# Problems posed in the
# 2020 Miklós Schweitzer Memorial Competition in Mathematics

**22 October – 2 November 2020**

**1.** We say that two sequences $x, y\colon \mathbb{N} \to \mathbb{N}$ are *completely different* if $x(n) \neq y(n)$ holds for all $n \in \mathbb{N}$. Let $F$ be a function assigning a natural number to every sequence of natural numbers such that $F(x) \neq F(y)$ for any pair of completely different sequences $x$, $y$, and for constant sequences we have $F((k, k, \dots)) = k$. Prove that there exists $n \in \mathbb{N}$ such that $F(x) = x(n)$ for all sequences $x$.

**2.** Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous periodic function and $\alpha \in \mathbb{R}$ is irrational, then the sequence $\{n\alpha + f(n\alpha)\}_{n=1}^{\infty}$ is dense modulo 1 in $[0, 1]$.

**3.** An $n \times n$ matrix $A$ with integer entries is called *representative* if, for any integer vector $\mathbf{v}$, there is a finite sequence $0 = \mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_\ell = \mathbf{v}$ of integer vectors such that for each $0 \leq i < \ell$, either $\mathbf{v}_{i+1} = A\mathbf{v}_i$ or $\mathbf{v}_{i+1} - \mathbf{v}_i$ is an element of the standard basis (i.e. one of its entries is 1, the rest are all equal to 0). Show that $A$ is not representative if and only if $A^\top$ has a real eigenvector with all non-negative entries and non-negative eigenvalue.

**4.** Consider horizontal and vertical segments in the plane that may intersect each other. Let $n$ denote their total number. Suppose that we have $m$ curves starting from the origin that are pairwise disjoint except for their endpoints. Assume that each curve intersects exactly two of the segments, a different pair for each curve. Prove that $m = O(n)$.

**5.** Prove that for a nowhere dense, compact set $K \subset \mathbb{R}^2$ the following are equivalent:

(i) $K = \bigcup_{n=1}^{\infty} K_n$ where $K_n$ is a compact set with connected complement for all $n$.

(ii) $K$ does not have a nonempty closed subset $S \subseteq K$ such that any neighborhood of any point in $S$ contains a connected component of $\mathbb{R}^2 \setminus S$.

**6.** Does there exist an entire function $F\colon \mathbb{C} \to \mathbb{C}$ such that $F$ is not zero everywhere, $|F(z)| \leq e^{|z|}$ for all $z \in \mathbb{C}$, $|F(iy)| \leq 1$ for all $y \in \mathbb{R}$, and $F$ has infinitely many real roots.

**7.** Let $p(n) \geq 0$ for all positive integers $n$. Furthermore, $x(0) = 0$, $v(0) = 1$, and

$$x(n) = x(n-1) + v(n-1), \qquad v(n) = v(n-1) - p(n)x(n) \qquad (n = 1, 2, \dots).$$

Assume that $v(n) \to 0$ in a decreasing manner as $n \to \infty$. Prove that the sequence $x(n)$ is bounded from above if and only if $\sum_{n=1}^{\infty} n \cdot p(n) < \infty$.

**8.** Let $\mathbb{F}_p$ denote the $p$-element field for a prime $p > 3$ and let $S$ be the set of functions from $\mathbb{F}_p$ to $\mathbb{F}_p$. Find all functions $D\colon S \to S$ satisfying

$$D(f \circ g) = \big(D(f) \circ g\big) \cdot D(g)$$

for all $f, g \in S$. Here, $\circ$ denotes function composition, so $(f \circ g)(x)$ is the function $f(g(x))$, and $\cdot$ denotes multiplication, so $(f \cdot g)(x) = f(x)g(x)$.

**9.** Let $D \subseteq \mathbb{C}$ be a compact set with at least two elements and consider the space $\Omega = \bigtimes_{n=0}^{\infty} D$ with the product topology. For any sequence $(d_n)_{n=0}^{\infty} \in \Omega$ let $f_{(d_n)}(z) = \sum_{n=0}^{\infty} d_n z^n$, and for each point $\zeta \in \mathbb{C}$ with $|\zeta| = 1$ we define $S = S(\zeta, (d_n))$ to be the set of complex numbers $w$ for which there exists a sequence $(z_k)$ such that $|z_k| < 1$, $z_k \to \zeta$, and $f_{(d_n)}(z_k) \to w$. Prove that on a residual set of $\Omega$, the set $S$ does not depend on the choice of $\zeta$.

**10.** Let $f$ be a polynomial of degree $n$ with integer coefficients and $p$ a prime for which $f$, considered modulo $p$, is a degree-$k$ irreducible polynomial over $\mathbb{F}_p$. Show that $k$ divides the degree of the splitting field of $f$ over $\mathbb{Q}$.

**11.** Given a real number $p > 1$, a continuous function $h\colon [0,\infty) \to [0,\infty)$, and a smooth vector field $Y\colon \mathbb{R}^n \to \mathbb{R}^n$ with $\operatorname{div} Y = 0$, prove the following inequality:

$$\int_{\mathbb{R}^n} h(|x|)|x|^p \leq \int_{\mathbb{R}^n} h(|x|)|x + Y(x)|^p.$$

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The Schweitzer Miklós Competition is open for all students who currently study in Hungary and also to Hungarian citizens studying abroad, who do not hold an MSc (or equivalent) degree in mathematics, computer science or related fields obtained in 2019 or earlier. (Those who graduated in 2020 are eligible to participate.)

Please submit solutions to the problems **in separate files** (PDF or JPG) by **12:00 (noon) CET, 2 November 2020**, via email to the following address: schweitzer.miklos@gmail.com. Please include your full name, affiliation, and email address when submitting.

Problems are to be solved individually, cooperation of any form is strictly forbidden. Please do not pose or discuss any of these problems at any online forum before the end of the competition, and let us know immediately if you notice that this happens.