Motivation

- **Constraint satisfaction problem**: natural way to express many combinatorical, mathematical and real world problems

\[
\begin{align*}
3x + 2y - z - 2u &= 1 \\
2x - 2y + 4z + u &= -2 \\
-x + \frac{1}{2}y - z - u &= 0 \\
x + y - 3u &= 1
\end{align*}
\]

- **Real benefits** from understanding limitations and better algorithms
- **Fruitful collaboration** between computer science, logic, graph theory and universal algebra, **new research directions**
Goals

- Computational complexity theory
  - Basic definitions and methods only
  - \( \textbf{P}, \textbf{NP} \) and \( \textbf{NP} \)-complete complexity classes

- Constraint satisfaction problem
  - Introduced for relational structures
  - Basic reduction theorems
  - Connection with relational clones

- Algebraic approach
  - Polymorphisms and compatible algebras,
  - Bounded width algorithm
  - Few subpowers algorithm
  - Applications in universal algebra

- Theory of absorption (ask Libor Barto)
\[ W = \bigcup_{n=0}^{\infty} \{0,1\}^n \] is the set of \textbf{words} over the \textbf{alphabet} \{0,1\}. The \textbf{length} of a word \( x \in W \) is \(|x|\).

**Definition**

\( f : W \to W \) is \textbf{computable in polynomial time} if there exist an algorithm and \( c, d \in \mathbb{N} \) such that for any word \( x \in W \) the algorithm stops in at most \(|x|^c + d\) steps and computes \( f(x) \).

- **algorithm**: computer program with infinite memory (Turing machine)
- **encoding of mathematical objects**: binary or ASCII text
- **polynomial bound**: masks all differences between various machines and encodings
- **examples**: basic arithmetic of natural numbers, factoring of polynomials over \( \mathbb{Q} \), linear programming over finite fields
**Definition**

**Decision problem** or **membership problem** is a non-empty proper set $L \subset W$ of words. The problem $L \subset W$ is **solvable in polynomial time** if its characteristic function

$$f(x) = \begin{cases} 
1 & \text{if } x \in L, \\
0 & \text{otherwise}
\end{cases}$$

is computable in polynomial time. \(\mathbf{P}\) is the class of polynomial time solvable decision problems.

- “PRIMES is in \(\mathbf{P}\)” by M. Agrawal, N. Kayal and N. Saxena
- **Examples**: class of bipartite graphs, solvable sets of linear equations
The complexity class **NP**

**Definition**

A decision problem \( L \subseteq W \) is **solvable in nondeterministic polynomial time** if there is a polynomial time computable map \( f: W \times W \rightarrow \{0, 1\} \) and \( c, d \in \mathbb{N} \) such that

1. if \( x \in L \), then there is \( y \in W \) so that \( |y| \leq |x|^c + d \) and \( f(x, y) = 1 \),
2. if \( x \notin L \), then for all \( y \in W \) we have \( f(x, y) = 0 \).

**NP** is the class of nondeterministic poly time solvable decision problems.

- \( f \) is the **verifier**, \( y \) is the **certificate**
- **informally**: \( x \in L \) iff there exists a short certificate \( y \in W \) that can be verified by a polynomial time algorithm
- **example**: \( L = \{3\text{-colorable graphs}\} \), certificate is \( g: G \rightarrow \{0, 1, 2\} \), verifier checks if adjacent vertices are assigned different colors
P versus NP problem

- $P \subseteq NP$, but we do not know if $P = NP$
- **P versus NP problem**: one of the Millennium Prize Problems proposed by the Clay Mathematics Institute, One million dollar prize

- We do not know if $NP = coNP$ where $coNP = \{ W \setminus L | L \in NP \}$
- **Integer factorization**: in $NP \cap coNP$ but probably not in $P$, decision problem encoded as “does $n$ have a prime factor smaller than $k$?”
Polynomial time reduction, the $\textbf{NP}$-complete class

**Definition**

$K \subset W$ is **polynomial time reducible** to $L \subset W$ if there is a polynomial time computable map $f : W \rightarrow W$ so that $x \in K \iff f(x) \in L$. They are **polynomial time equivalent**, if mutually reducible to each other.

- Polynomial time reducibility is a quasi order.
- Factoring out by polynomial time equivalence we get a poset.
- Minimal element is $\mathbb{P}$, we have joins, (what are the exact properties?)

**Definition**

$L \subset W$ is $\textbf{NP}$-complete if every $\textbf{NP}$-problem is poly time reducible to $L$.

- Boolean formula satisfiability (SAT, 3-SAT)
- Graph 3-coloring, solvable sudoku, graphs with Hamiltonian path, etc.
- Ladner’s theorem: if $\textbf{NP} \neq \mathbb{P}$, then there are intermediate classes.
Relational structures

Definition

\( \mathbb{A} = (A; R) \) is a **relational structure**, where for each relational symbol \( \varrho \in R \) of **arity** \( n \in \mathbb{N} \) we have a relation \( \varrho^A \subseteq A^n \). **Directed graph** is a relational structure \( \mathbb{G} = (G; \to) \) with a single binary relation \( \to^{\mathbb{G}} \subseteq G^2 \).

Definition

A **homomorphism** from \( \mathbb{A} = (A; R) \) to \( \mathbb{B} = (B; R) \) is a map \( f : A \to B \) that **preserves tuples**, i.e.

\[(a_1, \ldots, a_n) \in \varrho^A \implies (f(a_1), \ldots, f(a_n)) \in \varrho^B.\]

We write \( \mathbb{A} \to \mathbb{B} \) if there is a homomorphism from \( \mathbb{A} \) to \( \mathbb{B} \).

- **isomorphism**: bijective and both \( f \) and \( f^{-1} \) are homomorphisms
Definition

For a finite relational structure $\mathbb{B}$ we define

$$\text{CSP}(\mathbb{B}) = \{ A \mid A \rightarrow \mathbb{B} \}.$$  

- $\text{CSP}(\triangle)$ is the class of 3-colorable graphs  
- $\text{CSP}(\bullet)$ is the class of bipartite graphs

Dichotomy Conjecture (T. Feder, M. Y. Vardi, 1993)

For every finite structure $\mathbb{B}$ the membership problem for $\text{CSP}(\mathbb{B})$ is in $\mathbf{P}$ or $\mathbf{NP}$-complete.

The dichotomy conjecture is proved for example when $\mathbb{B}$
- is an undirected graph (P. Hell, J. Nešetřil),  
- has at most 3 elements (A. Bulatov)  

Open for directed graphs.
Example: solving a system of equations

$$(\exists x, y, z \in \mathbb{Z}_5)(x + y = z \land x + x = y \land z = 1)$$

$$(\exists x, y, z \in \mathbb{Z}_5)((x, y, z) \in F_1 \land (x, x, y) \in F_1 \land z \in F_2),$$

where $F_1 = \{ (x, y, z) \in \mathbb{Z}_5^3 : x + y = z \}$ and $F_2 = \{1\}$.

$$(\exists f : \{1, 2, 3\} \to \mathbb{Z}_5)((f(1), f(2), f(3)) \in F_1 \land (f(1), f(1), f(2)) \in F_1 \land f(3) \in F_2)$$

$$(\exists f : A \to B,$$

where $A = (\{1, 2, 3\}; E_1, E_2)$, $B = (\mathbb{Z}_5; F_1, F_2)$

$E_1 = \{ (1, 2, 3), (1, 1, 2) \}$, $E_2 = \{3\}.$

$A \in \text{CSP}(B)$
Example: solving a system of equations

\[(\exists x, y, z \in \mathbb{Z}_5)(x + y = z \land x + x = y \land z = 1)\]

\[\Downarrow\]

\[(\exists x, y, z \in \mathbb{Z}_5)((x, y, z) \in F_1 \land (x, x, y) \in F_1 \land z \in F_2),\]
where \(F_1 = \{(x, y, z) \in \mathbb{Z}_5^3 : x + y = z\}\) and \(F_2 = \{1\}\).

\[\Downarrow\]

\[(\exists f : \{1, 2, 3\} \rightarrow \mathbb{Z}_5)((f(1), f(2), f(3)) \in F_1 \land (f(1), f(1), f(2)) \in F_1 \land f(3) \in F_2)\]

\[\Downarrow\]

\[\exists f : \mathbb{A} \rightarrow \mathbb{B},\]
where \(\mathbb{A} = (\{1, 2, 3\}; E_1, E_2)\), \(\mathbb{B} = (\mathbb{Z}_5; F_1, F_2)\)

\(E_1 = \{(1, 2, 3), (1, 1, 2)\}\), \(E_2 = \{3\}\).

\[\Downarrow\]

\(\mathbb{A} \in \text{CSP}(\mathbb{B})\)
Example: solving a system of equations

\[(\exists x, y, z \in \mathbb{Z}_5)(x + y = z \land x + x = y \land z = 1)\]

\[\updownarrow\]

\[(\exists x, y, z \in \mathbb{Z}_5)((x, y, z) \in F_1 \land (x, x, y) \in F_1 \land z \in F_2),\]
where $F_1 = \{ (x, y, z) \in \mathbb{Z}_5^3 : x + y = z \}$ and $F_2 = \{1\}$.

\[\updownarrow\]

\[(\exists f : \{1, 2, 3\} \to \mathbb{Z}_5)((f(1), f(2), f(3)) \in F_1 \land (f(1), f(1), f(2)) \in F_1 \land f(3) \in F_2)\]

\[\updownarrow\]

\[\exists f : A \to B,\]
where $A = (\{1, 2, 3\}; E_1, E_2)$, $B = (\mathbb{Z}_5; F_1, F_2)$

$E_1 = \{ (1, 2, 3), (1, 1, 2) \}$, $E_2 = \{3\}$.

\[\updownarrow\]

$A \in \text{CSP}(B)$
Example: solving a system of equations

\[(\exists x, y, z \in \mathbb{Z}_5)(x + y = z \land x + x = y \land z = 1)\]

\[\Downarrow\]

\[(\exists x, y, z \in \mathbb{Z}_5)((x, y, z) \in F_1 \land (x, x, y) \in F_1 \land z \in F_2),\]

where \(F_1 = \{ (x, y, z) \in \mathbb{Z}_5^3 : x + y = z \}\) and \(F_2 = \{1\}.\)

\[\Downarrow\]

\[(\exists f : \{1, 2, 3\} \rightarrow \mathbb{Z}_5)((f(1), f(2), f(3)) \in F_1 \land (f(1), f(1), f(2)) \in F_1 \land f(3) \in F_2)\]

\[\Downarrow\]

\[\exists f : \mathbb{A} \rightarrow \mathbb{B},\]

where \(\mathbb{A} = (\{1, 2, 3\}; E_1, E_2), \mathbb{B} = (\mathbb{Z}_5; F_1, F_2)\)

\(E_1 = \{ (1, 2, 3), (1, 1, 2) \}, \ E_2 = \{3\}.\)

\[\Downarrow\]

\(\mathbb{A} \in CSP(\mathbb{B})\)
Example: solving a system of equations

\[(\exists x, y, z \in \mathbb{Z}_5) (x + y = z \land x + x = y \land z = 1)\]

\[\Downarrow\]

\[(\exists x, y, z \in \mathbb{Z}_5) ((x, y, z) \in F_1 \land (x, x, y) \in F_1 \land z \in F_2),\]

where \(F_1 = \{(x, y, z) \in \mathbb{Z}_5^3 : x + y = z\}\) and \(F_2 = \{1\}\).

\[\Downarrow\]

\[(\exists f : \{1, 2, 3\} \rightarrow \mathbb{Z}_5) ((f(1), f(2), f(3)) \in F_1 \land (f(1), f(1), f(2)) \in F_1 \land f(3) \in F_2)\]

\[\Downarrow\]

\[\exists f : \mathbb{A} \rightarrow \mathbb{B},\]

where \(\mathbb{A} = (\{1, 2, 3\}; E_1, E_2), \mathbb{B} = (\mathbb{Z}_5; F_1, F_2)\)

\(E_1 = \{(1, 2, 3), (1, 1, 2)\}, E_2 = \{3\}\).

\[\Downarrow\]

\(\mathbb{A} \in \text{CSP}(\mathbb{B})\)
Adding the equality relation

Lemma

Let $C = (B; R \cup \{\varepsilon\})$ be the extension of $B = (B; R)$ with the relation $\varepsilon = \{(b, b) \mid b \in B\}$. Then $\text{CSP}(B)$ is poly time equivalent with $\text{CSP}(C)$.

Proof.

- $\text{CSP}(B)$ is polynomial time reducible to $\text{CSP}(C)$
  - for $A = (A; R)$ we construct $A' = (A'; R \cup \{\varepsilon\})$ such that $A \in \text{CSP}(B) \iff A' \in \text{CSP}(C)$
  - let $A' = A$ and $\varepsilon^A = \emptyset$

- $\text{CSP}(C)$ is polynomial time reducible to $\text{CSP}(B)$
  - for $A = (A; R \cup \{\varepsilon\})$ we construct $A' = (A'; R)$ such that $A \in \text{CSP}(C) \iff A' \in \text{CSP}(B)$
  - let $\vartheta$ be the equivalence relation generated by $\varepsilon^A$
  - let $A' = A/\vartheta$ and $\varrho^{A'} = (\varrho^A)/\vartheta$
  - if $f : A \to C$, then $\vartheta \subseteq \ker f$ and $f' : A' \to B$, $f'(x/\vartheta) = f(x)$ works
  - if $g : A' \to B$, then $g' : A \to C$, $g'(x) = g(x/\vartheta)$ works
Adding projections and products (and intersections)

Lemma

Let $C = (B; R \cup \{\gamma\})$ be the extension of $B = (B; R)$ with a relation $\gamma$ that is a projection of $\beta \in R^B$. Then CSP($B$) is polynomial time equivalent with CSP($C$).

Idea: We let $A' = (A'; R)$ be the same as $A = (A; R \cup \{\gamma\})$, except for each tuple $(a_1, \ldots, a_k) \in \gamma^A$ we add $n - k$ new elements $x_{k+1}, \ldots, x_n$ to $A'$ and add the tuple $(a_1, \ldots, a_k, x_{k+1}, \ldots, x_n)$ to the relation $\beta^{A'}$.

Lemma

Let $C = (B; R \cup \{\gamma\})$ be the extension of $B = (B; R)$ with a relation $\gamma = \alpha \times \beta$ for $\alpha, \beta \in R^B$. Then CSP($B$) is polynomial time equivalent with CSP($C$).

Idea: We let $A' = (A'; R)$ be the same as $A = (A; R \cup \{\gamma\})$, except for each tuple $(a_1, \ldots, a_{n+m}) \in \gamma^A$ we add the tuple $(a_1, \ldots, a_n)$ to $\alpha^{A'}$ and the tuple $(a_{n+1}, \ldots, a_{n+m})$ to $\beta^{A'}$. 
Relational clones

**Definition**

A set $\Gamma$ of relations over a fixed set is a **relational clone** if it contains the equality relation and is closed under intersections, projections, products. The relational clone generated by $\Gamma$ is denoted by $\langle \Gamma \rangle$.

**Theorem**

Let $\mathbb{B} = (B; R)$ and $\mathbb{C} = (B; S)$ be finite relational structures on the same base set. If $\langle R^\mathbb{B} \rangle \subseteq \langle S^\mathbb{C} \rangle$, then $\text{CSP}(\mathbb{B})$ is poly time reducible to $\text{CSP}(\mathbb{C})$.

**Proof.**

Let $\mathbb{D} = (B; R \cup S)$ be the extension of both $\mathbb{B}$ and $\mathbb{C}$. Clearly, $\text{CSP}(\mathbb{B})$ is polynomial time reducible to $\text{CSP}(\mathbb{D})$. Since $R^\mathbb{B} \subseteq \langle S^\mathbb{C} \rangle$, we get by the previous lemmas, that $\text{CSP}(\mathbb{D})$ is poly time equivalent with $\text{CSP}(\mathbb{C})$.

We can assign an algorithmic complexity class to finitely generated relational clones (or functional clones)!
Reduction to cores

→ is a quasi order on the set of finite structures of same signature

**Lemma**

If $B \leftrightarrow C$, then $\text{CSP}(B) = \text{CSP}(C)$.

**Lemma**

Let $C$ be a minimal member of the $\leftrightarrow$ class of a finite structure $B$. Then

- every endomorphism of $C$ is an automorphism,
- $C$ is uniquely determined up to isomorphism, and
- $C$ is isomorphic to a substructure of $B$.

We say that $C$ is a core if it has no proper endomorphism

**Theorem**

Let $B$ be a finite relational structure and $C$ be its core. Then $\text{CSP}(B) = \text{CSP}(C)$.
Adding the singleton constant relations to cores

**Lemma**

Let $C = (B; \mathcal{R} \cup \{ \delta_b \mid b \in B \})$ be the extension of a core $B = (B; \mathcal{R})$ with $\delta_b = \{ b \}$ for $b \in B$. Then $\text{CSP}(B)$ is poly time equivalent with $\text{CSP}(C)$.

**Sketch of proof.**

Fix an ordering $B = \{ b_1, \ldots, b_k \}$ and consider the $k$-ary relation

$$\sigma^B = \{(f(b_1),\ldots,f(b_k)) \mid f : B \to B\}.$$ 

$\sigma^B$ is in $\langle \mathcal{R}^B \rangle$, so we may assume, that $\mathcal{R}$ already contains $\sigma$ and the equality relation $\varepsilon$. For $A = (A; \mathcal{R})$ define $A' = (A \cup B; \mathcal{R})$ as

$$\varepsilon^{A'} = \varepsilon^A \cup \{(a, b) \mid b \in B, a \in \delta^A_b\},$$

$$\sigma^{A'} = \sigma^A \cup \{(b_1,\ldots,b_k)\},$$

and

$$\varrho^{A'} = \varrho^A \text{ for all } \varrho \in \mathcal{R} \setminus \{\sigma, \varepsilon\}.$$
Finding solutions, limiting the signature

**Theorem**

If \( \text{CSP}(\mathbb{B}) \) is in \( \mathbf{P} \), then there exists a polynomial time algorithm that, for a given \( \mathbb{A} \), finds a homomorphism \( f: \mathbb{A} \rightarrow \mathbb{B} \) or proves that no such homomorphism exists.

**Theorem (T. Feder, M. Y. Vardi, 1993)**

For every finite relational structure \( \mathbb{B} \) there exists a directed graph \( \mathbb{G} \) such that \( \text{CSP}(\mathbb{B}) \) is polynomial time equivalent with \( \text{CSP}(\mathbb{G}) \).

Original proof “destroys algebraic structure”, there is a new proof by J. Bulin, D. Delić, M. Jackson and T. Niven, that preserve most linear idempotent Maltsev conditions (but not Maltsev operations).

**Theorem**

For every finite relational structure \( \mathbb{B} \) there exists a structure \( \mathbb{C} \) with only binary relations so that \( \text{CSP}(\mathbb{B}) \) is poly time equivalent with \( \text{CSP}(\mathbb{G}) \).
Finite duality

- set of finite relational structures modulo $\leftrightarrow$ is a partially ordered set
- isomorphic to the set of core isomorphism types
- minimal [maximal] element: 1-element structure, with empty [full] relations
- join: disjoint union, meet: direct product,
- satisfies distributive laws, join irreducible $=$ connected
- Heyting algebra (relatively pseudocomplemented)
- exponentiation: $B^A$ is defined on $B^A$ as $(f_1, \ldots, f_n) \in \varrho_B^A$ iff 
  $$(a_1, \ldots, a_n) \in \varrho_A \implies (f_1(a_1), \ldots, f_n(a_n)) \in \varrho_B.$$ 

- $B \land A \leq C \iff C^B \times A = (C^B)^A$ has a loop $\iff A \leq C^B$
- if $B$ is join irreducible with lower cover $C$, then $(B, C^B)$ is a dual pair

**Theorem (J. Nešetřil, C. Tardif, 2010)**

*Let $B$ be a finite connected core structure. Then $B$ has a dual pair $D$, i.e. $\text{CSP}(B) = \{ A \mid D \not\rightarrow A \}$, if and only if $B$ is a tree.*
A **polymorphism** of $\mathcal{B}$ is a homomorphism $p : \mathcal{B}^n \rightarrow \mathcal{B}$, that is an $n$-ary map that preserves the relations of $\mathcal{B}$, e.g. for a directed graph $\mathcal{B} = (B; \rightarrow)$

$$a_1 \rightarrow b_1, \ldots, a_n \rightarrow b_n \implies p(a_1, \ldots, a_n) \rightarrow p(b_1, \ldots, b_n).$$

$\text{Pol(}\mathcal{B}) = \{ p \mid p : \mathcal{B}^n \rightarrow \mathcal{B} \}$ is the **clone of polymorphisms**.

- if $\langle \mathcal{B} \rangle \subseteq \langle \mathcal{C} \rangle$ then $\text{CSP(}\mathcal{B}\rangle$ is poly time reducible to $\text{CSP(}\mathcal{C}\rangle$
- $\{ \langle \mathcal{B} \rangle \mid \text{CSP(}\mathcal{B}\rangle \in \mathbf{P} \}$ is a filter in the poset of finitely generated relational clones, $\{ \langle \mathcal{B} \rangle \mid \text{CSP(}\mathcal{B}\rangle \in \mathbf{NP}$-complete $\}$ is an ideal
- $\text{CSP(}\mathcal{B}\rangle$ is in $\mathbf{P}$ if $\mathcal{B}$ has nice polymorphisms

**Question**

Which polymorphisms guarantee that $\text{CSP(}\mathcal{B}\rangle$ is in $\mathbf{P}$?
Definition

Let $\mathbf{B} = (B; \mathcal{F})$ be a finite algebra. $(V, \mathcal{C})$ is an instance for $\text{CSP}(\mathbf{B})$ if

- $V$ is a finite set of variables,
- $\mathcal{C}$ is a finite set of constraints,
- where each constraint $(S, R) \in \mathcal{C}$ has
  - a scope $S \subseteq V$, and
  - a constraint relation $R \leq B^S$.

A map $f : V \rightarrow B$ is a solution if $f|_S \in R$ for all $(S, R) \in \mathcal{C}$. We define $\text{CSP}(\mathbf{B})$ to be the set of all solvable instances.

Theorem

$\text{CSP}(\mathbf{B})$ is polynomial time equivalent with a subproblem of $\text{CSP}(\mathbf{B})$ where all constraint relations must be in $\mathcal{R}^\mathbf{B}$ (after suitable ordering of elements).
Local and global reducibility

**Definition**

A relational structure $\mathcal{B} = (B; \mathcal{R})$ and an algebra $\mathcal{B} = (B; \mathcal{F})$ are **compatible** if $\mathcal{F}^B \subseteq \text{Pol}(\mathcal{B})$, or alternatively, $\mathcal{R}^B \subseteq \text{Inv}(\mathcal{B})$.

**Definition**

$\text{CSP}(\mathcal{B})$ is **locally polynomial time reducible** to $\text{CSP}(\mathcal{C})$ if for every relational structure $\mathcal{B}$ compatible with $\mathcal{B}$ there is a structure $\mathcal{C}$ compatible with $\mathcal{C}$ such that $\text{CSP}(\mathcal{B})$ is polynomial time reducible to $\text{CSP}(\mathcal{C})$.

Subtle difference between local and regular reducibility.

**Theorem**

*Let $\mathcal{B}$ and $\mathcal{C}$ be finite algebras. If $\mathcal{V}(\mathcal{B}) \subseteq \mathcal{V}(\mathcal{C})$, then $\text{CSP}(\mathcal{B})$ is locally polynomial time reducible to $\text{CSP}(\mathcal{C})$.***
Taylor terms

**Theorem (D. Hobby, R. McKenzie)**

For a locally finite variety $\mathcal{V}$ the followings are equivalent:

- $\mathcal{V}$ omits type 1 (tame congruence theory),
- $\mathcal{V}$ has a **Taylor** term operation:

  $t(x, x, \ldots, x) \approx x$,
  $t(x, -, \ldots, -) \approx t(y, -, \ldots, -)$,
  $t(-, x, \ldots, -) \approx t(-, y, \ldots, -)$,
  $$
  \vdots
  $$
  $t(-, -, \ldots, x) \approx t(-, -, \ldots, y)$.

**Theorem (W. Taylor, 1977)**

Every idempotent, locally finite variety without a Taylor term contains a two-element algebra in which every operation is a projection.
The **NP**-complete case

**Theorem (A. Bulatov, P. Jeavons, A. Krokhin)**

*If $\mathbb{B}$ is a finite core relational structure without a Taylor polymorphism, then $\text{CSP}(\mathbb{B})$ is **NP**-complete.*

**Proof.**

- $\mathbb{B}$ is a core, so we may assume $\{b\}$ is a relation for all $b \in B$
- the algebra $\mathbb{B} = (B; \text{Pol}(\mathbb{B}))$ is idempotent
- the variety $\mathcal{V}(\mathbb{B})$ contains a two-element trivial algebra $\mathbb{C}$
- 3-SAT is poly equivalent with $\text{CSP}(\mathbb{C})$ for some $\mathbb{C}$ compatible with $\mathbb{C}$
- $\text{CSP}(\mathbb{C}) \leq \text{CSP}(\mathbb{D}) \leq \text{CSP}(\mathbb{B})$ for some $\mathbb{D}$ compatible with $\mathbb{B}$

**Algebraic Dichotomy Conjecture**

*If $\mathbb{B}$ is a core and has a Taylor polymorphism, then $\text{CSP}(\mathbb{B})$ is in **P**.*
CSP for semilattice algebras

Theorem

Let $B$ be a finite algebra with a semilattice term operation. Then $\text{CSP}(B)$ is solvable in polynomial time.

Sketch of proof.

- Take an instance $(V; C)$ for $\text{CSP}(B)$
- Add the $(\{x\}, B)$ constraint for each variable $x \in V$
- For each scope $S$ create a single constraint relation $R_S \leq B^S$
- Modify the instance until $\pi_x(R_S) = R_{\{x\}}$ for all scope $S$ and $x \in S$:
  - $R'_{\{x\}} = R_{\{x\}} \cap \pi_x(R_S)$
  - $R'_S = \{ f \in R_S \mid f(x) \in R_{\{x\}} \}$.
- The new instance has the same set of solutions as the original
- Define $g : V \to B$, $g(x) = \bigwedge R_{\{x\}} \in R_{\{x\}}$
- For $(S, R_S)$ and $x \in S$ we have $f_x \in R_S$ with $f_x(x) = g(x)$
- Take $f = \bigwedge_{x \in S} f_x \in R_S$ and verify that $g|_S = f$, so $g$ is a solution
An instance \((V; C)\) for CSP(B) is \((k, l)\)-consistent, if
- for each scope \(S\) it has a unique constraint \((S; R_S)\),
- it contains a constraint for each scope \(S \subseteq V, |S| \leq l\), and
- \(\pi_S(R_T) = R_S\) whenever \(S \subseteq T\) are scopes and \(|S| \leq k\).

For every instance \((V; C)\) for CSP(B) a \((k, l)\)-consistent instance \((V, C')\) can be computed in polynomial time that has the same set of solutions.

Let \(B\) be a finite algebra with a \(k\)-ary near-unanimity term operation. Then CSP(B) is solvable in polynomial time.

- apply the \((k - 1, k)\) local consistency algorithm
- the instance has a solution iff all constraint relations are nonempty
Bounded width

Definition
A relational structure $\mathcal{B}$ has **bounded width** if there exist $k \leq l$ such that every nonempty $(k, l)$-consistent instance for CSP($\mathcal{B}$) has a solution.

If $\mathcal{B}$ has bounded width, then CSP($\mathcal{B}$) is solvable in polynomial time by the local consistency algorithm.

Theorem (B. Larose, L. Zádori, 2007)
*If a core relational structure $\mathcal{B}$ has bounded width, then the corresponding algebra $\mathcal{B} = (B; \text{Pol}(\mathcal{B}))$ generates congruence meet-semidistributive variety.*

Theorem (L. Barto, M. Kozik, 2009)
*A core relational structure $\mathcal{B}$ has bounded width if and only if the corresponding algebra $\mathcal{B} = (B; \text{Pol}(\mathcal{B}))$ generates congruence meet-semidistributive variety.*
**CSP for Maltsev algebras**

**Definition**

Let $B$ be an algebra with a Maltsev term $p$, and $n \in \mathbb{N}$.

- **index** is an element of $\{1, \ldots, n\} \times B^2$,
- an index $(i, a, b)$ is **witnessed** in $Q \subseteq B^n$ if there exist $f, g \in Q$ so that $f_1 = g_1, \ldots, f_{i-1} = g_{i-1}$ and $f_i = a, g_i = b$
- a **compact representation** of a subpower $R \leq B^n$ is a subset $Q \subseteq R$ that witnesses the same set of indices as $R$ and $|Q| \leq 2|B|^2 \cdot n$.

**Lemma**

*The compact representation $Q$ of $R \leq B^n$ generates $R$ as a subalgebra.*

- Idea: take $f \in R$ and its best approximation $g \in \text{Sg}(Q)$
- let $i$ be the smallest index where $f_i \neq g_i$
- take witnesses $f', g' \in Q$ for the index $(i, f(i), g(i))$
- but then $p(f', g', g)$ is a better approximation of $f$
Lemma

The 2-projections of $R \leq B^n$ are polynomial time computable from the compact representation of $R$.

- Idea: generate as usual, but keep track of representative tuples only

Lemma

For $c_1, \ldots, c_k \in B$ the compact representation of the subpower $R' = \{ f \in R | f_1 = c_1, \ldots, f_k = c_k \}$ is poly time computable from that of $R$.

- Idea: we prove it for $k = 1$ and use induction
- take $f, g \in R'$ witnesses for $(i, a, b)$ in $R'$
- then we have witnesses $f', g' \in Q$ for $(i, a, b)$, and
- $h \in \text{Sg}(Q)$ such that $h_1 = c$ and $h_i = a$
- thus $h, p(h, f', g') \in \text{Sg}(Q)$ witness $(i, a, b)$ in $R'$
Lemma

For $c \in B$ and $k \leq n$ the compact representation of the subpower $R' = \{ f \in R | f_k = c \}$ is polynomial time computable from that of $R$.

Lemma

For $1 \leq k, l \leq n$ the compact representation of the subpower $R' = \{ f \in R | f_k = f_l \}$ is polynomial time computable from that of $R$.

Theorem

Let $B$ be a finite Maltsev algebra. Then the compact representation of the product, projection and intersection of subpowers is computable in polynomial time from the compact representations of the arguments.

Theorem (A. Bulatov, V. Dalmau, 2006)

Let $B$ be a finite algebra with a Maltsev term operation. Then $\text{CSP}(B)$ is solvable in polynomial time.
CSP for few subpower algebras

**Definition**

A finite algebra $B$ has **few subpowers**, if the number of subalgebras of $B^n$ is bounded by $2^{n^a+b}$ for some fixed numbers $a, b \in \mathbb{N}$.

**Theorem (Berman, Idziak, Marković, McKenzie, Valeriote, Willard)**

A finite algebra $B$ has few subpowers iff it has a $k$-edge term operation

$$p(y, y, x, x, \ldots, x) \approx x,$$
$$p(x, y, y, x, \ldots, x) \approx x,$$
$$p(x, x, y, x, \ldots, x) \approx x,$$
$$\ldots$$
$$p(x, x, x, x, \ldots, y) \approx x.$$

**Theorem (Idziak, Marković, McKenzie, Valeriote, Willard, 2010)**

Let $B$ be a finite algebra with a $k$-edge term operation. Then $\text{CSP}(B)$ can be solved in polynomial time.
Theorem (McKenzie, Maróti, 2007)

For a locally finite variety has a Taylor term if and only if it has a weak near-unanimity term operation:

\[ t(y, x, \ldots, x) \equiv \cdots \equiv t(x, \ldots, x, y) \quad \text{and} \quad t(x, \ldots, x) \equiv x. \]

Theorem (L. Barto, M. Kozik, 2009)

Let \( G \) be a core directed graph with no sources and sinks. If \( G \) has a weak near-unanimity polymorphism, then it is a disjoint union of directed circles. So the dichotomy conjecture holds for digraphs with no sources and sinks.

The next classical theorem is now an easy corollary:

Theorem (P. Hell, J. Nešetřil, 1990)

Let \( G \) be an undirected graph. If \( G \) is bipartite, then \( \text{CSP}(G) \) is in \( P \), otherwise it is \( \text{NP} \)-complete.
Siggers term

**Theorem (M. Siggers, 2008; K. Kearnes, P. Markovic, R. McKenzie)**

A locally finite variety has a Taylor term if and only if it has Siggers term

\[ t(x, x, x, x) \approx x \quad \text{and} \quad t(x, y, z, y) \approx t(y, z, x, x). \]

**Proof.**

- let \( G = F_3(\mathcal{V}) \) be the 3-generated free algebra
- let \( \mathcal{G} = (G; \rightarrow) \) be the digraph \( \mathcal{G} = \text{Sg}(\{(x, y), (y, z), (z, x), (y, x)\}) \) whose edge relation is generated by these edges

\[ z \rightarrow y \rightarrow x \rightarrow z \]

- no sources and sinks: \( t(x, y, z) \rightarrow t(y, z, x) \rightarrow t(z, x, y) \rightarrow t(x, y, z) \)
- because of the generating edges, the core of \( \mathcal{G} \) must contain a loop
- the loop edge is \( t(((x, y), (y, z), (z, x), (x, z))) \) for some term \( t \).
Further algebraic results

**Theorem (E. Aichinger, P. Mayr, R. McKenzie, 2011)**

There are countable many Maltsev clones on a finite set. The same holds for clones with an edge operation.

**Theorem**

A finite algebra generates a congruence meet-semidistributive variety if and only if it has a ternary and a 4-ary weak near-unanimity operation $s$ and $t$ such that $t(x, x, y) \approx s(x, x, x, y)$.

**Theorem (L. Barto, 2011)**

If a finite relational structure has Jónsson polymorphisms, then it has a near-unanimity polymorphism.

**Theorem (L. Barto)**

If a finite relational structure has Gumm polymorphisms (congruence modularity), then it has an edge polymorphism.
Thank you!
The Constraint Satisfaction Problem
Theorem (M. Maróti)

Suppose, that each algebra $B \in \mathcal{B}$ has a congruence $\beta \in \text{Con}(B)$ such that $B / \beta$ has few subpowers and each $\beta$ block has bounded width. Then we can solve the constraint satisfaction problem over $\mathcal{B}$ in polynomial time.

Proof Overview.

- Take an instance $\mathcal{A} = \{ B_i, R_{ij} \mid i, j \in V \}$ and $\beta_i \in \text{Con}(B_i)$
- Consider extended constraints that not only limit the projection of the solution set to the $\{i, j\}$ coordinates, but also to $\prod_{v \in V} B_v / \beta_v$
- Use extended (2,3)-consistency algorithm
- Obtain a solution modulo the $\beta$ congruences so that the restriction of the problem to the selected congruence blocks is (2,3)-consistent.
- By the bounded width theorem there exists a solution.