1. The Lindelöf number $L(X)$ of a topological space $X$ is the least infinite cardinal $\lambda$ with the property that every open covering of $X$ has a subcovering of cardinality at most $\lambda$. Prove that if every non-countably infinite subset of a first countable space $X$ has a point of condensation, then $L(X) = \sup L(A)$, where $A$ runs over the separable closed subspaces of $X$.

(A point of condensation of a subset $H \subseteq X$ is a point $x \in X$ such that any neighbourhood of $x$ intersects $H$ in a non-countably infinite set.)

2. Write $t(G)$ for the number of complete quadrilaterals in the graph $G$ and $e_G(S)$ for the number of edges spanned by a subset $S$ of vertices of $G$. Let $G_1$, $G_2$ be two (simple) graphs on a common underlying set $V$ of vertices, $V' \subseteq V$, and assume that $|e_{G_1}(S) - e_{G_2}(S)| \leq n^2/1000$ holds for any subset $S \subseteq V$. Prove that $t(G_1) - t(G_2) < n^4/1000$.

3. Prove that there is a constant $c > 0$ such that for any $n > 1$ there exists a planar graph $G$ with $n$ vertices such that every straight-edged plane embedding of $G$ has a pair of edges with ratio of lengths at least $cn$.

4. Determine all totally multiplicative and non-negative functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ with the property that if $a, b \in \mathbb{Z}$ and $k \neq 0$, then there exist integers $q$ and $r$ such that $a = qb + r$ and $f(r) < f(b)$.

5. Let $G$ be a non-solvable finite group and let $\varepsilon > 0$. Show that there exist a positive integer $k$ and a word $w \in F_k$ such that $w$ assumes the value 1 with probability less than $\varepsilon$ when its $k$ arguments are considered to be independent and uniformly distributed random variables with values in $G$. (We write $F_k$ for the free group generated by $k$ elements.)

6. Is it true that if the perfect set $E \subset [0, 1]$ is of zero Lebesgue measure then those functions in $C^1[0, 1]$ which are one-to-one on $E$ form a dense subset of $C^1[0, 1]$?
(We use the metric

$$d(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)| + \sup_{x \in [0, 1]} |f'(x) - g'(x)|$$

to define the topology in the space $C^1[0, 1]$ of continuously differentiable real functions on $[0, 1]$.)
7. Suppose that the closed subset $K$ of the sphere

$$S^2 \quad \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \}$$

is symmetric with respect to the origin and separates any two antipodal points in $S^2 \setminus K$. Prove that for any positive $\varepsilon$ there exists a homogeneous polynomial $P$ of odd degree such that the Hausdorff distance between

$$Z(P) = \{ (x, y, z) \in S^2 : P(x, y, z) = 0 \}$$

and $K$ is less than $\varepsilon$.

8. Prove that for any $0 < \delta < 2\pi$ there exists a number $m > 1$ such that for any positive integer $n$ and unimodular complex numbers $z_1, \ldots, z_n$ with $z_1^\delta + \cdots + z_n^\delta = 0$ for all integer exponents $1 < \nu < m$, any arc of length $\delta$ of the unit circle contains at least one of the numbers $z_1, \ldots, z_n$.

9. Let $F$ be a smooth (i.e., $C^\infty$) closed surface. Call a continuous map $f : F \to \mathbb{R}^2$ an almost-immersion if there exists a smooth closed embedded curve $\gamma$ (possibly disconnected) in $F$ such that $f$ is smooth and of maximal rank (i.e., rank 2) on $F \setminus \gamma$, and each point $p \in \gamma$ admits local coordinate charts $(x, y)$ and $(u, v)$ about $p$ and $f(p)$, respectively, such that the coordinates of $p$ and $f(p)$ are zero and the map $f$ is given by $(x, y) \mapsto (u, v)$. $u = |x|$, $v = y$.

Determine the genera of those smooth, closed, connected, orientable surfaces $F$ that admit an almost-immersion in the plane with the curve $\gamma$ having a given positive number $n$ of connected components.

10. Let $\mathcal{N}_p$ stand for a $p$ dimensional random variable of standard normal distribution. For $a \in \mathbb{R}^p$, let $H_p(a)$ stand for the expectation $E|\mathcal{N}_p + a|$. For $p > 1$, prove that

$$H_p(a) \quad (p - 1) \int_0^\infty H_1 \left( \frac{|a|}{\sqrt{1 + r^2}} \right) \frac{r^{p-2}}{(1 + r^2)^{\frac{p}{2}}} \, dr.$$

The deadline for submitting solutions is 12:00 (CET) 8 November, 2004. If the participant relies on knowledge not included in the standard curriculum, then (s)he should cite the exact source. For further information see the homepage http://www.cs.elte.hu/~schw04.