Special Session on Semigroups, Algorithms and Universal Algebra

AMS Meeting, Louisville, March 20–21, 1998

PROBLEMS SUBMITTED

Algorithmic Problems

The problems in “input/output” format are decision problems. What is wanted is a proof of existence/non-existence of an algorithm (Turing machine) that accepts the indicated input data and always outputs the required output data.

(1) (Mark Sapir) Input: (1) A finite set of equations in a finite type, including Jónsson equations, so that the variety $V$ axiomatized by this set is congruence-distributive. (2) A finite algebra $A$ of the same type. Output: Decide if $A$ is embeddable into a finite simple algebra in $V$. [This is undecidable for non-congruence-distributive varieties.]

(2) (George McNulty) Say that a finite algebra $A$ is expandably residually finite if $A$ has finite type and every finite expansion $A'$ (obtained by adding finitely many operations to $A$) has $V(A')$ residually finite. Is the class of expandably residually finite algebras recursive? Find an intrinsic characterization of this class of algebras.

(3) (Steve Seif) Input: A finite semigroup $S$. Output: Decide if the term-equivalence problem for $S$ is in $P$. (If $P = NP$, this is easy.)
(4) ("the luncheon group") Input: (1) Finite algebra $A$ in a finite language. 
(2) $n \in \omega$. Output: Decide if $V(A)$ is residually $\leq n$.

The next nine problems were submitted by Ralph McKenzie.

(5) Is there a recursive class $\mathcal{A}$ of finite algebras of finite type such that 
$\text{typ}(V(F)) = \{4\}$ implies $F \in \mathcal{A}$ and $\text{typ}(V(F)) = \{5\}$ implies $F \not\in \mathcal{A}$? 
I.e., is the property $\text{typ}(V(F)) = \{4\}$ recursively separable from 
the property $\text{typ}(V(F)) = \{5\}$? Same question for type-sets $\{1, 3\}$ and 
$\{1, 5\}$, and for type-sets $\{1, 3\}$ and $\{1, 2, 3\}$. [The "type-sets" involved 
here are those defined in D. Hobby, R. McKenzie, *The Structure of Finite Algebras*. With the answers to these questions, we could complete a list of all the recursive properties of finite algebras $F$ which are 
determined by the type-set of $V(F)$.

(6) Is the class of all finite, equationally finitely-based algebras, of finite 
type, a recursively enumerable class?

(7) Input: (1) A finite algebra $F$ of finite type. (2) A finite list of equations 
holding in $F$ showing that $V(F)$ is congruence-modular. Output: Decide 
if $V(F)$ is finitely axiomatizable.

Output: Decide if $V(S)$ is finitely axiomatizable.

(9) Write $\kappa(F)$ for the least cardinal exceeding the size of every subdirectly 
irreducible algebra in $V(F)$, if there is such a cardinal, and put $\kappa(F) = \infty$ 
if there is no such cardinal. Here are three problems. Input: A finite 
algebra $F$ of finite type with $5 \not\in \text{typ}(V(F))$. (i) Output: decide if $V(F)$ 
is finitely axiomatizable. (ii) Output: decide if $\kappa(F) < \omega$. (iii) Output: 
Decide if $\kappa(F) < \infty$.

(10) In connection with the preceding problems, does there exist a finite 
algebra (possibly with infinitely many operations) with $5 \not\in \text{typ}(V(F))$ 
and $\omega \leq \kappa(F) < \infty$?

(11) Input: A finite algebra $F$ in a finite language. Output: Decide if $F$ has 
finitely based quasi-equations.
(12) Input: A finite algebra $F$ in a finite language. Output: Decide if there exists some finitely axiomatizable, locally finite quasi-variety $\mathcal{K}$ containing $\text{SP}(F)$.

(13) Input: A finite algebra $F$ in a finite language. Output: Decide if $F$ has a term operation $t(x_1, \ldots, x_n)$ satisfying the near-unanimity equations $t(x, \ldots, x, y, x, \ldots, x) \approx x$, for some $n \geq 3$. [If this is undecidable, then it is undecidable to determine if $\text{SP}(F)$ admits a natural duality in the sense of Davey and Priestley.]

Other Problems

(1) (Victor Gorbunov, submitted by Mark Sapir) Given a finite algebra, is it true that it either is finitely based or has no independent basis of equations? (Same problem for quasi-equations.)

(2) (B.H. Neumann, submitted by Constantine Tsinakis) Can every orderable group be embedded into a divisible orderable group?

(3) (R.W. Quackenbush, submitted by Ralph McKenzie) Prove or disprove: If $A$ is a finite algebra in a finite language and $V(A)$ is residually finite, then $V(A)$ has a finite residual bound.

(4) (J. Ježek, P. Markovic, M. Maroti, R. McKenzie) A tournament is a groupoid $(A, \cdot)$ satisfying $xy = yx \in \{x, y\}$ for all $x, y \in A$. Let $\mathcal{T}$ be the variety generated by all tournaments, $\mathcal{T}_n$ be the variety generated by all $n$-element tournaments. Conjecture: Every subdirectly irreducible algebra in $\mathcal{T}$ is a tournament. Problem: Is $\mathcal{T}$ inherently nonfinitely based? Problem: Is $\mathcal{T}_n$ residually finite?

(5) (Ralph Freese) Conjecture: A finitely presented lattice is infinite iff it has an element which is join-irreducible but not completely join-irreducible, or meet-irreducible but not completely meet-irreducible.

(6) (Garrett Birkhoff, 1942; submitted by Jonathan Farley) Conjecture: Let $P, Q, R$ be finite posets. If $Q^P \cong R^P$ then $Q \cong R$. (Here $Q^P$ is the poset of all monotone mappings from $P$ into $Q$, ordered as a subset of the Cartesian power $Q^P$.)
(7) (S. Bulman-Fleming, submitted by Matthew Gould and Mick Adams) For a finite poset P, the Aizenstät-Howie property is the property that every non-bijection order-preserving selfmap is a composition of idempotent order-preserving selfmaps. Results of Aizenstät and Howie (individually) imply that the following posets have the A-II property: (a) finite chains; (b) finite antichains; (c) finite antichains with 0 or 1 adjoined; (d) finite antichains with 0 and 1 adjoined. Conjecture: These are the only finite posets that have the A-II property. [Reference: Adams, Bulman-Fleming, Gould, and Wildsmith, Proc. Royal Soc. Edinburgh, circa 1994.]

(8) (Matthew Valeriote) Characterize those finite posets P such that the variety generated by the order primal algebra A(P) (A) satisfies some nontrivial congruence equation; or (B) omits types 1 and 5.

(9) (Dejan Delic) Problem:
1. Does DOPC imply DPC for M-algebras?
2. Give a characterization of those finite M-algebras A such that V(A) has DOPC (DPC).
3. Does SP(A) finitely axiomatizable imply V_{sf}(A^\wedge) finitely axiomatizable?

(10) (George McNulty) Is there a finite algebra A which is not dualizable but A ∈ SP(B) for some finite dualizable algebra B?

(11) (George McNulty) Describe the equational theory of
\[ \langle \omega, x + y, x \cdot y, x!, C(x, y), 0, 1 \rangle \]
where C(x, y) is the binomial coefficient.

(12) (Steve Seif) Let X be a set and let S be a semigroup of transformations of X. Then S is said to act primitively on X if the unary algebra (X; S) is simple (in the universal algebra sense; that is, if the only S-compatible equivalence relations on X are the diagonal and universal relations). Assuming that X has more than 2 elements, does there exist such a primitive action if S satisfies the following “nilpotence” condition: for all s ∈ S, there exists a positive integer n such that s^n is a constant function? If S is finite, the answer is “no.”
(13) (Alden Pixley, submitted by Ralph McKenzie) Assume that $V$ is a residually finite variety that is not congruence-distributive. Must there exist a finite algebra in $V$ with non-distributive congruence lattice?