1. Prove that there exists a function $f : [\omega_1]^2 \to \omega_1$ such that
(i) $f(\alpha, \beta) < \min(\alpha, \beta)$ whenever $\min(\alpha, \beta) > 0$; and
(ii) if $\alpha_0 < \alpha_1 < \ldots < \alpha_i < \ldots < \omega_1$, then $\sup\{\alpha_i : i < \omega\} = \sup\{f(\alpha_i, \alpha_j) : i, j < \omega\}$.

2. Let n red and n blue subarcs of a circle be given such that each red subarc intersects each blue subarc. Prove that there is a point which is covered by at least n of the given (red or blue) subarcs.

3. Prove that for every integer $n \geq 3$ there exists $N(n)$ with the following property. Whenever P is a set of at least $N(n)$ points of the plane such that any three points of P determines a nondegenerate triangle containing at most one point of P in its interior, then P contains the vertices of a convex n-gon whose interior does not contain any point of P.

4. Let $a_1 < a_2 < a_3$ be positive integers. Prove that there are integers x_1, x_2, x_3 such that $\sum_{i=1}^{3} |x_i| > 0$, $\sum_{i=1}^{3} a_i x_i = 0$, and

$$\max_{1 \leq i \leq 3} |x_i| \leq \frac{2}{\sqrt{3}} \sqrt{a_3} + 1.$$

Prove that $2/\sqrt{3}$ cannot be replaced by a smaller number.

5. Prove that for every $\varepsilon > 0$ there exists a positive integer n and there are positive numbers $a_1, \ldots, a_n$ such that for every $\varepsilon < x < 2\pi - \varepsilon$ we have

$$\sum_{k=1}^{n} a_k \cos kx < -\frac{1}{\varepsilon} \left| \sum_{k=1}^{n} a_k \sin kx \right|.$$

6. Suppose the real line is decomposed into two uncountable Borel sets. Prove that a suitable translated copy of the first set intersects the second in an uncountable set.

7. Let $H(D)$ denote the space of functions holomorphic on the disc $D = \{z : |z| < 1\}$, endowed with the topology of uniform convergence on each compact subset of D. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then we shall denote $S_n(f, z) = \sum_{k=0}^{n} a_k z^k$. A function $f \in H(D)$ is called *universal* if, for every continuous function $g : \partial D \to \mathbf{C}$ and for every $\varepsilon > 0$, there are partial sums $S_{n(j)}(f, z)$ approximating g uniformly on the arc $\{e^{it} : 0 \leq t \leq 2\pi - \varepsilon\}$. Prove that the set of universal functions contains a dense G_δ subset of $H(D)$.

8. Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be a map such that the image of every compact set is compact, and the image of every connected set is connected. Prove that f is continuous.

9. Let M be a closed, orientable 3-dimensional differentiable manifold, and let G be a finite group of orientation preserving diffeomorphisms of M. Let P and Q denote the set of those points of M whose stabilizer is nontrivial (that is, contains a nonidentity element of G) and noncyclic, respectively. Let $\chi(P)$ denote the Euler characteristic of P. Prove that the order of G divides $\chi(P)$, and Q is the union of $-2\frac{\chi(P)}{|G|}$ orbits of G.

10. Joe generates 4 independent random numbers in $(0,1)$ according to the uniform distribution. He shows one of the numbers to Bill, who has to guess whether the number shown is one of the extremal numbers (that is, the smallest or the greatest) of the four numbers or not. Can Joe have a deterministic strategy such that no matter what Bill's method is, the probability of the right guess of Bill is at most $1/2$?