Problems of the Miklós Schweitzer Memorial Competition, 2009.

1. On every card of a deck of cards a regular 17-gon is displayed with all sides and diagonals, and the vertices are numbered from 1 through 17. On every card all edges (sides and diagonals) are colored with a color $1, 2, \ldots, 105$ such that the following property holds: for every 15 vertices of the 17-gon the 105 edges connecting these vertices are colored with different colors on at least one of the cards. What is the minimum number of cards in the deck?

2. Let $p_1, \ldots, p_k$ be prime numbers, and let S be the set of those integers whose all prime divisors are among $p_1, \ldots, p_k$. For a finite subset A of the integers let us denote by $\mathcal{G}(A)$ the graph whose vertices are the elements of A, and the edges are those pairs $a, b \in A$ for which $a - b \in S$. Does there exist for all $m \geq 3$ an m-element subset A of the integers such that

(i) $\mathcal{G}(A)$ is complete?

(ii) $\mathcal{G}(A)$ is connected, but all vertices have degree at most 2?

3. Prove that there exist positive constants c and n_0 with the following property. If A is a finite set of integers, $|A| = n > n_0$, then

$$|A - A| - |A + A| \leq n^2 - cn^{8/5}.$$

4. Prove that the polynomial

$$f(x) = \frac{x^n + x^m - 2}{x^{\gcd(m,n)} - 1}$$

is irreducible over $\mathbb{Q}$ for all integers $n > m > 0$.

5. Let G be a finite non-commutative group of order $t = 2^n m$, where n, m are positive and m is odd. Prove, that if the group contains an element of order 2^n, then

(i) G is not simple;

(ii) G contains a normal subgroup of order m.

6. A set system (S, L) is called a Steiner triple system, if $L \neq \emptyset$, any pair $x, y \in S$, $x \neq y$ of points lie on a unique line $\ell \in L$, and every line $\ell \in L$ contains exactly three points. Let (S, L) be a Steiner triple system, and let us denote by xy the thrid point on a line determined by the points $x \neq y$. Let A be a group whose factor by its center $C(A)$ is of prime power order. Let $f, h : S \to A$ be maps, such that $C(A)$ contains the range of f, and the range of h generates A. Show, that if

$$f(x) = h(x)h(y)h(x)h(xy)$$

holds for all pairs $x \neq y$ of points, then A is commutative, and there exists an element $k \in A$, such that $f(x) = kh(x)$ for all $x \in S$.

7. Let H be an arbitrary subgroup of the diffeomorphism group $\mathsf{Diff}^\infty(M)$ of a differentiable manifold M. We say that a $\mathcal{C}^\infty$ vector field X is *weakly tangent* to the group H, if there exists a positive integer k and a $\mathcal{C}^\infty$-differentiable map $\varphi :]-\varepsilon, \varepsilon[^k \times M \to M$ such that

(i) for fixed $t_1, \dots, t_k$ the map

$$\varphi_{t_1,\dots,t_k} : x \in M \mapsto \varphi(t_1, \dots, t_k, x)$$

is a diffeomorphism of M, and $\varphi_{t_1,\dots,t_k} \in H$;

(ii) $\varphi_{t_1,\dots,t_k} \in H = \mathsf{Id}$ whenever $t_j = 0$ for some $1 \le j \le k$;

(iii) for any $\mathcal{C}^\infty$-function $f : M \to \mathbb{R}$

$$Xf = \left.\frac{\partial^k (f \circ \varphi_{t_1,\dots,t_k})}{\partial t_1 \dots \partial t_k}\right|_{(t_1,\dots,t_k)=(0,\dots,0)}.$$

Prove, that the commutators of $\mathcal{C}^\infty$ vector fields that are weakly tangent to $H \subset \mathsf{Diff}^\infty(M)$ are also weakly tangent to H.

8. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of measurable subsets of the real line which covers almost every point infinitely often. Prove, that there exists a set $B \subset \mathbb{N}$ of zero density, such that $\{A_n\}_{n\in B}$ also covers almost every point infinitely often. (The set $B \subset \mathbb{N}$ is of zero density if $lim_{n\to\infty} \frac{\#\{B\cap\{0,\dots,n-1\}\}}{n} = 0$.)

9. Let $P \subseteq \mathbb{R}^m$ be a non-empty compact convex set and $f : P \to \mathbb{R}_+$ be a concave function. Prove, that for every $\xi \in \mathbb{R}^m$

$$\int_P \langle \xi, x\rangle f(x)dx \le \left[\frac{m+1}{m+2}\sup_{x\in P}\langle \xi, x\rangle + \frac{1}{m+2}\inf_{x\in P}\langle \xi, x\rangle\right] \cdot \int_P f(x)d(x).$$

10. Let $U \subset \mathbb{R}^n$ be an open set, and let $L : U\times\mathbb{R}^n \to \mathbb{R}$ be a continuous, in its second variable first order positive homogeneous, positive over $U \times (\mathbb{R}^n \setminus \{0\})$ and of C^2-class Langrange function, such that for all $p \in U$ the Gauss-curvature of the hyper surface

$$\{v \in \mathbb{R}^n \mid L(p, v) = 1\}$$

is nowhere zero. Determine the extremals of L if it satisfies the following system

$$\sum_{k=1}^n y^k \partial_k \partial_{n+i} L = \sum_{k=1}^n y^k \partial_i \partial_{n+k} L \qquad (i \in \{1, \dots, n\})$$

of partial differetial equations, where $y^k(u, v) := v^k$ for $(u, v) \in U \times \mathbb{R}^k$, $v = (v^1, \dots, v^k)$.

11. Denote by H_n the linear space of $n \times n$ self-adjoint complex matrices, and by P_n the cone of positive-semidefinite matrices in this space. Let us consider the usual inner product on H_n

$$\langle A, B\rangle = \mathrm{tr} AB \qquad (A, B \in H_n)$$

and its derived metric. Show that every $\phi : P_n \to P_n$ isometry (that is a not necessarily surjective, distance preserving map with respect to the above metric) can be expressed as

$$\phi(A) = UAU^* + X \qquad (A \in H_n)$$

or

$$\phi(A) = UA^T U^* + X \qquad (A \in H_n)$$

where U is an $n \times n$ unitary matrix, X is a positive-semidefinite matrix, and ^{T} and * denote taking the transpose and the adjoint, respectively.

12. Let $Z_1, Z_2, \ldots, Z_n$ be d-dimensional independent random (column) vectors with standard normal distribution, $n - 1 > d$. Furthermore let

$$\overline{Z} = \frac{1}{n} \sum_{i=1}^{n} Z_i, \quad S_n = \frac{1}{n-1} \sum_{i=1}^{n} (Z_i - \overline{Z})(Z_i - \overline{Z})^\top$$

be the sample mean and the corrected empirical covariance matrix. Consider the standardized samples $Y_i = S_n^{-1/2}(Z_i - \overline{Z})$, $i = 1, 2, \ldots, n$. Show that

$$\frac{E|Y_1 - Y_2|}{E|Z_1 - Z_2|} > 1,$$

and that the ratio does not depend on d, only on n.