Problems of the Miklós Schweitzer Memorial Competition, 2015.

1. Let $K$ be a closed subset of the closed unit ball in $\mathbb{R}^3$. Suppose there exists a family of chords $\Omega$ of the unit sphere $S^2$, with the following property: for every $X, Y \in S^2$, there exist $X', Y' \in S^2$, as close to $X$ and $Y$ correspondingly, as we want, such that $X'Y' \in \Omega$ and $X'Y'$ is disjoint from $K$. Verify that there exists a set $H \subset S^2$, such that $H$ is dense in the unit sphere $S^2$, and the chords connecting any two points of $H$ are disjoint from $K$.

2. Let $\{x_n\}$ be a Van Der Corput series, that is, if the binary representation of $n$ is $\sum a_i 2^i$ then $x_n = \sum a_i 2^{-i-1}$. Let $V$ be the set of points on the plane that have the form $(n, x_n)$. Let $G$ be the graph with vertex set $V$ that is connecting any two points $(p, q)$ if there is a rectangle $R$ which lies in parallel position with the axes and $R \cap V = \{p, q\}$. Prove that the chromatic number of $G$ is finite.

3. Let $A$ be a finite set and $\rightarrow$ be a binary relation on it such that for any $a, b, c \in A$, if $a \neq b$, $a \rightarrow c$ and $b \rightarrow c$ then either $a \rightarrow b$ or $b \rightarrow a$ (or possibly both). Let $B$, $B \subset A$ be minimal with the property: for any $a \in A \setminus B$ there exists $b \in B$, such that either $a \rightarrow b$ or $b \rightarrow a$ (or possibly both). Suppose that $A$ has at most $k$ elements that are pairwise not in relation $\rightarrow$, prove that $B$ has at most $k$ elements.

4. Let $a_n$ be a series of positive integers with $a_1 = 1$ and for any arbitrary prime number $p$, the set $\{a_1, a_2, \cdots, a_p\}$ is complete remainder system modulo $p$. Prove that $\lim_{n \to \infty} \frac{a_n}{n} = 1$.

5. Let $f(x) = x^n + x^{n-1} + \cdots + x + 1$ for an integer $n > 1$. For which $n$ are there polynomials $g, h$ with real coefficients and degrees smaller than $n$ such that $f(x) = g(h(x))$.

6. Let $G$ be the permutation group of a finite set $\Omega$. Consider $S \subset G$ such that $1 \in S$ and for any $x, y \in \Omega$ there exists a unique element $\sigma \in S$ such that $\sigma(x) = y$. Prove that, if the elements of $S \setminus \{1\}$ are conjugate in $G$, then $G$ is 2-transitive on $\Omega$.

7. We call a bar of width $w$ on the surface of the unit sphere $S^2$, a spherical segment, centered at the origin, which has width $w$ and is symmetric with respect to the origin. Prove that there exists a constant $c > 0$, such that for any positive integer $n$ the surface $S^2$ can be covered with $n$ bars of the same width so that any point is contained in no more than $c \sqrt{n}$ bars.

8. Prove that all continuous solutions of the functional equation

$$
(f(x) - f(y)) \left( f \left( \frac{x + y}{2} \right) - f \left( \sqrt{xy} \right) \right) = 0 , \ \forall x, y \in (0, +\infty)
$$

are the constant functions.

9. For a function $u$ defined on $G \subset \mathbb{C}$ let us denote by $Z(u)$ the neighborhood of unit radius of the set of roots of $u$. Prove that for any compact set $K \subset G$
there exists a constant $C$ such that if $u$ is an arbitrary real harmonic function on $G$ which vanishes in a point of $K$ then:

$$\sup_{z \in K} |u(z)| \leq C \sup_{Z(u) \cap G} |u(z)|.$$

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable, strictly convex function. Let $H$ be a Hilbert space and $A, B$ be bounded, self adjoint linear operators on $H$. Prove that, if $f(A) - f(B) = f'(B)(A - B)$ then $A = B$.

11. For $[0, 1] \subset E \subset [0, +\infty)$ where $E$ is composed of a finite number of closed interval, we start a two dimensional Brownian motion from the point $x < 0$ terminating when we first hit $E$. Let $p(x)$ be the probability of the finishing point being in $[0, 1]$. Prove that $p(x)$ is increasing on $[-1, 0)$.