ALGEBRAIC BOL LOOPS

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Abstract. In this paper, we study the category of algebraic Bol loops over an algebraically closed field of definition. On the one hand, we apply techniques from the theory of algebraic groups in order to prove structural theorems for this category. On the other hand, we present some examples showing that these loops lack some nice properties of algebraic groups; for example, we construct local algebraic Bol loops which are not birationally equivalent to global algebaric loops.

1. Introduction

It is well known that every loop L may be realized using the so called *Baer correspondence*. Let G be a group, $H \leq G$ a subgroup and M a subset of G such that $1 \in M$ and the following holds: for all $g \in G$, M is a system of representatives of the right cosets of H^g in G. In other words, every element $x \in G$ can be uniquely decomposed as $x = h^g m$ with $h \in H$ and $m \in M$. In this case we can define a product $* : M \times M \to M$ by $m * n = k$ where $m, n, k \in M$ and $Hmn = Hk$. The triple (G, H, M) is called the *loop folder* of $L = (M, *)$ and G is the *enveloping group* of L. For any loop L we can construct a loop folder (G, H, M) such that $L \cong (M, *)$. Let M be the set of right multiplication maps $R_x : y \to yx$ of L, $G = \langle M \rangle$ and the H the stabilizer of 1 in G. Then (G, H, M) is indeed a loop folder with the extra properties that M generates G and H is core-free. The group generated by the right multiplications of L is the *right multiplication group* $\mathrm{RMlt}(L)$ os L.

Let G be an algebraic group over an algebraically closed field k with closed subgroup H and closed subset M and assume that for each conjugate H^g of H in G, the map

$$H^g \times M \to G, \qquad (h, m) \mapsto hm$$

is a biregular morphism. Then the triple (G, H, M) is an *algebraic loop folder* and the corresponding loop L is a *strongly algebraic* loop.

There is a more natural definition of the concept of algebraic loops, see [9]. A loop L is *algebraic* if L is an algebraic variety over an algebraically closed field k with regular morphisms

$$m : L \times L \to L, \quad \phi : L \times L \to L, \quad \psi : L \times L \to L,$$

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such that the identities

$$x = m(e,x) = m(x,e) = m(y,\phi(y,x)) = m(\psi(x,y),y) \tag{1}$$

hold for all $x, y \in L$ and some fixed $e \in L$. In this case $m(x,y) = x \cdot y$ is the loop product and $\psi(x,y) = x/y$, $\phi(x,y) = y\backslash x$ are the right and left divisions, respectively.

If the morphisms m, ψ, ϕ are well defined rational maps from $L \times L \to L$ such that the identities (1) hold on a Zariski-open subset of $L \times L$ then we shall call L *local algebraic loops.* If only the regular morphism $m : L \times L \to L$ is defined such that $x = m(e,x) = m(x,e)$ then we shall call L a *weakly algebraic loop.*

In this paper, we examine the class of *algebraic right Bol loops,* that is, algebraic loops which satisfy the *right Bol identity*

$$((xy)z)y = x(y(zy)).$$

We explain the relations between the classes of algebraic, stongly algebraic and local algebraic Bol loop. We will show some structure theorems and give many examples.

2. Algebraic vs. strongly algebraic loops

One of the main questions in the theory of algebraic loops for a given class of loops is the equivalence of the notion of algebraic and strongly algebraic loops.

It is known that via the *localization process*, any algebraic group determines a formal group this section. This method works for the class of local algebraic loops, as well, see [10]. A formal algebraic loop over the field k a system

$$\boldsymbol{\mu}(\boldsymbol{X},\boldsymbol{Y}) = (\mu^i(X_1,\dots,X_n,Y_1,\dots,Y_n)), \qquad i = 1,\dots,n$$

of formal power series in $2n$ variables over k such that the identities

$$\boldsymbol{\mu}(\boldsymbol{X},\boldsymbol{0}) = \boldsymbol{\mu}(\boldsymbol{0},\boldsymbol{X}) = \boldsymbol{X}$$

hold. The integer n is the dimension of the formal loop. If the formal loop $\boldsymbol{\mu}$ is the localization of a local Bol loop, then it clearly satisfies the *formal Bol identity*

$$\boldsymbol{\mu}(\boldsymbol{X},\boldsymbol{\mu}(\boldsymbol{\mu}(\boldsymbol{Y},\boldsymbol{Z}),\boldsymbol{Y})) = \boldsymbol{\mu}(\boldsymbol{\mu}(\boldsymbol{\mu}(\boldsymbol{X},\boldsymbol{Y}),\boldsymbol{Z}),\boldsymbol{Y}).$$

Moreover, any algebraic automorphism of an algebraic loop induces an automorphism of the associated formal loop.

A finite dimensional $\mathbb{R}$-vector space B with trilinear operation $(.,.,.)$ and bilinear operation $[.,.]$ is a *Bol algebra* if

$$(x,y,y) = 0, (x,y,z) + (y,z,x) + (z,x,y) = 0$$
$$((x,a,b),y,z) + (x,(y,a,b),z) + (x,y,(z,a,b)) = ((x,y,z),a,b)$$
$$([x,y],a,b) = [(x,a,b),y] - [x,(y,a,b)] + ([a,b],x,y) + [[a,b],[x,y]]$$

holds for all $x, y, z, a, b \in B$. L. Sabinin [16] developed a complete theory for local differentiable Bol loops. [16, 5.34 Proposition] says that local differentiable right Bol loops are functorially equivalent to Bol algebras. This functorial equivalence works perfectly between finite dimensional Bol k-algebras and formal Bol loops over fields of characteristic 0. In particular, the automorphisms of a formal Bol loop over a field of characteristic 0 correspond biuniquely to linear automorphisms of the tangent Bol k-algebra.

Let X be a variety and G be a group consisting of algebraic tranformations of X. We define connectedness and dimension of G as in [14].

Lemma 2.1. *Let L be a global algebraic Bol loop over an algebraically closed field k of characteristic 0 and G a connected group consisting of algebraic automorphisms of L. Then G is biregularly isomorphic to a closed subgroup of $GL_n(k)$ where $n = \dim(L)$. In particular, G is finite dimensional and has a unique structure of an algebraic transformation group on L.*

Proof. Let α be an algebraic automorphism of L and denote by $\boldsymbol{\mu}(\boldsymbol{X}, \boldsymbol{Y})$ the formal Bol loop associated to L. As $\alpha(e) = e$, it has a localization $\boldsymbol{\alpha}(\boldsymbol{T})$ which is a formal automorphism of $\boldsymbol{\mu}$. The action of α on the tangent Bol algebra is given by the Jacobian $(\frac{\partial \boldsymbol{\alpha}^i}{\partial T^j}(\mathbf{0}))$ of $\boldsymbol{\alpha}$. Hence, we have an algebraic embedding φ of G into $GL_n(k)$. Let us define the action of G on $GL_n(k)$ by $X^g = X\varphi(g)$. By [14, Lemma 2], the orbit of 1 is a locally closed subvariety of $GL_n(k)$. On the one hand, this orbit is precisely $\mathrm{Im}\varphi$. On the other hand, $\overline{\mathrm{Im}\varphi} = \mathrm{Im}\varphi$ by [5, Proposition 7.4.A]. □

The main result of this sections is the following.

Theorem 2.2. *Let L be a connected algebraic Bol loop over a field k of characteristic 0. Then the right multiplication group $\mathrm{RMlt}(L)$ of L is a connected algebraic group; in particular, L is a strongly algebraic loop.*

Proof. For any $x \in L$, we define the algebraic transformation $\alpha_x = (R_x^{-1}, L_x R_x)$ on $L \times L$. Let G be the group generated by the connected algebraic family $\{\alpha_x \mid x \in L\}$, then G is itself connected. It is easy to see that any element (β_1, β_2) of G can be uniquely extended to an autotopism $(\beta_1, \beta_2, \beta_3)$ of L. Hence, the stabilizer $G_{(e,e)}$ of $(e, e) \in L \times L$ is contained in $\mathrm{Aut}(L)$. We show that G is finite dimensional of dimension at most $n^2 + 2n$ where $n = \dim(L)$. Let $\{\varphi_t \mid t \in T\}$ an injective family of elements of G with connected variety T of dimension $N > n^2 + 2n$. By [14, Lemma 2], $X = \{\varphi_t(e, e) \mid t \in T\}$ is a locally closed subvariety of $L \times L$. The set $\{t \in t \mid \varphi_t(e, e) = (e, e)\}$ is a closed subvariety of T, let T_0 be a connected component of maximal dimension. As $\dim T_0 + \dim X = \dim T$ and $\dim X \leq 2n$, we have $\dim T_0 > n^2$. However, $\{\varphi_t \mid t \in T_0\}$ is a connected injective algebraic family in $\mathrm{Aut}(L)$, hence a subset of $GL_n(k)$ by Lemma 2.1, a contradiction. The main theorem of [14] implies the claimed result. □

Clearly, if $\mathrm{RMlt}(L)$ is an algebraic transformation group on L, then L can be given by the algebraic loop folder (G, H, K) where $G = \mathrm{RMlt}(L)$, $H = \mathrm{RInn}(L)$ and $K = \{R_x \mid x \in L\}$. Indeed, the decomposition $G \to H \times K$, $g \mapsto hR_x$ with $x = e^g$, $h = gR_x^{-1}$ is a biregular bijection between G and $H \times K$. This implies that in this case, L is strongly algebraic. Conversely, let L be given by a connected algebraic loop folder (G, H, K). We do not destroy the algebraic property of the folder by assuming that H does not contain a proper normal subgroup of G. Then, by identifying L with the coset space G/H, G can be seen as an algebraic transformation group acting on L. Moreover, every right translation of L will be contained in G. Since K is connected, it generates a closed connected subgroup of G, hence $\mathrm{RMlt}(L)$ is a connected algebraic transformation group.

Corollary 2.3. *Let L be an algebraic Bol loop over an algebraically closed field k of characteristic* 0*. Then L is a strongly algebraic loop.*

Unfortunately, Lemma 2.1 does not hold when $\mathrm{char}(k) > 0$. More precisely, a connected group of automorphisms of L can have infinite dimension. The rest of the proof works fine. Therefore we have the following

Conjecture. *Let L be a connected algebraic Bol loop over an algebraically closed field k. Then* $\mathrm{RMlt}(L)$ *is an algebraic transformation group. In particular, every algebraic Bol loop is strongly algebraic.*

From the proof of Theorem 2.2 follows that in order to show the strong algebraic property, it is sufficient to study the right inner mapping group of an algebraic Bol loop.

Proposition 2.4. *Let L be an algebraic Bol loop over an algebraically closed field k and assume that the right inner mapping group $H = \mathrm{RInn}(L)$ of L is finite dimensional. Then L is strongly algebraic.*

3. Simple algebraic and local algebraic Bol loops

Throughout this section k denotes an algebraically closed field. As all known algebraic Bol loops are strongly algebraic, in the following examples, we will often skip the adjective "strongly".

It is known that given any algebraic group G with closed normal subgroup N, one can give the abstract group G/N the structure of (affine) algebraic group, see [5, Section 11 and 12]. This problem is rather subtle already for algebraic groups, and in general the solution is not known for algebraic loops. The next theorem gives a solution for strongly algebraic loops, that is, for loops given by algebraic loop folders. The normality condition for loop folders was given in [1, 2.6]. A subfolder (G_0, H_0, K_0) corresponds to a normal subloop if and only if

(NC) for each $g \in G$, $k_0 \in K_0$ and $k \in K$, $k_0 k = l_0 k'$ for some $l_0 \in H^g \cap G_0$ and $k' \in K$.

In particular, $K_0 K = H_0 K$ holds.

Theorem 3.1. *Let (G, H, K) be an algebraic loop folder with corresponding loop L. Let N be a closed normal subloop of L. Then there is an algebraic loop folder $(\bar{G}, \bar{H}, \bar{K})$ such that the corresponding algebraic loop $\bar{L}$ is isomorphic to the abstract factor loop L/N. Moreover, the natural homomorphism $L \to \bar{L} = L/N$ is a regular morphism. The algebraic loop $\bar{L}$ is unique up to algebraic isomorphism.*

Proof. We assume w.l.o.g. that $\mathrm{core}_G(H) = 1$ and identify the homogenous space G/H with L. Let H_1 denote the stabilizer of the the closed set $N \subseteq L$; $H_1 \leq G$ is closed by [5, Proposition 8.2]. $G_0 = \mathrm{core}_G(H_1) = \cap_{g \in G} H_1^g$ is an intersection of closed sets, hence is a closed normal subgroup of G. Write $H_0 = G_0 \cap H$, $K_0 = G_0 \cap K$ for the closed subsets of G. (G_0, H_0, K_0) is the normal subfolder corresponding to the abstract loop homomorphism

$L \to L/N$. By the normality condition (NC), $K_0K = H_0K$, thus, $G_0K = H_0K_0K = H_0K$. As $H_0 \cap K = 1$, this means that the subset $K_1 = G_0K$ of G is biregularly isomorphic to the subvariety $H_0 \times K$ of $H \times K$. In particular, G_0K is closed in G, since the varieties G and $H \times K$ are biregularly equivalent.

Let φ be the natural homomorphism $G \to \hat{G} = G/G_0$ and define $\bar{H} = \varphi(H)$ and $\bar{K} = \varphi(K)$. The loop homomorphism $L \to L/N$ corresponds to an abstract folder homomorphism $\varphi : (G, H, K) \to (\bar{G}, \bar{H}, \bar{K})$. In order to see that L/N is algebraic, we have to show that $\bar{H}, \bar{K}$ are closed in $\hat{G}$. Indeed, the respective preimages $H_1 = G_0H$ and $K_1 = G_0K$ of $\hat{H}$ and $\bar{K}$ are closed in G. As $\hat{G} = G/G_0$ is endowed with the quotient topology (cf. [5, Section 12]), $\hat{H}$, $\hat{K}$ are closed. This completes the proof. □

The (strongly) algebraic loop L is said to be *simple* if it has no proper closed normal subloops. The most important example of strongly algebraic Bol loops is the Paige loop $M(k)$, for the definition see [15]. It is known that $M(k)$ is a nonassociative simple Moufang loop, its multiplication group is the projective orthogonal group $P\Omega_8^+(k)$.

In the remainder of this section, we give examples of simple algebraic Bol loops. Most of the examples are constructed from an exact factorization $G = AB$ of the group G. Briefly said, $G = AB$ is an exact factorization of G if A, B are subgroups such that $G = AB$ and $A \cap B = 1$. Then the triple $(G \times G, A \times B, K)$ is a Bol loop folder with $K = \{(x, x^{-1}) \mid x \in G\}$; for details see [11]. This construction gives many proper simple Bol loops. However, the known conditions for the simplicity of the associated loop were rather complicated. We now give a sufficient condition which covers almost all known cases.

Proposition 3.2. *Let $G = AB$ be an exact factorization of the group G and let L be the corresponding Bol loop. Assume that*

(i) $G' = G$,
(ii) $Z(G) = 1$,
(iii) $\mathrm{core}_G(A) = \mathrm{core}_G(B) = 1$,
(iv) *A is maximal in G, and*
(v) *the normal closure of B in G is G.*

Then L is a simple loop.

Proof. By (i) and (iii), the folder $(G \times G, A \times B, K)$ is faithful and any loop homomorphism corresponds to a surjective morphism $\varphi : (G \times G, A \times B, K) \to (\bar{G}, \bar{H}, \bar{K})$ of loop folders; let (G_0, H_0, K_0) be the kernel of φ, then $G_0 \lhd G$. We can assume without loss of generality that $\mathrm{core}_{\bar{G}}(\bar{H}) = 1$, that is, $\mathrm{core}_{G\times G}(G_0(A \times B)) = G_0$. Since $[G_0, G \times 1] \le G \times 1$, we can write $[G_0, G \times 1] = U \times 1$ with a normal subgroup U of G. Assume first $U = 1$, then by $Z(G) = 1$, we have $G_0 = 1 \times V$ with a normal subgroup V of G. As $K_0 = G_0 \cap K = 1$, $G_0 = H_0K_0 = H_0 \le H$ and $V \le B$, which contradicts to $\mathrm{core}_G(B) = 1$.

We can therefore assume $U \ne 1$. By the maximality of A, $G = AU$ and

$$G \times 1 \le G \times B = (U \times 1)(A \times B) \le G_0(A \times B).$$

Then $G \times 1 \leq \mathrm{core}_{G\times G}(G_0(A \times B)) = G_0$, which implies that G_0 has the form $G_0 = G \times V$ for some $V \lhd G$. Then $\bar{G} = (G \times G)/N \cong G/V$ and $\bar{K} = KG_0/G_0 = (G \times G)/N = \bar{G}$ by $KG_0 = G \times G$. Moreover, as $G_0(A \times B) = G \times VB$, we have $\bar{H} \cong VB/V$. Now, the triple $(G/V, VB/V, G/V)$ is a loop folder if and only if $VB/V = 1$, that is, if and only if $B \leq V$. By assumption (v), $V = G$ and the image of φ is trivial. □

If G is an algebraic group and A, B are closed subgroups then the resulting loop will be a strongly algebraic Bol loop.

Simple algebraic Bol loops from exact factorizations. The next construction yields a non-Moufang simple algebraic Bol loop. Let G be the semidirect product of $SL_2(k)$ and k^2. We can represent G by 3×3 matrices:

$$G = \left\{ \begin{pmatrix} a_{11} & a_{12} & x_1 \\ a_{21} & a_{22} & x_2 \\ 0 & 0 & 1 \end{pmatrix} \mid a_{ij}, x_i \in k, a_{11}a_{22} - a_{12}a_{21} = 1 \right\}.$$

Clearly, G is a connected algebraic group of dimension 5 with $Z(G) = 1$. Since $SL_2(k)$ acts irreducibly on k^2, the only connected normal subgroup of G is $N_0 = k^2$.

We define the subgroups

$$A = SL_2(k) = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a_{ij} \in k, a_{11}a_{22} - a_{12}a_{21} = 1 \right\}$$

and

$$B = \left\{ \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} \mid x_1, x_2 \in k \right\} \cong k^2$$

of G. We have $AB = G$, $A \cap B = 1$ and $\mathrm{core}_G(A) = \mathrm{core}_G(B) = 1$, that is, A and B do not contain proper normal subgroups of G.

By Proposition 3.2, the Bol loop L corresponding to this exact factorization is simple. Notice that this is true for *any* field k; even for $k = \mathbb{F}_2$ and $k = \mathbb{F}_3$, when $SL_2(k)$ (and G) are solvable. In these two cases, the proof needs some more attention since $SL_2(k)' \neq SL_2(k)$. Furthermore, we notice that the loop corresponding to the case $k = \mathbb{F}_2$ has order 24, it is isomorphic to the loop given in [11, Example II].

The local hyperbolic plane loop. By Weil's theorem [17], any local algebraic group is birationally equivalent to an algebraic group. In this section, we construct a local algebraic Bol loop and prove that it is not birationally equivalent to a global algebraic loop.

The translations of the hyperbolic plane are defined as products of two central symmetries; the set of hyperbolic translations forms a sharply transitive set on the hyperbolic plane, the associated loop is the classical simple Bruck loop. An elegant representation of this loop was given in [8] by the operation

$$x \cdot y = \frac{x + y}{1 + x\bar{y}}$$

on the unit disc $\{z \in \mathbb{C} \mid |z| < 1\}$. Formal expansion using $x = x_1 + ix_2$, $y = y_1 + iy_2$ gives the formal operation

$$(x_1, x_2) \cdot (y_1, y_2) = (z_1, z_2)$$

with

$$(2) \qquad \begin{cases} z_1 = \dfrac{x_1 + x_1^2 y_1 + y_1 + x_1 y_1^2 + 2y_1 x_2 y_2 + x_2^2 y_1 - y_2^2 x_1}{1 + 2x_1 y_1 + 2x_2 y_2 + x_1^2 y_1^2 + x_2^2 y_2^2 + x_1^2 y_2^2 + x_2^2 y_1^2}, \\ z_2 = -\dfrac{-x_1^2 y_2 - 2x_1 y_1 y_2 + x_2 y_1^2 - x_2 - x_2^2 y_2 - y_2 - y_2^2 x_2}{1 + 2x_1 y_1 + 2x_2 y_2 + x_1^2 y_1^2 + x_2^2 y_2^2 + x_1^2 y_2^2 + x_2^2 y_1^2}. \end{cases}$$

This operation defines a simple local algebraic right Bruck loop on k^2 for any field k. The unit element is $(0, 0)$ and the inverse of (x_1, x_2) is $(-x_1, -x_2)$. Straightforward calculation gives that the right inner map $R_{(y_1,y_2),(z_1,z_2)}$ is

$$(x_1, x_2) \mapsto (ax_1 + bx_2, -bx_1 + ax_2),$$

where

$$a = \frac{z_2^2 y_2^2 + 2z_2 y_2 - z_2^2 y_1^2 + 4z_1 z_2 y_1 y_2 - z_1^2 y_2^2 + z_1^2 y_1^2 + 2z_1 y_1 + 1}{1 + 2z_1 y_1 + 2z_2 y_2 + z_1^2 y_1^2 + z_2^2 y_2^2 + z_2^2 y_1^2 + z_1^2 y_2^2},$$

$$b = \frac{2z_1 z_2 y_2^2 + 2z_1^2 y_1 y_2 - 2z_2^2 y_1 y_2 - 2z_1 z_2 y_1^2 - 2y_1 z_2 + 2z_1 y_2}{1 + 2z_1 y_1 + 2z_2 y_2 + z_1^2 y_1^2 + z_2^2 y_2^2 + z_2^2 y_1^2 + z_1^2 y_2^2}.$$

Moreover, $a^2 + b^2 = 1$ holds identically. Thus, the right inner maps are contained in a 1-dimensional algebraic group H acting on k^2.

We claim that this local loop is not birationally equivalent to an algebraic loop. Let us assume that $(L, \cdot)$ is an algebraic loop such that $\alpha : k^2 \to L$ is a birational isomorphism. Then $\mathrm{RInn}(L)$ has the structure of a 1-dimensional algebraic transformation group on L. By Proposition 2.4, $G = \mathrm{RMlt}(L)$ is a 3-dimensional algebraic transformation group. Moreover, as L is a simple Bruck loop, G is a simple group, hence $G \cong PSL(2, k)$. Any simple Bruck loop can be given by a loop folder (G, H, K) where $H = C_G(\sigma)$ and $K = \{g \in G \mid g^\sigma = g^{-1}\}$ for an involutorial automorphism σ of G, cf. [4]. It is easy to check that $PSL_2(k)$ has no such automorphism. This proves that (2) indeed defines a proper local algebraic Bol loop.

4. Algebraic solvable Bol loops

In this section, we investigate the relation between solvable (strongly) algebraic groups and and algebraic loop folders (G, H, K) with solvable group G. We first show that the Jordan decomposition is well-defined in the class of power-associative strongly algebraic loops.

Proposition 4.1. *Let L be a connected power-associative strongly algebraic loop. If $x \in L$, there exist unique elements $s, u \in L$ such that: $R_x = R_s R_u$, s and u are contained in a closed Abelian subgroup of L, R_s is semisimple and R_u is unipotent in* $\mathrm{RMlt}(L)$. *If $\varphi : L \to \bar{L}$ is a morphism of strongly algebraic loops then $\varphi(x)_s = \varphi(x_s)$ and $\varphi(x)_u = \varphi(x_u)$.*

Proof. Let L be given by a faithful algebraic loop folder (G, H, K) with $G = \mathrm{RMlt}(L)$. Since L is power-associative and K closed in G, R_x is contained in a closed Abelian subgroup U of G such that $U \subseteq K$. Let $R_x = s_0 u_0$

be the unique Jordan decomposition of R_x in U. As U is contained in K, there are unique elements $s, u \in L$ such that $s_0 = R_s$, $u_0 = R_u$. Finally, the set $\{y \in L \mid R_y \in U\}$ is a closed Abelian subgroup of L, which contains s, u. The last assertion follows from the fact that morphisms of strongly algebraic loops are equivalent to morphisms of algebraic loop folders, see Theorem 3.1. □

Now, we are able to prove the Lie-Kolchin theorem for strongly algebraic Bol loops.

Theorem 4.2. *Let L be a connected strongly algebraic Bol loop and assume that* $\mathrm{RMlt}(L)$ *is solvable. Then L has a closed connected solvable normal subloop L_u consisting of the unipotent elements of L. The factor loop L/L_u is a torus. In particular, L is solvable.*

Proof. Let L be given by the faithful algebraic loop folder (G, H, K) with $G = \mathrm{RMlt}(L)$. Let U be the unipotent radical of G and put $U_1 = HU$. We claim that $U_1 \cap K \subseteq U$. Take an abritrary $R_x \in U_1 \cap K$, $R_x = R_s R_u$ its Jordan decomposition with $R_s, R_u \in U_1 \cap K$. By the Lie-Kolchin theorem, we have the decomposition $H = H_s H_u$ of the solvable algebraic group H; thus, $U_1 = H_s U$ and H_s is a maximal torus in U_1. This implies that R_s is conjugate to an element of H_s, hence $R_s = 1$ and $R_x = R_u \in U$. Clearly, $(U_1, U_1 \cap H, U_1 \cap K)$ determines a closed subfolder of (G, H, K). Moreover, as it satisfies the normality condition (NC), the corresponding subloop N is normal in L. By $U_1 \cap K = U \cap K$, N consists precisely of the unipotent elements of L. The factor loop L/N has the algebraic loop folder $(G/HU, 1, G/HU)$, thus $L/N \cong G/HU$ is a torus.

In order to show the solvability of L, it remains to deal with the case when L consists of unipotent elements. Then $K \subseteq G_u$ and $G = \langle K \rangle = G_u$ can be assumed. In this case, H is contained in a proper closed normal subgroup M of G and the surjective morphism $(G, H, K) \to (G/M, 1, G/M)$ of algebraic loop folders corresponds to a surjective morphism $L \to G/M$ of algebraic loops. □

After this result, it is natural to ask about the structure of algebraic Bol loop folders (G, H, K) where G is a connected unipotent group. We are able to handle only a rather special situtation.

Proposition 4.3. *Let k be an algebraically closed field of characteristic 0. Let L be a connected strongly algebraic Bol loop over k with loop folder (G, H, K). Assume that G is a connected unipotent group of nilpotency class 2. Then* $\dim(Z(L)) > 0$. *In particular, L is nilpotent.*

Proof. Since G is 2-divisible and has nilpotency class 2, the operation $x+y = x^{1/2} y x^{1/2}$ defines a commutative algebraic group on G with closed connected subgroups H and K. Thus, G can be coordinatized such that

$$\begin{aligned} H &= \{(x_1, \ldots, x_{n_1}, 0, \ldots, 0) \mid x_i \in k\}, \\ K &= \{(0, \ldots, 0, x_1, \ldots, x_{n_2}) \mid x_i \in k\}, \\ & n = n_1 + n_2 = \dim G, \end{aligned}$$

and the 1-parameter subgroups have the form $\{tg \mid t \in k\}$. If $x \in K \cap Z(G)$ then $tx \in K \cap Z(G)$ for all $t \in k$.

Let us first assume that $K \cap Z(G) \neq 1$ and (G, H, K) is faithful, that is, $G = \mathrm{RMlt}(L)$. Take an arbitrary element $R_x \in K \cap Z(G)$. Then $R_x R_y = R_y R_x$ implies $xy = yx$ for all $y \in L$. As L is 2-divisible, that is, $R_x = (R_{x^{1/2}})^2$ for some $R_{x^{1/2}} \in K \cap Z(G)$, we have $R_x R_y = R_{x^{1/2}} R_y R_{x^{1/2}} = R_{(x^{1/2}y)x^{1/2}}$. Thus, $x \in N_\mu = N_\rho$. These properties imply $x \in Z(L)$; in particular, $\dim(Z(L)) = \dim(K \cap Z(G)) > 0$.

Let us now suppose that (G, H, K) is an algebraic Bol loop folder such that

(i) G is connected with nilpotency class 2,
(ii) $K \cap G' = 1$,
(iii) the dimension of G is minimal.

Assumption (iii) implies $G = \mathrm{RMlt}(L)$ for the associated Bol loop L. Let M be a closed connected normal subgroup containing H and put $K_0 = M \cap K$. As (M, H, K_0) is an algebraic Bol loop folder satisfying (i) and (ii), M has to be commutative by $\dim M < \dim G$; hence, $M \cong k^{n-1}$. Fix an element $g_0 \in G \setminus M$ and denote the map $H \to G' \leq M$, $h \mapsto [h, g_0]$ by β. We define the subset

$$Y = \bigcup_{g \in G} H^g = \bigcup_{t \in k} H^{tg_0} = \{h[tg_0, h] \mid h \in H, t \in k\}$$

of M. As M is a vector group, we change to additive notation: $Y = \{h + t\beta(h) \mid h \in H, t \in k\}$. The map $\beta : H \to G'$ is additive and algebraic, hence linear. Moreover, $H \cap G' = 0$ by $\mathrm{core}_G(H) = 1$. Clearly, for any $y \in Y$ and $s \in k$, $sy \in Y$. Thus, Y determines a point set in the projective space $\mathbb{P}_{n-2}$ given by M. The homogenization of Y is

$$\bar{Y} = \{sh + t\beta(h) \mid h \in H, s, t \in k\},$$

which is a closed projective variety in $\mathbb{P}_{n-2}$. Indeed, it is the morphical image of the complete variety $\mathbb{P}_1 \times \mathbb{P}_{n_1-1}$, hence closed by [5, Proposition 6.1(c)]. The projective dimension of $\bar{Y}$ is n_1. The subspace K_0 of M determines an $(n_2 - 2)$-dimensional projective subspace of $\mathbb{P}_{n-2}$. Thus, $\mathrm{prdim}(\bar{Y}) + \mathrm{prdim}(K_0) = n - 2$, the projective varieties $\bar{Y}$ and K_0 have at least one projective point $x \in M$ in common. The elements of Y are conjugates to elements of H, therefore $Y \cap K_0 = \emptyset$ and $x \in \bar{Y} \setminus Y$. Since the elements of $\bar{Y} \setminus Y$ correspond to the parameter $s = 0$, $\bar{Y} \setminus Y \subseteq [g_0, H] \subseteq G'$. This contradicts to the assumption $K \cap G' = 1$. □

We notice that the above result does not hold if k is not algebraically closed, for examples see [7]. Moreover, it is not clear what the nilpotency class of L can be. The construction in [2, Example VII.5.3] gives a strongly algebraic Moufang loop of nilpotency class 3 such that the right multiplication group is nilpotent of class 2.

We formulate the following open question:

Problem. *Let L be a connected strongly algebraic Bol loop over an algebraically closed field k with loop folder (G, H, K) such that G is a connected unipotent group. Is it true that* $\dim(Z(L)) > 0$*?*

5. Constructions of solvable algebraic Bol loops

In this class of examples, we assume that G is an algebraic group over k which is a semidirect product of the connected algebraic groups A and B; $G = A \rtimes B$. Clearly, $G = AB$ is an exact factorization. Explicit calculation shows that the resulting Bol loop L is isomorphic to the split extension constructed by Johnson and Sharma [6]. In particular, if the action of B on A is not Abelian then L is non-Moufang and non-Bruck, see [6, Theorem 2].

Take $A = k^n$ and $B = T_n(k)$ the group of $n \times n$ upper triangular matrices; then $G = AB$ is solvable. The following proposition says that the associated Bol loop L is solvable.

Proposition 5.1. *Let $G = AB$ be an exact factorization and $N \leq A$ is normal in G. Define $K = \{(x, x^{-1}) \mid x \in G\} \subseteq G \times G$ and $\bar{K} = \{(xN, x^{-1}N) \mid xN \in G/N\} \subseteq G/N \times G/N$. Then the following hold:*

(i) *$\varphi : (G \times G, A \times B, K) \to (G/N \times G/N, A/N \times B, \bar{K})$ is a surjective morphism of loop folders. The kernel of φ is the normal subfolder $(N \times N, N \times 1, K_N)$ with $K_N = \{(x, x^{-1}) \mid x \in N\}$. The corresponding normal subloop is isomorphic to the group N.*

(ii) *If $N \leq Z(G) \cap A$ then $\ker \varphi \leq Z(L)$, where L is the Bol loop associated to the exact factorization $G = AB$.*

Proof. We first notice that it is meaningful to speak of the subgroup $A/N \times B$ of $G/N \times G/N$, since $B \cap N = 1$ implies that the image of B in G/N is isomorphic to B. Moreover, G/N is has an exact factorization with subgroups $A/N, B$. We leave to the reader to check that φ is indeed a morphism of loop folders with kernel $(N \times N, N \times 1, K_N)$. The corresponding loop is precisely the group N. This shows (i). To see (ii), assume $N \leq Z(G) \cap A$ and take an arbitrary element (n, n^{-1}) of the transversal belonging to $\ker \phi$. Then $(n, n^{-1})K = K$ implies that the corresponding loop element $x \in L$ is contained in the right and middle nucleus $N_\rho(L) = N_\mu(L)$. Furthermore, by $(n, n^{-1}) \in Z(G \times G)$, the associated loop element x commutes with all elements of L. Thus, $x \in Z(L)$. □

We mention that for solvable algebraic $G = AB$, the solvability of the corresponding Bol loop follows from Theorem 5.1, as well. However, Proposition 5.1 is also useful for the construction of non-Moufang nilpotent algebraic Bol loops. In fact, if A and B are nilpotent groups and $B \leq \mathrm{Aut}(A)$ is not Abelian, then by Proposition 5.1(ii), L is nilpotent.

Finally, we mention that many examples of nilpotent algebraic nonassociative Bruck and Moufang loops are known. For the Moufang case, see [2, Example VII.5.3]. For nilpotent algebraic Bruck loops, one has to consider the operation $x \circ y = x^{\frac{1}{2}} y x^{\frac{1}{2}}$ on any unipotent group G with $\mathrm{char}(k) \neq 2$, cf. [12, Section 12].

A local algebraic solvable Bol loop. We finish this section by constructing a local algebraic solvable Bol loop and show that it is not birationally equivalent to a global algebraic loop if $\mathrm{char}(k) = 0$.

Let k be an algebraically closed field of characteristic $\neq 2$ and define the operation

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1 + y_1, x_2 + y_2 + \frac{x_1 y_2 - y_1 x_2}{2 + y_1 + z_1})$$

on k^2. This defines a local algebraic Bol loop by formal calculation; the inverse of (x_1, x_2) is $(-x_1, -x_2)$. The right inner map $R_{(y_1,y_2),(z_1,z_2)}$ maps

$$(x_1, x_2) \mapsto (x_1, x_2 + \frac{2x_1(y_2 z_1 - y_1 z_2)}{(2 + x_1)(2 + x_1 + y_1)}).$$

This implies that all right inner maps are contained in the 1-parameter group of transformations

$$\{u_t : (x_1, x_2) \mapsto (x_1, x_2 + t\frac{x_1}{2 + x_1}) \mid t \in k\}.$$

Assume that this local algebraic loop is birationally equivalent to a 2-dimensional algebraic loop L. By Proposition 2.4, L is strongly algebraic with an algebraic loop folder (G, H, K) where $\dim G = 3$. Clearly, $Z(L)$ has dimension 0. We show that $\mathrm{RMlt}(L)$ is nilpotent of class 2; this contradicts to Proposition 4.3 if $\mathrm{char}(k) = 0$.

Lemma 5.2. *Let $(L, \cdot)$ be a right Bol loop and define the core $x + y = (yx^{-1})y$ of L. Then the following are equivalent.*

(i) *For all $x, y, z \in L$, $((x + y) + 1) + z = ((x + z) + 1) + y$.*
(ii) *The group M generated by the maps $P_y = L_y R_y$, $y \in L$, is Abelian.*
(iii) *The group Γ generated by the autotopisms $\alpha_y = (R_y^{-1}, L_y R_y, R_y)$, $y \in L$, is nilpotent of class at most 2.*
(iv) $\mathrm{RMlt}(L)$ *is nilpotent of class at most 2.*

Proof. We have

$$xP_yP_z = (((x + 1) + y) + 1) + z \quad \text{and} \quad xP_zP_y = (((x + 1) + z) + 1) + y,$$

hence (i) implies (ii). The projection pr_2 maps Γ onto M, the kernel consists of autotopisms of the form $(L_n, 1, L_n)$ with $n \in N_\lambda$. As for $n \in N_\lambda$, L_n centralizes $\mathrm{RMlt}(L)$, $\ker \mathrm{pr}_2 \leq Z(\Gamma)$. Thus, (ii) implies (iii). Since $\mathrm{RMlt}(L)$ is a homomorphic image of Γ, from (iii) follows (iv). Finally, $R_x + R_y = R_y R_x^{-1} R_y = R_{x+y}$ shows that $y \mapsto R_y$ is an embedding of $(L, +)$ into the core of $\mathrm{RMlt}(L)$. The identity in (i) can be easily shown for groups of nilpotency class 2. □

It is easy to check that

$$((x_1, x_2) \cdot (-y_1, -y_2)) \cdot (x_1, x_2) = (2x_1 - y_1, 2x_2 - y_2),$$

and the core of this local Bol loop satisfies the identity of Lemma 5.2(i). The same is true for the core of L, which implies that $\mathrm{RMlt}(L)$ is indeed nilpotent of class at most 2.

References

[1] M. Aschbacher. On Bol loops of exponent 2. *J. Algebra* **288,** No. 1, 99-136 (2005).
[2] R. H. Bruck. A survey of binary systems. Springer-Verlag (1958).
[3] S. Doro. Simple Moufang loops. *Math. Proc. Camb. Phil. Soc.*, **83**, (1978), 377-392.
[4] G. Glauberman. On loops of odd order. J. Algebra 1 1964 374–396.

[5] J. E. Humphreys. Linear algebraic groups. Springer-Verlag, New York-Heidelberg, 1975.
[6] K. V. Johnson and B. L. Sharma. On a family of Bol loops. Boll. Un. Mat. Ital. Suppl. 1980, no. 2, 119–126.
[7] M. K. Kinyon and G. P. Nagy. Bol loops in 2-step nilpotent groups. In preparation. 2008.
[8] E. Kolb and A. Kreuzer. Geometry of kinematic K-loops. *Abh. Math. Sem. Univ. Hamburg* 65 (1995), 189–197.
[9] G. P. Nagy. Algebraic commutative Moufang loops. Forum Math. 15 (2003), 37-62.
[10] G. P. Nagy. Tangential structure of formal Bruck loops. Publ. Math. Debrecen, 61/1-2 (2002), 87-118.
[11] G. P. Nagy. A class of simple proper Bol loops. Submitted. Available from http://arxiv.org/abs/math/0703919v1. 2007.
[12] P. T. Nagy and K. Strambach. Loops in group theory and Lie theory. de Gruyter Expositions in Mathematics, 35. Walter de Gruyter & Co., Berlin, 2002.
[13] H. O. Pflugfelder. Quasigroups and Loops: Introduction, *Sigma Series in Pure Mathematics* **7**, Heldermann Verlag Berlin (1990).
[14] C. P. Ramanujam. A note on automorphism groups of algebraic varieties. Math. Ann. 156 1964 25–33.
[15] L. J. Paige. A class of simple Moufang loops. Proc. Amer. Math. Soc. 7 (1956), 471–482.
[16] L. V. Sabinin. Smooth quasigroups and loops. Mathematics and its Applications, 492. Kluwer Academic Publishers, Dordrecht, 1999.
[17] A. Weil. On algebraic groups of transformations. Amer. J. Math. 77, (1955), 355-391.

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