1. For an arbitrary ordinal number $\alpha$ let $H(\alpha)$ denote the set of functions $f : \alpha \to \{-1, 0, 1\}$ that map all but finitely many elements of $\alpha$ to 0. Order $H(\alpha)$ according to the last difference, that is, for $f, g \in H(\alpha)$ let $f \prec g$ if $f(\beta) < g(\beta)$ holds for the maximum ordinal number $\beta < \alpha$ with $f(\beta) \neq g(\beta)$. Prove that the ordered set $(H(\alpha), \prec)$ is scattered (i.e. it does not contain a subset isomorphic to the set of rational numbers with the usual order), and that any scattered order type can be embedded into some $(H(\alpha), \prec)$.

2. Let $G$ be a simple $k$-edge-connected graph on $n$ vertices and let $u$ and $v$ be different vertices of $G$. Prove that there exist $k$ edge-disjoint paths from $u$ to $v$ each having at most $\frac{20n}{k}$ edges.

3. Put $A = \{\text{yes, no}\}$. A function $f : A^n \to A$ is called a decision function if

(a) the value of the function changes if we change all of its arguments; and
(b) the value does not change if we replace any of the arguments by the function value.

A function $d : A^n \to A$ is called a dictatorial function, if there is an index $i$ such that the value of the function equals its $i$th argument.

The democratic function is the function $m : A^3 \to A$ that outputs the majority of its arguments.

Prove that any decision function is a composition of dictatorial and democratic functions.

4. For a given natural number $n$, consider those sets $A \subseteq \mathbb{Z}_n$ for which the equation $xy = uv$ has no other solution in the residual classes $x, y, u, v \in A$ than the trivial solutions $x = u$, $y = v$ and $x = v$, $y = u$. Let $g(n)$ be the maximum of the size of such sets $A$. Prove that

$$\limsup_{n \to \infty} \frac{g(n)}{\sqrt{n}} = 1.$$ 

5. Denote by $\lambda(H)$ the Lebesgue outer measure of $H \subseteq [0,1]$. The horizontal and vertical sections of the set $A \subseteq [0,1] \times [0,1]$ are denoted by $A^v$ and $A_x$ respectively; that is, $A^v = \{x \in [0,1] : (x,y) \in A\}$ and $A_x = \{y \in [0,1] : (x,y) \in A\}$ for all $x, y \in [0,1]$.

(a) Is there a decomposition $A \cup B$ of the unit square $[0,1] \times [0,1]$ such that $A^v$ is the union of finitely many segments of total length less than $1/2$ and $\lambda(B_x) \leq 1/2$ for all $x, y \in [0,1]$?

(b) Is there a decomposition $A \cup B$ of the unit square $[0,1] \times [0,1]$ such that $A^v$ is the union of finitely many segments of total length not greater than $1/2$ and $\lambda(B_x) < 1/2$ for all $x, y \in [0,1]$?
6. Let $K \subseteq \mathbb{R}$ be compact. Prove that the following two statements are equivalent to each other.

(a) For each point $x$ of $K$ we can assign an uncountable set $F_x \subseteq \mathbb{R}$ such that

$$\text{dist}(F_x, F_y) \geq |x - y|$$

holds for all $x, y \in K$;

(b) $K$ is of measure zero.

7. Let the complex function $F(z)$ be regular on the punctuated disk $\{0 < |z| < R\}$. By a level curve we mean a component of the level set of $\text{Re } F(z)$, that is, a maximal connected set on which $\text{Re } F(z)$ is constant. Denote by $A(r)$ the union of those level curves that are entirely contained in the punctuated disk $\{0 < |z| < r\}$. Prove that if the number of components of $A(r)$ has an upper bound independent of $r$ then $F(z)$ can only have a pole type singularity at 0.

8. Prove that there exists an absolute constant $c$ such that any set $H$ of $n$ points of the plane in general position can be coloured with $c \cdot \log n$ colours in such a way that any disk of the plane containing at least one point of $H$ intersects some colour class of $H$ in exactly one point.

9. Let $M$ be a connected, compact $C^\infty$-differentiable manifold, and denote the vector space of smooth real functions on $M$ by $C^\infty(M)$. Let the subspace $V \leq C^\infty(M)$ be invariant under $C^\infty$-diffeomorphisms of $M$, that is, let $f \circ h \in V$ for every $f \in V$ and for every $C^\infty$-diffeomorphism $h : M \rightarrow M$. Prove that if $V$ is different from the subspaces $\{0\}$ and $C^\infty(M)$ then $V$ only contains the constant functions.

10. Let $X_1, X_2, \ldots$ be independent random variables of the same distribution such that their joint distribution is discrete and is concentrated on infinitely many different values. Let $a_n$ denote the probability that $X_1, \ldots, X_{n+1}$ are all different on the condition that $X_1, \ldots, X_n$ are all different ($n \geq 1$). Show that

(a) $a_n$ is strictly decreasing and tends to 0 as $n \rightarrow \infty$; and

(b) for any sequence $1 \leq f(1) < f(2) < \ldots$ of positive integers the joint distribution of $X_1, X_2, \ldots$ can be chosen such that

$$\limsup_{n \rightarrow \infty} \frac{a_{f(n)}}{a_n} = 1$$

holds.

The deadline for submitting solutions to the problems is November the 18th, 2002/12h (CET). If the participant uses some knowledge that is not contained in the standard curriculum, then (s)he should cite the exact source. For further information see the homepage http://www.cs.elte.hu/~schw02.