Polymorphisms of reflexive digraphs

Miklós Maróti and László Zádori

Bolyai Institute, University of Szeged, Hungary

**Definition**

- A **digraph** is a pair $G = (G; \rightarrow)$, where $G$ is the set of vertices and $\rightarrow \subseteq G^2$ is the set of edges.

- A **relational structure** is a tuple $G = (G; E_1, \ldots, E_k)$, where $G$ is the underlying set and $E_i \subseteq G^{n_i}$ is an $n_i$-ary relation.

**Definition**

A **homomorphism** from a digraph $G$ to $H$ is a map $f : G \to H$ that preserves edges

$$a \rightarrow b \text{ in } G \quad \implies \quad f(a) \rightarrow f(b) \text{ in } H.$$  

We write $G \to H$ if there exists a homomorphism from $G$ to $H$. 
**CONSTRAINT SATISFACTION PROBLEM (CSP)**

**Definition**
For a finite relational structure $\mathbb{H}$ we define

$$\text{CSP}(\mathbb{H}) = \{ G | G \rightarrow \mathbb{H} \}.$$ 

**Example**
- $\text{CSP}(\triangle)$ is the class of three-colorable graphs.
- $\text{CSP}(\bullet)$ is the class of bipartite graphs.
The membership problem for CSP(\(\mathcal{H}\))
- always decidable in nondeterministic polynomial time (NP)
- is decidable in polynomial time (P) for some \(\mathcal{H}\)

**Dichotomy Conjecture (Feder, Vardi, 1993)**

For every finite structure \(\mathcal{H}\) the membership problem for CSP(\(\mathcal{H}\)) is in P or \(\text{NP}\)-complete.

The dichotomy conjecture holds when \(\mathcal{H}\)
- is an undirected graph (Hell, Nešetřil), or
- has at most 3 elements (Bulatov), or
- a smooth directed graph (Barto, Kozik, Niven).

Open for directed graphs.
The membership problem for CSP($\mathbb{H}$)

- always decidable in nondeterministic polynomial time ($\text{NP}$)
- is decidable in polynomial time ($\text{P}$) for some $\mathbb{H}$

**Dichotomy Conjecture (Feder, Vardi, 1993)**

For every finite structure $\mathbb{H}$ the membership problem for CSP($\mathbb{H}$) is in $\text{P}$ or $\text{NP}$-complete.

The dichotomy conjecture holds when $\mathbb{H}$

- is an undirected graph (Hell, Nešetřil), or
- has at most 3 elements (Bulatov), or
- a smooth directed graph (Barto, Kozik, Niven).

Open for directed graphs.
The membership problem for $\text{CSP}(H)$

- always decidable in nondeterministic polynomial time ($\text{NP}$)
- is decidable in polynomial time ($\text{P}$) for some $H$

**Dichotomy Conjecture (Feder, Vardi, 1993)**

For every finite structure $H$ the membership problem for $\text{CSP}(H)$ is in $\text{P}$ or $\text{NP}$-complete.

The dichotomy conjecture holds when $H$

- is an undirected graph (Hell, Nešetřil), or
- has at most 3 elements (Bulatov), or
- a smooth directed graph (Barto, Kozik, Niven).

Open for directed graphs.
\[ \exists x, y, z \in \mathbb{Z}_5 (x + y = z \land x + x = y \land z = 1) \]

\[ \Leftrightarrow \]

\[ \exists x, y, z \in \mathbb{Z}_5 ((x, y, z) \in F_1 \land (x, x, y) \in F_1 \land z \in F_2) \]

where \( F_1 = \{ (x, y, z) \in \mathbb{Z}_5^3 : x + y = z \} \) and \( F_2 = \{1\} \).

\[ \Leftrightarrow \]

\[ \exists f : \{1, 2, 3\} \rightarrow \mathbb{Z}_5 ((f(1), f(2), f(3)) \in F_1 \land (f(1), f(1), f(2)) \in F_1 \land f(3) \in F_2) \]

\[ \Leftrightarrow \]

\[ \exists f : G \rightarrow H, \]

where \( G = (\{1, 2, 3\}; E_1, E_2) \), \( H = (\mathbb{Z}_5; F_1, F_2) \)

\( E_1 = \{ (1, 2, 3), (1, 1, 2) \} \), \( E_2 = \{3\} \).

\[ \Leftrightarrow \]

\( G \in \text{CSP}(H) \)
EXAMPLE: SOLVING A SYSTEM OF EQUATIONS

\[(\exists x, y, z \in \mathbb{Z}_5)(x + y = z \land x + x = y \land z = 1)\]

\[\updownarrow\]

\[(\exists x, y, z \in \mathbb{Z}_5)((x, y, z) \in F_1 \land (x, x, y) \in F_1 \land z \in F_2),\]

where \( F_1 = \{ (x, y, z) \in \mathbb{Z}_5^3 : x + y = z \} \) and \( F_2 = \{1\}. \)

\[\updownarrow\]

\[(\exists f : \{1, 2, 3\} \rightarrow \mathbb{Z}_5)((f(1), f(2), f(3)) \in F_1 \land (f(1), f(1), f(2)) \in F_1 \land f(3) \in F_2)\]

\[\updownarrow\]

\[\exists f : G \rightarrow H,\]

where \( G = (\{1, 2, 3\}; E_1, E_2) \) and \( H = (\mathbb{Z}_5; F_1, F_2) \)

\[E_1 = \{ (1, 2, 3), (1, 1, 2) \}, \quad E_2 = \{3\}. \]

\[\updownarrow\]

\[G \in \text{CSP}(H)\]
Example: solving a system of equations

\[(\exists x, y, z \in \mathbb{Z}_5)(x + y = z \land x + x = y \land z = 1)\]

\[\iff\]

\[(\exists x, y, z \in \mathbb{Z}_5)((x, y, z) \in F_1 \land (x, x, y) \in F_1 \land z \in F_2),\]

where \(F_1 = \{(x, y, z) \in \mathbb{Z}_5^3 : x + y = z\}\) and \(F_2 = \{1\}\).

\[\iff\]

\[(\exists f : \{1, 2, 3\} \to \mathbb{Z}_5)((f(1), f(2), f(3)) \in F_1 \land (f(1), f(1), f(2)) \in F_1 \land f(3) \in F_2)\]

\[\iff\]

\[\exists f : G \to H,\]

where \(G = (\{1, 2, 3\}; E_1, E_2)\), \(H = (\mathbb{Z}_5; F_1, F_2)\)

\(E_1 = \{(1, 2, 3), (1, 1, 2)\}\), \(E_2 = \{3\}\).

\[\iff\]

\(G \in \text{CSP}(H)\)
Example: solving a system of equations

\[(\exists x, y, z \in Z_5)(x + y = z \land x + x = y \land z = 1)\]

\[\mapsto\]

\[(\exists x, y, z \in Z_5)((x, y, z) \in F_1 \land (x, x, y) \in F_1 \land z \in F_2),\]

where \(F_1 = \{ (x, y, z) \in Z_5^3 : x + y = z \}\) and \(F_2 = \{1\}\).

\[\mapsto\]

\[(\exists f : \{1, 2, 3\} \rightarrow Z_5)((f(1), f(2), f(3)) \in F_1 \land (f(1), f(1), f(2)) \in F_1 \land f(3) \in F_2)\]

\[\mapsto\]

\[\exists f : G \rightarrow H,\]

where \(G = (\{1, 2, 3\}; E_1, E_2), H = (Z_5; F_1, F_2)\)

\(E_1 = \{ (1, 2, 3), (1, 1, 2) \}, E_2 = \{3\}\).

\[\mapsto\]

\(G \in CSP(H)\)
Example: solving a system of equations

\[(\exists x, y, z \in \mathbb{Z}_5)(x + y = z \land x + x = y \land z = 1)\]

\[\iff\]

\[(\exists x, y, z \in \mathbb{Z}_5)((x, y, z) \in F_1 \land (x, x, y) \in F_1 \land z \in F_2),\]

where \( F_1 = \{ (x, y, z) \in \mathbb{Z}_5^3 : x + y = z \} \) and \( F_2 = \{1\}. \)

\[\iff\]

\[(\exists f : \{1, 2, 3\} \to \mathbb{Z}_5)((f(1), f(2), f(3)) \in F_1 \land (f(1), f(1), f(2)) \in F_1 \land f(3) \in F_2)\]

\[\iff\]

\[\exists f : G \to H,\]

where \( G = (\{1, 2, 3\}; E_1, E_2), \quad H = (\mathbb{Z}_5; F_1, F_2) \)

\( E_1 = \{(1, 2, 3), (1, 1, 2)\}, \quad E_2 = \{3\}. \)

\[\iff\]

\( G \in \text{CSP}(H) \)
Lemma

If $H_1 \leftrightarrow H_2$, then $\text{CSP}(H_1) = \text{CSP}(H_2)$. In particular, if $r : H \to H$ is a retraction ($r^2 = r$), then $\text{CSP}(H) = \text{CSP}(H|_{r(H)})$.

Lemma

For every finite relational structure $H_1$ there exists $H_2$ such that

1. $H_2$ is a directed graph (with unary relations),
2. $H_2$ is a core, i.e., every endomorphism is bijective,
3. every singleton unary relation $\{a\}$ is in $H_2$, and

$\text{CSP}(H_1)$ is polynomial time equivalent to $\text{CSP}(H_2)$. 

CSP REDUCTIONS: CORES
CSP reductions: polymorphisms

Definition

A **polymorphism** of $\mathbb{H}$ is a homomorphism $p : \mathbb{H}^n \to \mathbb{H}$, that is an $n$-ary map that preserves edges

$$a_1 \to b_1, \ldots, a_n \to b_n \implies p(a_1, \ldots, a_n) \to p(b_1, \ldots, b_n).$$

$\text{Pol}(\mathbb{H}) = \{ p \mid p : \mathbb{H}^n \to \mathbb{H} \}$ is the **clone of polymorphisms**.

Lemma

*If $\mathbb{H}_1, \mathbb{H}_2$ have the same underlying set and $\text{Pol}(\mathbb{H}_1) \subseteq \text{Pol}(\mathbb{H}_2)$, then $\text{CSP}(\mathbb{H}_2)$ is polynomial time reducible to $\text{CSP}(\mathbb{H}_1)$.**

Question

Which polymorphisms guarantee that $\text{CSP}(\mathbb{H})$ is in $\mathsf{P}$?
**Definition**

A **polymorphism** of $\mathbb{H}$ is a homomorphism $p : \mathbb{H}^n \to \mathbb{H}$, that is an $n$-ary map that preserves edges

$$a_1 \to b_1, \ldots, a_n \to b_n \implies p(a_1, \ldots, a_n) \to p(b_1, \ldots, b_n).$$

$\text{Pol}(\mathbb{H}) = \{ p \mid p : \mathbb{H}^n \to \mathbb{H} \}$ is the **clone of polymorphisms**.

**Lemma**

*If $\mathbb{H}_1$, $\mathbb{H}_2$ have the same underlying set and $\text{Pol}(\mathbb{H}_1) \subseteq \text{Pol}(\mathbb{H}_2)$, then $\text{CSP}(\mathbb{H}_2)$ is polynomial time reducible to $\text{CSP}(\mathbb{H}_1)$.***

**Question**

Which polymorphisms guarantee that $\text{CSP}(\mathbb{H})$ is in $\text{P}$?
NICE POLYMORPHISMS

Theorem

CSP(\(H\)) is in \(\mathbf{P}\) if \(\operatorname{Pol}(H)\) contains one of the following:

- **a semilattice operation** (Jevons et. al.)
  
  \[ x \land y \approx y \land x, \quad x \land (y \land z) \approx (x \land y) \land z, \quad x \land x \approx x. \]

- **a near-unanimity operation**
  
  \[ p(y, x, \ldots, x) \approx p(x, y, x, \ldots, x) \approx \cdots \approx p(x, \ldots, x, y) \approx x, \]

- **a totally symmetric idempotent operation** (Dalmau, Pearson),
  
  \[ \{x_1, \ldots, x_n\} = \{y_1, \ldots, y_n\} \implies p(x_1, \ldots, x_n) \approx p(y_1, \ldots, y_n) \]

- **a Maltsev operation** (Bulatov, Dalmau)
  
  \[ p(x, y, y) \approx p(y, y, x) \approx x, \]
Theorem

\textit{CSP}(H) is in \textbf{P} if \textit{Pol}(H) contains one of the following:

- \textit{Edge operations} (Idziak, Marković, McKenzie, Valeriote, Willard)

\[
p(y, y, x, x, \ldots, x) \approx x,
\]
\[
p(x, y, y, x, \ldots, x) \approx x,
\]
\[
p(x, x, x, y, \ldots, x) \approx x,
\]
\[
\vdots
\]
\[
p(x, x, x, x, \ldots, y) \approx x.
\]

- \textit{Jónsson operations} (Barto, Kozik),

- \textit{Willard operations} (Barto, Kozik).
Weak near-unanimity

Theorem (McKenzie, Maróti)

For a locally finite variety $\mathcal{V}$ the followings are equivalent:

- $\mathcal{V}$ omits type 1 (tame congruence theory),
- $\mathcal{V}$ has a Taylor term,
- $\mathcal{V}$ has a weak near-unanimity operation:
  \[ p(y, x, \ldots, x) \approx \cdots \approx p(x, \ldots, x, y) \quad \text{and} \quad p(x, \ldots, x) \approx x. \]

Theorem (Bulatov, Larose, Zádori)

If $\mathcal{H}$ is a core and does not have a Taylor (or weak near-unanimity) polymorphism, then $\text{CSP}(\mathcal{H})$ is NP-complete.

Algebraic dichotomy conjecture

If $\mathcal{H}$ is a core and has a weak near-unanimity polymorphism, then $\text{CSP}(\mathcal{H})$ is in P.
**Theorem (Barto)**

If a finite relational structure has Jónsson polymorphisms

\[ x = d_0(x, y, z), \]
\[ d_i(x, y, x) = x \text{ for all } i, \]
\[ d_i(x, y, y) = d_{i+1}(x, y, y) \text{ for even } i, \]
\[ d_i(x, x, y) = d_{i+1}(x, x, y) \text{ for odd } i, \]
\[ d_n(x, y, z) = z, \]

then it has a near-unanimity polymorphism.

**Valeriote’s Conjecture**

If a finite relational structure has Gumm polymorphisms, then it has an edge polymorphism.
**Theorem (Larose, Zádori)**

*If a finite poset has Gumm polymorphisms*

\[
\begin{align*}
x & = d_0(x, y, z), \\
d_i(x, y, x) & = x \text{ for all } i, \\
d_i(x, y, y) & = d_{i+1}(x, y, y) \text{ for even } i, \\
d_i(x, x, y) & = d_{i+1}(x, x, y) \text{ for odd } i, \\
d_n(x, y, y) & = p(x, y, y), \text{ and} \\
p(x, x, y) & = y,
\end{align*}
\]

*then it has a near-unanimity polymorphism.*

**Theorem (Kun, Szabó)**

*There is a polynomial algorithm for checking if a poset has a near-unanimity polymorphism.*
Theorem (Larose, Loten, Zádori)

*If a finite symmetric reflexive digraph has Gumm polymorphisms, then it has a near-unanimity polymorphism.*

Theorem (Kazda)

*If a finite digraph has Maltsev polymorphism*

\[ p(y, x, x) \approx p(x, x, y) \approx y \]

*then it has majority polymorphism*

\[ m(x, x, y) \approx m(x, y, x) \approx m(y, x, x) \approx x. \]

Conjecture

*If a finite digraph has Gumm polymorphisms, then it has an near-unanimity polymorphism.*
**Main Result**

**Theorem**

*If a finite reflexive digraph $G$ has Gumm polymorphisms*

\[ x = d_0(x, y, z), \]
\[ d_i(x, y, x) = x \text{ for all } i, \]
\[ d_i(x, y, y) = d_{i+1}(x, y, y) \text{ for even } i, \]
\[ d_i(x, x, y) = d_{i+1}(x, x, y) \text{ for odd } i, \]
\[ d_n(x, y, y) = p(x, y, y), \text{ and} \]
\[ p(x, x, y) = y, \]

*then it has Jónsson polymorphisms (same as above, but $p(x, y, y) \approx y$).*
A digraph $G$ is **connected** if for all $a, b \in G$ there exists a path

$$a = a_0 \rightarrow a_1 \leftarrow a_2 \rightarrow a_3 \leftarrow \cdots \rightarrow a_n = b$$

with some pattern.

$G$ is **strongly connected** if for all $a, b \in G$ there exist paths

$$a = a_0 \rightarrow a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow \cdots \rightarrow a_n = b,$$

$$a = b_0 \leftarrow b_1 \leftarrow b_2 \leftarrow b_3 \leftarrow \cdots \leftarrow b_n = b.$$

$G$ is **extremely connected** if for all $a, b \in G$ there exist a path

$$a = a_0 \leftrightarrow a_1 \leftrightarrow a_2 \leftrightarrow a_3 \leftrightarrow \cdots \leftrightarrow a_n = b.$$
**Definition**

Let $G$, $H$ be digraphs and $f, g \in H^G$ be maps. We write $f \to g$ iff

\[
 a \to b \text{ in } G \implies f(a) \to g(b) \text{ in } H.
\]

**Lemma**

- *The set of homomorphisms from $G$ to $H$ is*

\[
 H^G = \{ f \in H^G \mid f \to f \}.
\]

- *If $G$ is reflexive, then the Cartesian power of $G$ is*

\[
 G^n = G\{ \circ \circ \cdots \circ \}.
\]

- *If $f \to g$ in $H^G^n$ and $f_1 \to g_1, \ldots, f_n \to g_n$ in $G^F$, then *

\[
 f(f_1, \ldots, f_n) \to g(g_1, \ldots g_n) \text{ in } H^F.
\]
**Theorem**

Let $G$ be a finite reflexive digraph admitting Gumm operations. If $G$ is [strongly, extremely] connected, then so is $G^G$.

**Proof.**

- Take a minimal counterexample $G$.
- $\{\text{id}\}$ is a [strong, extreme] component of $G^G$.
- If $G$ admits a ternary operation $d$ satisfying
  - $d(x, y, y) \approx x$, or
  - $d(x, y, x) \approx d(x, x, y) \approx x$,

  then $d(x, y, z)$ is the first projection.
- Use the Gumm identities (or Hobby-McKenzie operations for omitting types 1 and 5) to show that $G$ satisfies $x \approx y$. 


**Lemma**

Let $G$ be a finite reflexive digraph admitting Gumm operations. If $G$ is [strongly, extremely] connected, then so is the digraph

$$I_2(G) = \{ f \in G^{G^2} \mid f(x, x) \approx x \}.$$

of idempotent binary polymorphisms of $G$.

**Definition**

A digraph $K \leq G^H$ is a **idempotent $G$-subalgebra**, if it is closed under the idempotent polymorphisms of $G$.

**Corollary**

Let $G$ be a finite reflexive digraph admitting Gumm operations. If $G$ is strongly connected, then it is extremely connected.
**Definition**

An edge $f \rightarrow g$ in $\mathbb{G}^G$ is **refinable** if there exists $h \in \mathbb{G}^G$ such that

- $f \rightarrow h \rightarrow g$ and $f \neq h \neq g$, and
- $h(x) \in \{f(x), g(x)\}$ for all $x \in G$.

**Lemma**

If $f \rightarrow g$ is non-refinable, then $[[f \neq g]]$ is strongly connected.

**Proof.**

1. Take a nonconstant $\varphi : [[f \neq g]] \rightarrow \mathbb{G}^G|\{f,g\}$ homomorphism.
2. Look at the refinement $f \rightarrow h \rightarrow g$ where

$$h(x) = \begin{cases} (\varphi(x))(x), & \text{if } x \in [[f \neq g]] \\ f(x), & \text{otherwise.} \end{cases}$$
Getting Jónsson operations

**Theorem**

If a finite reflexive digraph $G$ has Gumm polymorphisms, then it has Jónsson polymorphisms.

**Proof.**

- The digraph $\{ f \in G^3 | f(x, y, x) = (x, y, x) \}$ is connected.
- Connect $\text{id}$ with $s(x, y, z) = (z, y, z)$ via non-refinable links.
- We want a connection $\text{id} = f_0, f_1, \ldots, f_n = s$ such that $x = \pi_1(f_0), \pi_1(f_1), \ldots, \pi_1(f_n) = z$ are Jónnson operations.
- Bad link: $[[f \neq g]]$ contains both $(a, a, b)$ and $(c, d, d)$, then $[[f \neq g]] \subseteq C^3$ for some strongly connected component $C$.
- $C$ is extremely connected, so we can replace the bad $(f, g)$ link with $f = h_0 \leftrightarrow h_1 \leftrightarrow \cdots \leftrightarrow h_n = g$.
- Refine all these links and we have no more bad links.
Let $G$ be a poset, or a reflexive symmetric digraph. If $f \rightarrow g$ is a non-refinable edge in $G^G$, then $|[f \neq g]| = 1$.

This is not true for reflexive digraphs:

```
1 2
\bullet \leftrightarrow \bullet
```

Dismantability is defined via “one-point elementary retractions”.

**Theorem**

*If a finite symmetric digraph $G$ admits Gumm operations, then it admits Jónsson operations.*
Decidability of NU

Theorem

If a finite reflexive digraph $G$ admits a sequence of Jónsson operations, then it has one with length at most $16 \cdot |G|^7$.

Corollary

Given a finite reflexive digraph, it is decidable in polynomial time if it admits a near-unanimity operation.

Proof.

Use CSP and the bounded width algorithm.

Theorem (Maróti)

Given a finite algebra $A$, it is decidable if it has a near-unanimity term.
Decidability of NU

Theorem

If a finite reflexive digraph $G$ admits a sequence of Jónsson operations, then it has one with length at most $16 \cdot |G|^7$.

Corollary

Given a finite reflexive digraph, it is decidable in polynomial time if it admits a near-unanimity operation.

Proof.

Use CSP and the bounded width algorithm.

Theorem (Maróti)

Given a finite algebra $A$, it is decidable if it has a near-unanimity term.
**Totally symmetric operations**

**Definition**
A digraph has the **fixed clique property**, if every endomorphism preserves some clique of the digraph.

**Lemma**
Every finite connected reflexive digraph that admits a near-unanimity operation has the fixed clique property.

**Lemma**
Every finite reflexive digraph that admits a near-unanimity operation also admits cyclic idempotent operations of all arities.

**Theorem**
Every finite reflexive digraph that admits an NU operation also admits totally symmetric idempotent operations of all arities.
Thank you!