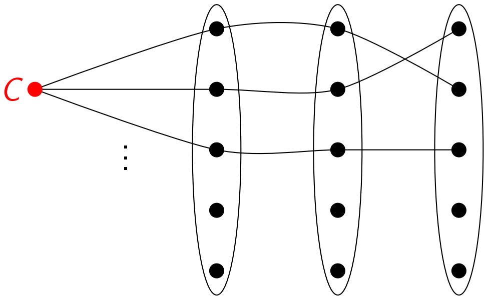

3-nets in PG$(2, \mathbb{K})$ coordinatized by a group

Classes of lines: $\mathcal{A}, \mathcal{B}, \mathcal{C}$.

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A 3-net realizes the group G, if for all $a, b, c \in G$ we have

$$a \cdot b = c \quad \Longleftrightarrow \quad \alpha(a), \beta(b), \gamma(c) \text{ meet in one point.}$$

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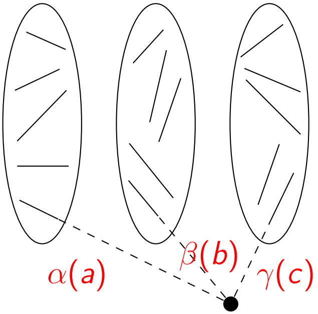

Classes of points: $\mathcal{A}, \mathcal{B}, \mathcal{C}$.
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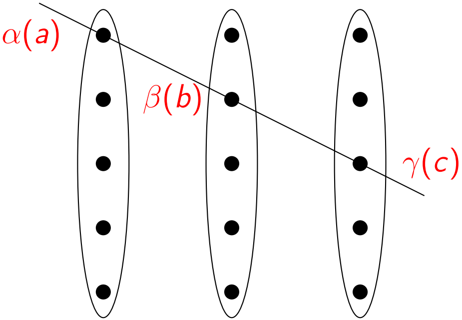
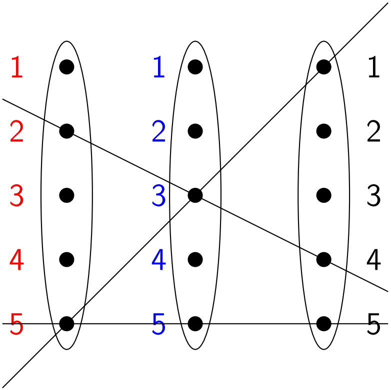
A dual 3-net realizes a group G, if for all $a, b, c \in G$ we have

$$a \cdot b = c \quad \iff \quad \alpha(a), \beta(b), \gamma(c) \text{ are collinear points.}$$



$Q = \{1, 2, 3, 4, 5\}$



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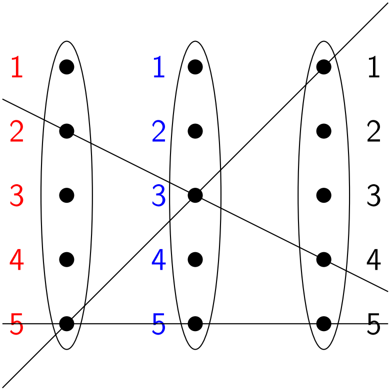

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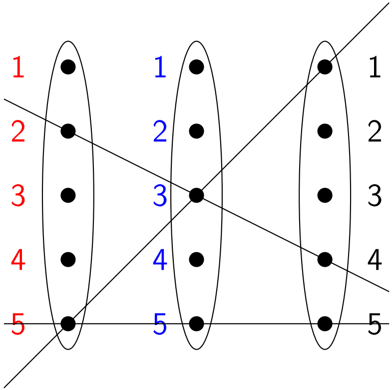

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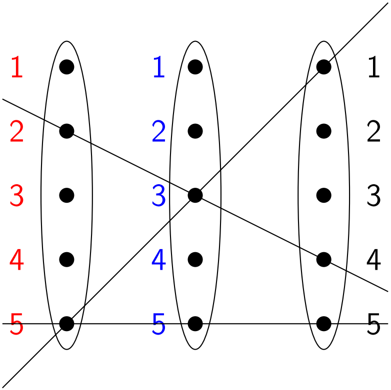

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Latin square

Latin square $\longleftrightarrow$ $(Q, *)$ quasigroup.

Cross-ratio

Theorem (Korchmáros, Nagy, Pace, 2013)

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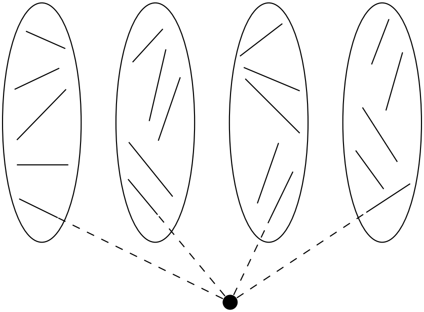

Cross-ratio

Theorem (Korchmáros, Nagy, Pace, 2013)

A dual 4-net in $\mathrm{PG}(2, \mathbb{K})$ has a constant cross ratio, that is, for any line intersecting the components, the cross-ratio of the four intersection points is constant.

Transversal line

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Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be a 3-net of order n in $\mathrm{PG}(2, \mathbb{K})$. Assume that ℓ is a transversal. Then there is a scalar κ such that for all $P \in \ell \cap \lambda$ ($P = m_1 \cap m_2 \cap m_3$, $m_1 \in \lambda_1$, $m_2 \in \lambda_2$, $m_3 \in \lambda_3$), the cross-ratio of the lines ℓ, m_1, m_2, m_3 is κ.

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Theorem (B., Korchmáros, Nagy, 2014)

Let $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$ a dual 3-net of order n in $\mathrm{PG}(2, \mathbb{K})$. Assume that Λ is in perspective position with respect to point T. Then there is a scalar κ such that for all lines ℓ through T, the cross-ration of the points $T, \ell \cap \Lambda_1, \ell \cap \Lambda_2, \ell \cap \Lambda_3$ is κ.

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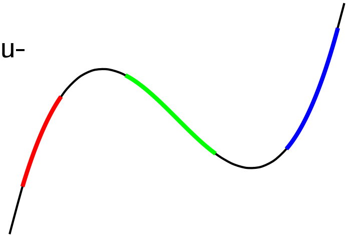

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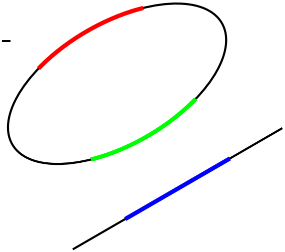

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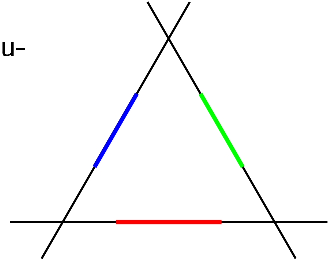

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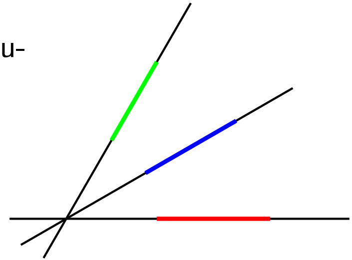

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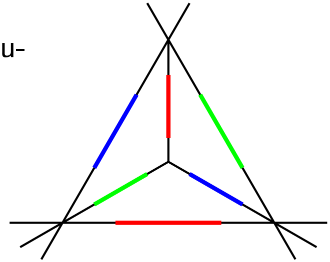

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- **Regular**: its components lie on three lines.
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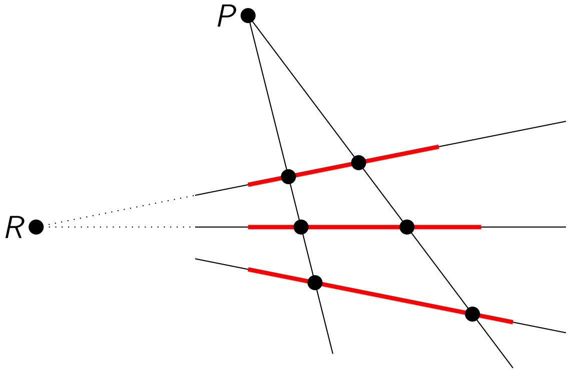
  - **Pencil type**: the three lines are concurrent.
- **Tetrahedron type**: its components lie on the sides of a non-degenerate quadrangle (sides and diagonals).

Regular and tetrahedron type dual 3-nets

Theorem (B., Korchmáros, Nagy, 2014)

Any regular dual 3-net in perspective position is of pencil type.



Pencil type dual 3-net doesn't exist in zero characteristic.

In positive characteristic they only exist when the order of the dual 3-net is divisible by the characteristic.

Theorem (B., Korchmáros, Nagy, 2014)

No regular dual 3-net in perspective position exists in zero characteristic. This holds for dual 3-nets in positive characteristic whenever the order of the 3-net is smaller than the characteristic.

Theorem (B., Korchmáros, Nagy, 2014)

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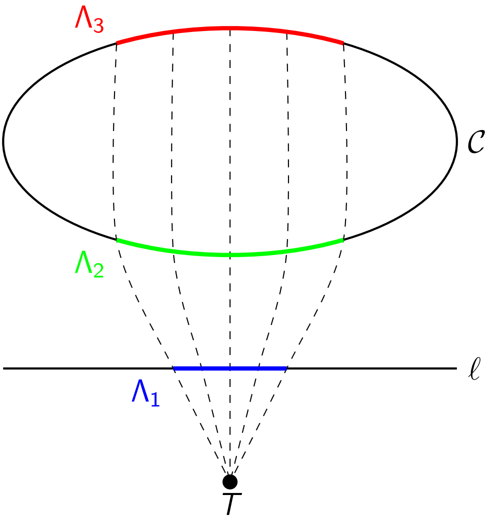
Proposition (B., Korchmáros, Nagy, 2014)

Let $\mathbb{K}$ be an algebraically closed field whose characteristic is zero or greater than n. Then no dual 4-net of order n embedded in $PG(2, \mathbb{K})$ has a derived dual 3-net which is either triangular or of tetrahedron type.

Conic-line type dual 3-nets



Conic-line type dual 3-nets

Proj. coord. system: $T=(0,0,1)$, $\ell\colon Z=0$, $\mathcal{C}\colon XY=Z^2$.
It can be shown, the Λ has a parametrization with n^{th} root of unity:

$$\Lambda_2=\{(c,c^{-1}),(c\xi,c^{-1}\xi^{-1}),\dots,(c\xi^{n-1},c^{-1}\xi^{-n+1})\},$$

where $c\in\mathbb{K}^*$ and ξ is a n^{th} root of unity in $\mathbb{K}$.
The $u\colon (x,y)\mapsto(-x,-y)$ perspectivity takes Λ_2 to Λ_3:

$$\Lambda_3=\{(-c,-c^{-1}),(-c\xi,-c^{-1}\xi^{-1}),\dots,(-c\xi^{n-1},-c^{-1}\xi^{-n+1})\}.$$

If n is even, then $\xi^{n/2}=-1$. If n is odd then

$$\Lambda_1=\{(c^{-2}),(c^{-2}\xi),\dots,(c^{-2}\xi^{n-1})\}.$$

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Lemma

For n odd, the above $(\Lambda_1, \Lambda_2, \Lambda_3)$ conic-line type dual 3-net is in perspective position with center T.

Theorem (B., Korchmáros, Nagy, 2014)

Let $\mathbb{K}$ be an algeraically closed field of characteristic zero or greater than n. Then every conic-line type dual 3-net of order n in $\mathrm{PG}(2, \mathbb{K})$ in perspective position is projectively equivalent to the example given on the previous slide.

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Let $\mathbb{K}$ be an algeraically closed field of characteristic zero or greater than n. Then no dual 4-net of order n in $\mathrm{PG}(2, \mathbb{K})$ has a derived dual 3-net of conic-line type.

Proper algebraic dual 3-net

$\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$ lies on Γ irreducible cubic curve.
Suppose: $\mathrm{char}(\mathbb{K}) \notin \{2, 3\}$.

shape of Γ	# of infl. points	canonical form
nonsingular	9	$Y^2 = X(X-1)(X-c)$
node	3	$Y^2 = X^3$
cusp	1	$Y^2 = X^3 + X^2$

j-invariant

The j-invariant classifies elliptic curves up to isomorphism.

j-invariant

If a cubic curve Γ can be trasformed into the form

$$Y^2 = X(X-1)(X-c)$$

then the j-invariant of the curve is

$$j(\Gamma) = 2^8 \frac{(c^2 - c + 1)^3}{c^2(c-1)^2}.$$

Theorem (B., Korchmáros, Nagy, 2014)

Let $\mathbb{K}$ be an algebraically closed field of characteristic different from 2 and 3. Let $\Lambda = (\Lambda_1, \Lambda_2 \Lambda_3)$ be a dual 3-net of order $n \geq 7$ in $PG(2, \mathbb{K})$ which lies on an irreducible cubic curve Γ. If Γ is singular or is nonsingular with $j(\Gamma) \neq 0$ then Λ is not in perspective position. If $j(\Gamma) = 0$ then there are at most three point T_1, T_2, T_3 such that Λ is in perspective position with center T_i.

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Proposition (B., Korchmáros, Nagy, 2014)

Let $\mathbb{K}$ be an algebraically closed field, such that $\mathrm{char}(\mathbb{K}) \notin \{2, 3\}$. Then no dual 4-net of order $n \geq 7$ in $PG(2, \mathbb{K})$ has a derived dual 3-net lying on a plane cubic.

Summary

Theorem (B., Korchmáros, Nagy, 2014)

Let Γ be a dual 3-net of order n coordinatized by a group. Assume that Λ is embedded in a projective plane $PG(2, \mathbb{K})$ over an algebraically closed field witk $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > n$. If Λ is in perspective position and $n \neq 8$ then one of the following two cases occur:

(i) A component of Λ lies on a line whilethe other two lie on a nonsingular conic.

(ii) Λ is contained in a nonsingular cubic curve $\mathcal{C}$ with zero $j(\mathcal{C})$-invariant, and Λ is in perspective position with at most three center.

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Theorem (B., Korchmáros, Nagy, 2014)

Let Γ be a dual 3-net of order n coordinatized by a group. Assume that Λ is embedded in a projective plane $PG(2, \mathbb{K})$ over an algebraically closed field witk $\text{char}(\mathbb{K}) = 0$ or $\text{char}(\mathbb{K}) > n$. If Λ is in perspective position and $n \neq 8$ then one of the following two cases occur:

(i) A component of Λ lies on a line whilethe other two lie on a nonsingular conic.

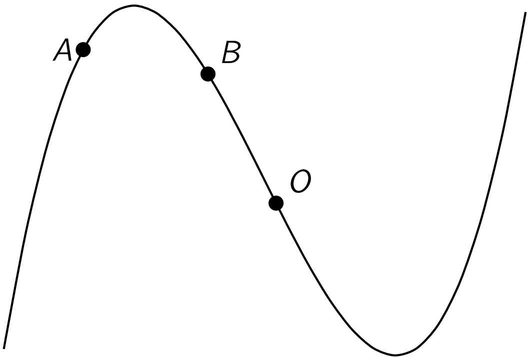
(ii) Λ is contained in a nonsingular cubic curve $\mathcal{C}$ with zero $j(\mathcal{C})$-invariant, and Λ is in perspective position with at most three center.

Thank you for your attention!

Cubic curve

Addition on cubic curve

Let Γ be a cubic curve and let Γ^* be the set of its smooth points. Let $O \in \Gamma^*$ be a fixed point. In this case we can define the sum of $A, B \in \Gamma^*$ points:



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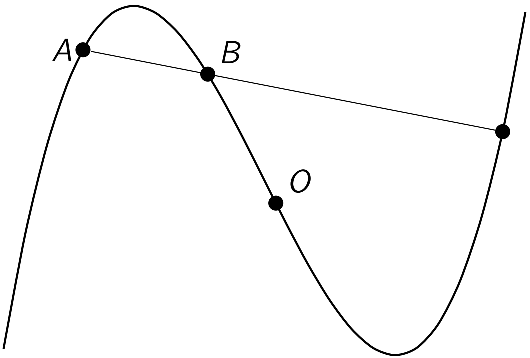

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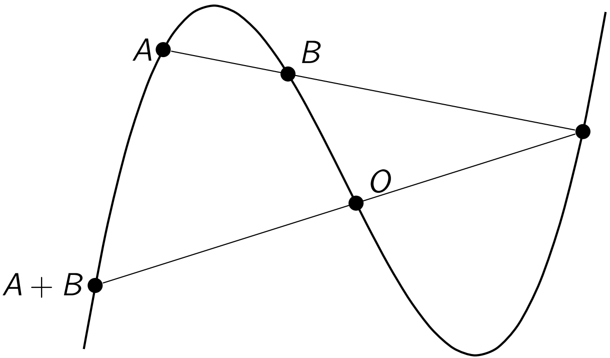

Cubic curve

Theorem

Let Γ be a cubic curve and let Γ^* be the set of its smooth points. Let $O \in \Gamma^*$ be a fixed point. Then $(\Gamma^*, +, O)$ is an abelian group.

Theorem

1. If Γ: $Y = X^3$, then $(\Gamma^*, +) \cong (K, +)$.
2. If Γ: $Y^2 = X^3$, then $(\Gamma^*, +) \cong (K, +)$.
3. If Γ: $Y^2 = X^3 + X^2$, then $(\Gamma^*, +) \cong (K^*, \cdot)$.

Classification theorem

Theorem

In $\mathrm{PG}(2,\mathbb{K})$ defined over an algebraically closed field $\mathbb{K}$ of characteristic $p \geq 0$, let $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$ be a dual 3-net of order $n \geq 4$ which realizes a group G. If either $p = 0$ or $p > n$ then one of the following holds.

(i) G is cyclic or direct product of two cyclic groups and Λ is algebraic.

(ii) G is dihedral and Λ is of tetrahedron type.

(iii) G is the quaternion group of order 8.

(iv) G has order 12 and is isomorphic to A_4.

(v) G has order 24 and is isomorphic to S_4.

(vi) G has order 60 and is isomorphic to A_5.