Problems of the Miklós Schweitzer Memorial Competition, 2015.

1. Let K be a closed subset of the closed unit ball in $\mathbb{R}^3$. Suppose there exists a family of chords Ω of the unit sphere S^2, with the following property: for every $X, Y \in S^2$, there exist $X', Y' \in S^2$, as close to X and Y correspondingly, as we want, such that $X'Y' \in \Omega$ and $X'Y'$ is disjoint from K. Verify that there exists a set $H \subset S^2$, such that H is dense in the unit sphere S^2, and the chords connecting any two points of H are disjoint from K.

2. Let $\{x_n\}$ be a Van Der Corput series,that is,if the binary representation of n is $\sum a_i 2^i$ then $x_n = \sum a_i 2^{-i-1}$.Let V be the set of points on the plane that have the form (n, x_n).Let G be the graph with vertex set V that is connecting any two points (p, q) if there is a rectangle R which lies in parallel position with the axes and $R \cap V = \{p, q\}$.Prove that the chromatic number of G is finite.

3. Let A be a finite set and $\rightarrow$ be a binary relation on it such that for any $a, b, c \in A$, if $a \neq b, a \rightarrow c$ and $b \rightarrow c$ then either $a \rightarrow b$ or $b \rightarrow a$ (or possibly both). Let B, $B \subset A$ be minimal with the property: for any $a \in A \setminus B$ there exists $b \in B$, such that either $a \rightarrow b$ or $b \rightarrow a$ (or possibly both). Supposing that A has at most k elements that are pairwise not in relation $\rightarrow$, prove that B has at most k elements.

4. Let a_n be a series of positive integers with $a_1 = 1$ and for any arbitrary prime number p, the set $\{a_1, a_2, \cdots, a_p\}$ is a complete remainder system modulo p. Prove that $\lim_{n\to\infty} \dfrac{a_n}{n} = 1$.

5. Let $f(x) = x^n + x^{n-1} + \cdots + x + 1$ for an integer $n \geq 1$. For which n are there polynomials g, h with real coefficients and degrees smaller than n such that $f(x) = g(h(x))$.

6. Let G be the permutation group of a finite set Ω.Consider $S \subset G$ such that $1 \in S$ and for any $x, y \in \Omega$ there exists a unique element $\sigma \in S$ such that $\sigma(x) = y$.Prove that,if the elements of $S \setminus \{1\}$ are conjugate in G,then G is $2-$transitive on Ω.

7. We call a bar of width w on the surface of the unit sphere $\mathbb{S}^2$, a spherical segment, centered at the origin, which has width w and is symmetric with respect to the origin. Prove that there exists a constant $c > 0$, such that for any positive integer n the surface $\mathbb{S}^2$ can be covered with n bars of the same width so that any point is contained in no more than $c\sqrt{n}$ bars.

8. Prove that all continuous solutions of the functional equation

$$(f(x) - f(y))\left(f\left(\frac{x+y}{2}\right) - f\left(\sqrt{xy}\right)\right) = 0\ ,\ \forall x, y \in (0, +\infty)$$

are the constant functions.

9. For a function u defined on $G \subset \mathbb{C}$ let us denote by $Z(u)$ the neignborhood of unit raduis of the set of roots of u. Prove that for any compact set $K \subset G$

there exists a constant C such that if u is an arbitrary real harmonic function on G which vanishes in a point of K then:

$$\sup_{z\in K}|u(z)| \le C \sup_{Z(u)\cap G}|u(z)|.$$

10. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuously differentiable,strictly convex function.Let H be a Hilbert space and A,B be bounded,self adjoint linear operators on H.Prove that,if $f(A)-f(B)=f'(B)(A-B)$ then $A=B$.

11. For $[0,1]\subset E\subset[0,+\infty)$ where E is composed of a finite number of closed interval,we start a two dimensional Brownian motion from the point $x<0$ terminating when we first hit E.Let $p(x)$ be the probability of the finishing point being in $[0,1]$.Prove that $p(x)$ is increasing on $[-1,0)$.