Problems of the Miklós Schweitzer Memorial Competition, 2008.

1. Let $\mathcal{H} \subset P(X)$ be a system of subsets of $X$ and $\kappa > 0$ be a cardinal number such that every $x \in X$ is contained in less than $\kappa$ members of $\mathcal{H}$. Prove, that there exists an $f : X \to \kappa$ coloring, such that every nonempty $A \in \mathcal{H}$ has a “unique” point, that is, an element $x \in A$ such that $f(x) \neq f(y)$ for all $x \neq y \in A$.

2. Let $t \geq 3$ be an integer, and for $1 \leq i < j \leq t$ let $A_{ij} = A_{ji}$ be an arbitrary subset of an $n$-element set $X$. Prove, that there exist $1 \leq i < j \leq t$ for which \[ |(X \setminus A_{ij}) \cup \bigcup_{k \neq i, j} (A_{ik} \cap A_{jk})| \geq \frac{t-2}{2t-2} n. \]

3. A bipartite graph on the sets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ of vertices (that is the edges are of the form $x_iy_j$) is called tame if it has no $x_iy_jx_ky_{\ell}$ path ($i, j, k, \ell \in \{1, \ldots, n\}$) where $j < \ell$ and $i + j > k + \ell$. Calculate the infimum of those real numbers $\alpha$ for which there exists a constant $c = c(\alpha) > 0$ such that for all tame graphs $e \leq c \cdot n^{\alpha}$, where $e$ is the number of edges and $n$ is half of the number of vertices.

4. Let $A$ be a subgroup of the symmetric group $S_n$, and $G$ be a normal subgroup of $A$. Show, that if $G$ is transitive, then $|A : G| \leq 5^{n-1}$.

5. Let $A$ be an infinite subset of the set of natural numbers, and denote by $\tau_A(n)$ the number of divisors of $n$ in $A$. Construct a set $A$ for which \[ \sum_{n \leq x} \tau_A(n) = x + O(\log \log x), \]
and show that there is no set for which the error term is $o(\log \log x)$ in the above formula.

6. Can you draw circles on the plane so that every line intersects at least one of them but no more than 100 of them?

7. Let $f : \mathbb{R}^1 \to \mathbb{R}^2$ be a continuous function such that $f(x) = f(x + 1)$ for all $x$, and let $t \in [0, 1/4]$. Prove, that there exists $x \in \mathbb{R}$, such that the vector from $f(x - t)$ to $f(x + t)$ is perpendicular to the vector from $f(x)$ to $f(x + 1/2)$. 

8. Let $S$ be the above Sierpinski triangle. What can we say about the Hausdorff dimension of the elevation sets $f^{-1}(y)$ for typical continuous real functions defined on $S$? (A property is satisfied for typical continuous real functions on $S$ if the set of functions not having this property is of the first Baire category in the metric space of continuous $S \to \mathbb{R}$ functions with the supremum norm.)

9. For a given $\alpha > 0$ let us consider the regular, non-vanishing $f(z)$ maps on the unit disc $\{|z| < 1\}$ for which $f(0) = 1$ and $\Re z \frac{f'(z)}{f(z)} > -\alpha$ ($|z| < 1$). Show that the range of

$$g(z) = \frac{1}{(1 - z)^{2\alpha}}$$

contains the range of all other such functions. Here we consider that regular branch of $g(z)$ for which $g(0) = 1$.

10. Let $V$ be the set of non-collinear pairs of vectors in $\mathbb{R}^3$, and $H$ be the set of lines passing through the origin. Is it true that for every continuous map $f : V \to H$ there exists a continuous map $g : V \to \mathbb{R}^3 \setminus \{0\}$ such that $g(v) \in f(v)$ for all $v \in V$?

11. Let $\xi_1, \ldots, \xi_n$ be (not necessarily independent) random variables with normal distribution for which $E\xi_j = 0$ and $E\xi_j^2 \leq 1$ for all $1 \leq j \leq n$. Prove, that

$$E \left( \max_{1 \leq j \leq n} \xi_j \right) \leq \sqrt{2 \log n}.$$