A NOTE ON THE GROUP OF PROJECTIVITIES OF FINITE PLANES

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Abstract. In this short note we show that the group of projectivities of a projective plane of order 23 cannot be isomorphic to the Mathieu group M_{24}. By a result of T. Grundhöfer [5], this implies that the group of projectivities of a non-desarguesian projective plane of finite order n is isomorphic either to the alternating group A_{n+1} or to the symmetric group S_{n+1}.

1. Introduction

Any projective plane Π can be coordinatized by a planar ternary ring (R, T), see [6]. There is a natural bijection between the set of points of an arbitrary line ℓ and the set $R \cup \{\infty\}$. Let G denote the group of projectivities of Π; then G acts 3-transitively on the point set of ℓ. Equivalently, we can consider the group G of projectivities as a permutation group acting on $R \cup \{\infty\}$.

The fundamental theorem of projective planes says that Π is pappian if and only if G is sharply 3-transitive. In [5], T. Grundhöfer has shown that the group of projectivities of a non-desarguesian projective plane Π of finite order n is either the alternating group A_{n+1}, or the symmetric group S_{n+1}, or $n = 23$ and G is the Mathieu group M_{24}. In this paper, we show that the latter case cannot occur. Our proof uses computer calculations.

2. Coordinate loops and their multiplication groups

For a loop $(L, \cdot, 1)$, we denote by L_x, R_x the left and right translation maps by x, respectively. These maps generate the multiplication group $\mathrm{Mlt}(L)$ of L. The stabilizer of the unit element $1 \in L$ is the inner mapping group $\mathrm{Inn}(L)$ of L. The set of left (or right) translations form a sharply transitive set of permutations. Moreover, for any $x, y \in L$, $L_x R_y L_x^{-1} R_x^{-1} \in \mathrm{Inn}(L)$.

The next result was already noticed by A. Drápal [1] in a slightly weaker form.

Lemma 2.1. *The Mathieu group M_{22} does not contain the multiplication group of a loop.*

Proof. Assume $\mathrm{Mlt}(L) \leq G = M_{22}$, then G contains two sharply transitive subsets U, V of order 22 such that $1 \in U, V$ and for all $u \in U, v \in V$, $uvu^{-1}v^{-1} \in H = G_1$. For any $c \in N_{S_{22}}(G)$ there is an element $w \in V$ such

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that $H^{wc} = H$. Then, the pair $c^{-1}Uc, c^{-1}w^{-1}Vc$ has the same properties as U, V: the commutator element

$$c^{-1}(uw^{-1}vu^{-1}(w^{-1}v)^{-1})c = c^{-1}w^{-1}(wuw^{-1}u^{-1})(uvu^{-1}v^{-1})wc$$

is indeed contained in $H^{wc} = H$. Thus, U can be taken modulo $\mathrm{Aut}(M_{22}) = N_{S_{22}}(G)$. Up to conjugacy by $\mathrm{Aut}(M_{22})$ there are 3 fixed point free elements in G represented by

(1 2 15 14 17 11)(3 8 19 22 9 13)(4 10)(5 7 18)(6 16 12)(20 21),
(1 2 20 3 18 21 9 22)(4 6 19 8 5 11 7 17)(10 15 16 14)(12 13), and
(1 2 9 16 18 22 8 15 10 11 6)(3 7 5 19 17 14 12 21 4 20 13).

These three elements generate G, so they describe the action of G we work with. Pick $1 \neq a \in U$. By the previous remark we may assume that a is one of the given 3 elements. Note that $1^a = 2$. By sharp transitivity of U there are $b, c \in U$ with $1^b = 3$ and $1^c = 4$.

Let F denote the set of fixed point free elements of G and for $X \subset G$ define the set $S_X = \{g \in F \mid xgx^{-1}g^{-1} \in H \ \forall x \in X\}$. Note that if X is a subset of U, then S_X contains V. In particular, for any $i, j \in \{1, \ldots, 22\}$, there is $v \in S_X$ with $i^v = j$. However, a straightforward computer calculation (see the remark below) shows that for any a as above and $b, c \in F$ with $ab^{-1}, bc^{-1}, ca^{-1} \in F$, $1^b = 3$, $1^c = 4$, there exist $i, j \in \{1, \ldots, 22\}$ such that no element of $S_{\{a,b,c\}}$ maps i to j. This proves the lemma. □

With given planar ternary ring (R, T), one can introduce binary operations $x + y = T(1, x, y)$, $x \cdot y = T(x, y, 0)$ in such a way that $(R, +, 0)$ and $(R^* = R \setminus \{0\}, \cdot, 1)$ are loops.

Lemma 2.2. *Let G be the group of projectivities of the projective plane Π. Then, the 2-point stabilizer $G_{0,\infty}$ contains the multiplication group $\mathrm{Mlt}(R^*, \cdot)$ of the multiplicative loop $(R^*, \cdot)$.*

Proof. Easy calculation shows that for any $a \in R^*$, the projectivities

$$\alpha = ([1]\ (0)\ [1,0]\ (\infty)\ [a,0]\ (0)\ [1]),$$
$$\beta = ([1]\ (0,0)\ [a]\ (0)\ [1])$$

map the point $(1, y)$ of $[1]$ to $(1, a \cdot y)$ and $(1, y \cdot a)$, respectively. Moreover, α and β leave the points $(1, 0)$ and (∞) fixed. □

Our main result completes the solution of the conjecture in [2, p. 160].

Theorem 2.3. *The group of projectivities of a non-desarguesian projective plane of finite order n contains the alternating group A_{n+1}.*

Proof. By [5], we only have to exclude the case $n = 23$ and $G = M_{24}$. However, if this case would exist, then by Lemma 2.2, M_{22} would contain the multiplication group of a loop, which contradicts Lemma 2.1. □

We conclude this note with two remarks. First, we notice that both the alternating and the symmetric group can happen to be the group of projectivities of a non-desarguesian finite projective plane, see [4] and the references therein. The second remark concerns the computer calculation in the proof of Lemma 2.1. Let a be one of the 3 possibilities from above, then the number of possibilities for $b \in F$ with $ab^{-1} \in F$ and $1^b = 3$ is 3214, 3290, or 3318, respectively. The sizes of the sets $S_{a,b}$ are between 355 and 538. In the majority of the cases there is a pair $i, j \in \{1, \dots, 22\}$ such that no element from $S_{a,b}$ maps i to j. In the remaining cases one determines the possibilities for c, and shows that the transitivity property of $S_{a,b,c}$ fails again.

The computation takes about 40 minutes on an average home PC. The algorithm was implemented twice independently in the computer algebra systems GAP [3] and Magma [7].

References

[1] A. Drápal. Multiplication groups of loops and projective semilinear transformations in dimension two. J. Algebra 251 (2002), no. 1, 256–278.
[2] P. Dembowski. Finite geometries. Springer-Verlag, Berlin-New York, 1968.
[3] GAP Group. GAP — Groups, Algorithms, and Programming. University of St Andrews and RWTH Aachen, 2002, Version 4r3.
[4] T. Grundhöfer. Die Projektivitätengruppen der endlichen Translationsebenen. J. Geom. 20 (1983), no. 1, 74–85.
[5] T. Grundhöfer. The groups of projectivities of finite projective and affine planes. Eleventh British Combinatorial Conference (London, 1987). Ars Combin. 25 (1988), A, 269–275.
[6] D. R. Hughes and F. C. Piper. Projective planes. Springer-Verlag, New York-Berlin, 1973.
[7] Wieb Bosma, John Cannon, and Catherine Playoust. The Magma algebra system. I. The user language. J. Symbolic Comput., 24(3-4):235-265, 1997.

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