Problems of the Miklós Schweitzer Memorial Competition, 2013.

1. Let q be a positive integer. Prove there exists a constant C_q such that the following inequality holds for any finite set A of integers:

$$|A + qA| \geq (q + 1)|A| - C_q.$$

2. Prove there exists a constant k_0 such that for any $k \geq k_0$, the equation

$$a^{2n} + b^{4n} + 2013 = ka^n b^{2n}$$

has no positive integer solutions a, b, n.

3. Find for which positive integers n the A_n alternating group has a permutation which is contained in exactly one 2-Sylow subgroup of A_n.

4. Let A be an Abelian group with n elements. Prove that there are two subgroups in $\mathrm{GL}(n, \mathbb{C})$, isomorphic to S_n, whose intersection is isomorphic to the automorphism group of A.

5. A subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is said to have the γ property with respect to a scalar product $\langle \cdot, \cdot \rangle$ given on $\mathfrak{g}$ if $X \in \mathfrak{h}$ implies $\langle [X, Y], X \rangle = 0$ for all $Y \in \mathfrak{g}$. Prove that the maximum dimension of γ-property subalgebras of a given 2 step nilpotent Lie algebra with respect to a scalar product is independent of the selection of the scalar product.

6. Let $\mathcal{A}$ be a C^* algebra with a unit element and let $\mathcal{A}_+$ be the cone of the positive elements of $\mathcal{A}$ (this is the set of such self adjoint elements in $\mathcal{A}$ whose spectrum is in $[0, \infty)$. Consider the operation

$$x \circ y = \sqrt{x} y \sqrt{x}, \ \ x, y \in \mathcal{A}_+$$

Prove that if for all $x, y \in \mathcal{A}_+$ we have

$$(x \circ y) \circ y = x \circ (y \circ y),$$

then $\mathcal{A}$ is commutative.

7. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an additive function (that is $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$) for which $x \mapsto f(x)f(\sqrt{1 - x^2})$ is bounded of some nonempty subinterval of $(0, 1)$. Prove that f is continuous.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and strictly increasing function for which

$$f^{-1}\left(\frac{f(x) + f(y)}{2}\right)(f(x) + f(y)) = (x + y)f\left(\frac{x + y}{2}\right)$$

for all $x, y \in \mathbb{R}$(f^{-1} denotes the inverse of f). Prove that there exist real constants $a \neq 0$ and b such that $f(x) = ax + b$ for all $x \in \mathbb{R}$.

9. Prove that there is a function $f : (0, \infty) \to (0, \infty)$ which is nowhere continuous and for all $x, y \in (0, \infty)$ and any rational α we have

$$f\left(\left(\frac{x^\alpha + y^\alpha}{2}\right)^{\frac{1}{\alpha}}\right) \leq \left(\frac{f(x)^\alpha + f(y)^\alpha}{2}\right)^{\frac{1}{\alpha}}.$$

Is there such a function if instead the above relation holds for every $x, y \in (0, \infty)$ and for every irrational α?

10. Consider a Riemannian metric on the vector space $\mathbb{R}^n$ which satisfies the property that for each two points a, b there is a single distance minimising geodesic segment $g(a, b)$. Suppose that for all $a \in \mathbb{R}^n$, the Riemannian distance with respect to a, $\rho_a : \mathbb{R}^n \to \mathbb{R}$ is convex and differentiable outside of a. Prove that if for a point $x \neq a, b$ we have

$$\partial_i \rho_a(x) = -\partial_i \rho_b(x), \ \ i = 1, \cdots, n$$

then x is a point on $g(a, b)$ and conversely.

11. (a) Consider an ellipse in the plane. Prove that there exists a Riemannian metric which is defined on the whole plane, and with respect to which the ellipse is a geodesic. Prove that the Gaussian curvature of any such Riemannian metric takes a positive value.

(b) Consider two nonintersecting, simple closed smooth curves in the plane. Prove that if there is a Riemmanian metric defined on the whole plane and the two curves are geodesics of that metric, then the Gaussian curvature of the metric vanishes somewhere.

12. There are n tokens in a pack. Some of them (at least one, but not all) are white and the rest are black. All tokens are extracted randomly from the pack, one by one, without putting them back. Let X_i be the ratio of white tokens in the pack before the i^{th} extraction and let

$$T = \max\{|X_i - X_j| : 1 \leq i \leq j \leq n\}.$$

Prove that $\mathbb{E}(T) \leq H(\mathbb{E}(X_1))$, where $H(x) = -x \ln x - (1 - x) \ln(1 - x)$.