Problems of the Miklós Schweitzer Memorial Competition, 2001.

1. Let $f : 2^S \to \mathbb{R}$ be a function defined on the subsets of a finite set S. Prove that if $f(A) = f(S \setminus A)$ and $\max\{f(A), f(B)\} \geq f(A \cup B)$ for all subsets A, B of S, then f assumes at most $|S|$ distinct values.

2. Let $\alpha \leq -2$ be an integer. Prove that for every pair β_0, β_1 of integers there exists a uniquely determined sequence $0 \leq q_0, \dots, q_k < \alpha^2 - \alpha$ of integers, such that $q_k \neq 0$ if $(\beta_0, \beta_1) \neq (0, 0)$ and
$$\beta_i = \sum_{j=0}^{k} q_j (\alpha - i)^j, \text{ for } i = 0, 1.$$

3. How many minimal left ideals does the full matrix ring $M_n(K)$ of $n \times n$ matricies over a field K have?

4. Find the units of $R = \mathbb{Z}[t][\sqrt{t^2 - 1}]$.

5. Prove that if the function f is defined on the set of positive real numbers, its values are real, and f satisfies the equation
$$f\left(\frac{x+y}{2}\right) + f\left(\frac{2xy}{x+y}\right) = f(x) + f(y)$$
for all positive x, y, then
$$2f(\sqrt{xy}) = f(x) + f(y)$$
for every pair x, y of positive numbers.

6. Let $I \subset \mathbb{R}$ be a nonempty open interval, $\varepsilon \geq 0$ and $f : I \to \mathbb{R}$ a function satisfying the
$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon t(1-t)|x - y|$$
inequality for all $x, y \in I$ and $t \in [0, 1]$. Prove that there exists a convex $g : I \to \mathbb{R}$ function, such that the function $\ell := f - g$ has the ε-Lipschitz property, that is
$$|\ell(x) - \ell(y)| \leq \varepsilon |x - y| \qquad \text{for all } x, y \in I.$$

7. Let $e_1, \dots, e_n$ be semilines on the plane starting from a common point. Prove, that if there is no $u \not\equiv 0$ harmonic function on the whole plane that vanishes on the set $e_1 \cup \dots \cup e_n$, then there exists a pair i, j of indices such that no $u \not\equiv 0$ harmonic function on the whole plane exists that vanishes on $e_i \cup e_j$.

8. Let H be a complex Hilbert space. The bounded linear operator A is called *positive*, if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. Let $\sqrt{A}$ be the positive square root of A, i.e. the uniquelly determined positive operator satisfying $(\sqrt{A})^2 = A$. On the set of positive operators we introduce the
$$A \circ B = \sqrt{A}B\sqrt{B}$$
operation. Prove, that for a given pair A, B of positive operators the identity
$$(A \circ B) \circ C = A \circ (B \circ C)$$
holds for all positive operator C if and only if $AB = BA$.

9. Let H be the hyperbolic plane, $I(H)$ be the isometry group of H, and $O \in H$ be a fixed starting point. Determine those continuous $\sigma : H \to I(H)$ mappings, that satisfy the following three conditions:

   (a) $\sigma(O) = \mathrm{id}$, and $\sigma(X)O = X$ for all $X \in H$;

   (b) for every $X \in H \setminus \{O\}$ point, the $\sigma(X)$ isometry is a paracyclic shift, i.e. every member of a system of paracycles through a common infinitely far point is left invariant.

   (c) for any pair $P, Q \in H$ of points there exists a point $X \in H$ such that $\sigma(X)P = Q$.

   Prove, that the $\sigma : H \to I(H)$ mappings satisfying the above conditions are differentiable with the exception of a point.

10. (not yet translated)

11. (not yet translated)

Unofficial translation by Miklós Maróti