1. Let $p$ be prime. Denote by $N(p)$ the number of integers $x$ for which $1 \leq x \leq p$ and 

$$x^p \equiv 1 \pmod{p}.$$ 

Prove that there exist numbers $c < 1/2$ and $p_0 > 0$ such that 

$$N(p) \leq p^c$$ 

if $p \geq p_0$.

2. Let $G$ be a countably infinite, $d$-regular, connected, vertex-transitive graph. Show that there is a complete pairing in $G$.

3. Let $A_i, i = 1, 2, \ldots, t$ be distinct subsets of the base set $\{1, 2, \ldots, n\}$ complying to the following condition 

$$A_i \cap A_k \subseteq A_j$$ 

for any $1 \leq i < j < k \leq t$. Find the maximum value of $t$.

4. Prove that if $n \geq 2$ and $I_1, I_2, \ldots, I_n$ are prime ideals in a unitary commutative ring such that for any nonempty $H \subseteq \{1, 2, \ldots, n\}$ the set $\sum_{h \in H} I_h$ is a prime ideal, then 

$$I_2I_3I_4 \ldots I_n + I_1I_3I_4 \ldots I_n + \cdots + I_1I_2 \ldots I_{n-1}$$

is also a prime ideal.

5. Given the vectors $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ in the plane with the following properties: for every $1 \leq i \leq n$, $|v_i - w_i| \leq 1$, and for every $1 \leq i < j \leq n$, $|v_i - v_j| \geq 3$ and $v_i - w_i \neq v_j - w_j$. Prove that for sets $V = \{v_1, \ldots, v_n\}$ and $W = \{w_1, \ldots, w_n\}$, the set of $V + (V \cup W)$ must have at least $cn^{3/2}$ elements, for some universal constant $c > 0$.

6. Is there a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ for every $d \in \mathbb{R}$ we have $g_{m,d}(x) = f(x, mx+d)$ is strictly monotonic on $\mathbb{R}$ if $m \geq 0$, and not monotone on any non-empty open interval if $m < 0$?

7. Is there any sequence $(a_n)_{n=1}^\infty$ of non-negative numbers, for which $\sum_{n=1}^\infty a_n^2 < \infty$, but $\sum_{n=1}^\infty \left(\sum_{k=1}^\infty \frac{a_k}{k}\right)^2 = \infty$?

8. Let $D \subset \mathbb{R}^2$ be a finite Lebesgue measure of a connected open set and $u : D \to \mathbb{R}$ a harmonic function. Show that it is either a constant $u$ or for almost every $p \in D$ 

$$f'(t) = (\text{grad } u)(f(t)), \quad f(0) = p$$

has no initial value problem (differentiable everywhere) solution to $f : [0, \infty) \to D$.

9. For each $M$ $m$-dimensional closed $C^\infty$ set, assign a $G(m)$ in some euclidean space $\mathbb{R}^q$. Denote by $\mathbb{R}^q$ a $q$-dimensional real projective space. $AG(M) \subseteq \times \mathbb{R}^q$. The set consists of $(x,e)$ pairs for which $x \in M \subseteq \mathbb{P}^q$ and $e \subseteq \mathbb{R}^{q+1} = \mathbb{R}^q \times \mathbb{R}$ and $a(0,\ldots,0,1) \in \mathbb{R}^{q+1}$ in a stretched $(m+1)$-dimensional linear subspace. Prove that if $N$ is a $n$-dimensional closed set $C^\infty$, then $P = G(M \times M)$ and $Q = G(M) \times G(N)$ are cobordant, that is,
there exists a $(2m + 2n + 1)$-dimensional compact, flanged set $C^\infty$ with a disjoint union of $P$ and $Q$.

10. Consider the space $\{0, 1\}^\mathbb{N}$ with the product topology (where $\{0, 1\}$ is a discrete space). Let $T : \{0, 1\}^\mathbb{N} \to \{0, 1\}^\mathbb{N}$ be the left-shift, i.e., $(Tx)(n) = x(n + 1)$ for every $n \in \mathbb{N}$. Can a finite number of Borel sets be given: $B_1, \ldots, B_m \subset \{0, 1\}^\mathbb{N}$ such that

$$\{T^i(B_j) \mid i \in \mathbb{N}, 1 \leq j \leq m\}$$

the $\sigma$-algebra generated by the set system coincides with the Borel set system?

11. (Not yet translated.)