Minimal quasivarieties of semilattices with a group of automorphisms

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The variety of $F$-semilattices

Nashville

The quasivariety of $F$-semilattices

Tournaments

1996: \textbf{F-SEMILATTICES}

\begin{itemize}
  \item \textbf{Definition} \ F\text{-semilattice is an algebra } S = \langle S; \land, F \rangle \text{ where}
  \begin{itemize}
    \item $\langle S; \land \rangle$ is a semilattice,
    \item $F = \langle F; \cdot, -1, \text{id} \rangle$ is a group,
    \item $F$ acts on $\langle S; \land \rangle$ as automorphisms.
  \end{itemize}
  
  For a fixed group $F$ the class of $F$-semilattices is a variety.
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  \item Kearnes, Szendrei (97): Self-rectangulating varieties of type 5.
  \item Burris, Valeriote (1983): Expanding varieties by monoids of endomorphisms.
  \item Ježek (1991): Subdirectly irreducible $\mathbb{Z}$-semilattices.
  \item Ježek (1982): Simple $\mathbb{Z}^2$-semilattices.
\end{itemize}
Definition

\( F \)-semilattice is an algebra \( S = \langle S; \land, F \rangle \) where

- \( \langle S; \land \rangle \) is a semilattice,
- \( F = \langle F; \cdot, \cdot^{-1}, \text{id} \rangle \) is a group,
- \( F \) acts on \( \langle S; \land \rangle \) as automorphisms.

For a fixed group \( F \) the class of \( F \)-semilattices is a variety.

1996: **Canonical Embedding**

**Definition**

\[ \mathcal{P}(F) = \langle \mathcal{P}(F); \cap, F \rangle \text{ where } f(A) = A \cdot f^{-1} \text{ for all } A \subseteq F. \]

For \( s \in S \) the map \( \varphi_s : S \to \mathcal{P}(F) \), \( \varphi_s(x) = \{ f \in F \mid f(x) \geq s \} \) is a homomorphism that separate the points of \( S \).

**Lemma**

*Every subdirectly irreducible \( F \)-semilattice \( S \) is isomorphic to a subalgebra \( U \leq \mathcal{P}(F) \) where*

- \( M = \bigcap \{ A \in U \mid \text{id} \in A \} \in U \),
- \( M \) is a submonoid of \( F \),
- \( M \cdot A = A \) for all \( A \in U \).

*If \( S \) is finite, then \( M \) is a subgroup and \( U = \{ \emptyset \} \cup \{ Mf \mid f \in F \} \) is a flat semilattice.*
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*If \( S \) is finite, then \( M \) is a subgroup and \( U = \{ \emptyset \} \cup \{ Mf \mid f \in F \} \) is a flat semilattice.*
We assume that $F$ is commutative (open for general groups).

**Lemma (Maróti)**

*If a simple $F$-semilattice has a least element, then it is isomorphic to*

$$S_M = \{\emptyset\} \cup \{ Mf \mid f \in F \}$$

*for some subgroup $M \leq F$.***

**Lemma (Maróti)**

*If a simple $F$-semilattice does not have a least element, then it can be embedded into*

$$R_\beta = \langle \mathbb{R}; \min, F \rangle$$

*where $\beta : F \to \langle \mathbb{R}; + \rangle$ is a homomorphism and $f(a) = a - \beta(f)$.***
1996: Simple F-semilattices

Definition

\[ \beta : F \to \langle \mathbb{R}; + \rangle \text{ is dense if} \]

\[ (\forall \varepsilon > 0)(\exists f \in F)(0 < \beta(f) < \varepsilon). \]

Theorem

If F is commutative, then the simple F-semilattices are precisely:
- \( S_M \) where \( M \) is any subgroup of \( F \),
- \( Z_\alpha \), where \( \alpha : F \to \langle \mathbb{Z}; + \rangle \) is a surjective homomorphism,
- \( R_\beta \), where \( \beta : F \to \langle \mathbb{R}; + \rangle \) is a dense homomorphism.

These algebras are pairwise nonisomorphic.
The variety of F-semilattices

The quasivariety of F-semilattices

Nashville

Tournaments
The variety of F-semilattices

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Tournaments
1997–2002: **Tournaments**

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**Definition**

A *tournament* is a conservative commutative groupoid.

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Tournaments

1997-2002: **TOURNAMENTS**

**Definition**

A tournament is a conservative commutative groupoid.

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2001: **Entropic groupoids**

**Definition**

**Medial** identity: \((xy)(zu) = (xz)(yu)\), **entropic** identity: you can exchange variables at the same \((l, r)\) position.

**Theorem (Ježek, Maróti)**

- Decidable of a finite groupoid whether it satisfies all entropic identities.
- Undecidable of a finite partial groupoid whether it satisfies all entropic identities.
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2007: $\mathbb{Z}$-SEMILATTICES

$\mathbf{F} = \mathbb{Z}$, so $\mathbf{F} = \text{Sg}(\{f\})$ for some $f \in \mathbf{F}$.

**Definition**

- $\mathbf{A}_k = \{\emptyset\} \cup \{0, \ldots, k - 1\}$ flat semilattice, $f(\emptyset) = \emptyset$ and $f(i) = i + 1 \mod k$
- $\mathbf{A}_\infty = \{\emptyset\} \cup \mathbb{Z}$
- $\mathbf{B}_1^+ = \langle \mathbb{Z}, \min, f \rangle$, $f(i) = i + 1$
- $\mathbf{B}_1^- = \langle \mathbb{Z}, \max, f \rangle$, $f(i) = i + 1$
- $\mathbf{C}_1 = \mathbf{B}_1^+ \times \mathbf{B}_1^-$
- $\mathbf{B}_k^+, \mathbf{B}_k^-, \mathbf{C}_k$ spiral construction:

$$f(\langle x, i \rangle) = \begin{cases} 
\langle x, i + 1 \rangle & \text{if } i < k - 1, \\
\langle x + 1, 0 \rangle & \text{if } i = k - 1.
\end{cases}$$
2007: Minimal Quasivarieties

Theorem (Dziobiak, Ježek, Maróti)

The minimal quasivarieties of \( \mathbb{Z} \)-semilattices are precisely the quasivarieties generated by \( A_\infty, A_k, B_k^+, B_k^-, C_k \) for all \( k \geq 1 \).

These quasivarieties are pairwise distinct.

Lemma

The unary terms are of the form \( t(x) = \bigwedge_{h \in H} h(x) \) for some \( H \subseteq \mathbb{Z} \), so they are endomorphisms.

Lemma

Let \( Q \) be a minimal quasivariety, and \( A \in Q \) be a nontrivial algebra generated by \( a \in A \).

- If \( 0 \in A \) and \( t(a) = 0 \), then \( Q \models t(x) \approx 0 \).
- If \( t(a) = s(a) \), then \( Q \models t(x) \approx s(x) \).
Theorem (Dziobiak, Ježek, Maróti)

The minimal quasivarieties of $\mathbb{Z}$-semilattices are precisely the quasivarieties generated by $A_\infty, A_k, B_k^+, B_k^-, C_k$ for all $k \geq 1$. These quasivarieties are pairwise distinct.

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2007: Minimal varieties

Lemma

Let $Q$ be a minimal quasivariety of $\mathbb{Z}$-semilattices and $S$ be the one-generated free algebra. If $|S| = 1$, then $Q = Q(A_1)$. If $|S| > 1$, then $S$ is isomorphic to $A_{\infty}$, $A_k (k \geq 2)$, $B_k^+$, $B_k^-$ or $C_k$.

Theorem (Dziobiak, Ježek, Maróti)

The minimal varieties of $\mathbb{Z}$-semilattices are precisely the quasivarieties generated by $A_{\infty}$ and $A_k (k \geq 1)$.

Remark

There are $2^{\aleph_0}$ many subvarieties of the variety of $\mathbb{Z}$-semilattices.
Again, we have to assume that $F$ is commutative.

**Lemma**

Suppose, that $A$ is generated by $a \in A$, $Q = Q(A)$, and $B \in Q$ is generated by $b \in B$. Then

$$
\varphi : A \rightarrow B, \quad r(a) \mapsto r(b)
$$

is a surjective homomorphism.

**Theorem (I. Nagy)**

Suppose that $A$ is one-generated and $Q = Q(A)$. Then $Q$ is minimal if and only if every subalgebra of $A$ generated by a non-zero element is isomorphic to $A$. 
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2009: **Minimal Quasivarieties of $F$-Semilattices**

**Theorem (I. Nagy)**

*If $F$ is finite, then the minimal quasivarieties of $F$-semilattices are the quasivarieties generated by $A_H$ where $H$ is a subgroup of $F$.***

**Theorem (I. Nagy)**

*It is enough to describe all minimal quasivarieties that have no zero element. (Removal of the spiral, construction in both direction.)*

**Example**

$D_\alpha = \langle \{ (k + l\alpha, m + n\alpha) \in \mathbb{R}^2 \mid k + l\alpha \leq m + n\alpha \}; \langle \text{min}, \text{max} \rangle, \mathbb{Z}^2 \rangle$ generates a minimal quasivariety for all irrational number $\alpha$. 
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2009: Minimal quasivarieties of $F$-semilattices

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Tournaments
2007: Bi-tournaments

Definition
Every tournament \( \langle T; \land \rangle \) can be turned into a bi-tournament as
\[
x \land y = x \iff x \lor y = y.
\]

Open problem
*Is the variety generated by bi-tournaments is finitely axiomatizable?*

Candidate of 12 equations, one of which is
\[
g(f(g(x, y), f(f(x, y), z), g(x, y))), f(f(x, z), g(x, y))) = g(f(x, f(f(x, y), z), g(x, y))), f(f(x, z), g(x, y))).
\]
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The quasivariety of \( F \)-semilattices

Tournaments
Thank you!