Let $B = (B; F)$ be an algebra and $t$ be a unary idempotent polynomial of $B$. The 
retract of $B$ via $t$ is the algebra $t(B) = (\{t(f) : f \in F\})$. A template $B$ for the 
constraint satisfaction problem is a set of finite idempotent algebras of similar type 
closed under taking subalgebras, homomorphic images and retracts via idempotent 
unary polynomials, but containing only one algebra of each isomorphism type. An 
instance $A = \{B_i \in B \mid i \in V, I \in S\}$ of the constraint satisfaction problem $CSP(B)$ 
consists of a set $V$ of variables, a domain set $B_i \in B$ for each variable $i \in V$, a set $S \subseteq P(V)$ of constraint scopes, 
and a constraint relation $R_I \leq \prod_{i \in I} B_i$ for each scope $I \in S$. A solution of $A$ 
is a function $f \in \prod_{i \in V} B_i$ such that $f|_I \in R_I$ for each scope $I \in S$.

1. Consistent maps

Definition 1. Let $A$ be an instance of $CSP(B)$. A set $p = \{p_i : i \in V\}$ of maps is 
consistent with $A$ if for all $i \in V$ the map $p_i$ is a unary polynomial of $B_i$, and for 
every scope $I \in S$ and tuple $r \in R_I$ the tuple $p|_I(r) = \langle p_i(r_i) : i \in I \rangle$ is also in $R_I$. 
We say that $p$ is permutational, if each $p_i$ is a permutation, and it is idempotent, if 
$p_i(p_i(x)) = p_i(x)$ for all $i \in V$.

Clearly, every consistent set $p = \{p_i : i \in V\}$ of maps can be iterated to obtain 
an idempotent one $p' = \{p^k_i : i \in V\}$ where $k = (\max_{i \in I} |B_i|)!$ for example.

Definition 2. Let $A$ be an instance of $CSP(B)$ and $p = \{p_i : i \in V\}$ be a consistent 
set of idempotent unary polynomials. The retraction of $A$ via $p$ is the new instance $p(A)$ of $CSP(B)$ defined as 
$p(A) = \{p_i(B_i), p|_I(R_I) \mid i \in V, I \in S\}$.

It easily follows from the definitions that the relation 
$p|_I(R_I) = \{p|_I(r) \mid r \in R_I\} = R_I \cap \prod_{i \in I} p_i(B_i)$ 
is a subuniverse of $\prod_{i \in I} p_i(B_i)$.

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Lemma 3. Let $A$ be an instance of $\text{CSP}(B)$ and $p$ be a consistent set of idempotent unary polynomials. Then $A$ has a solution if and only if $p(A)$ does.

Proof. Since $p_i(B_i) \subseteq B_i$ and $p | _f(R_I) \subseteq R_I$, any solution of $p(A)$ is a solution of $A$. Conversely, if $f$ is a solution of $A$, then the function $p \circ f = \{ (p_i(f_i) : i \in V) \}$ is a solution of $p(A)$. □

Definition 4. Let $A$ be an instance of $\text{CSP}(B)$ and $t$ be a binary term such that $t(x,t(x,y)) = t(x,y)$. For an element $b \in B_i$ we put $t_b(x) = t(b,x)$, which is an idempotent polynomial of $B_i$. The decomposition of $A$ via $t$ is the new instance $t(A)$ of $\text{CSP}(B)$ defined as

$$t(A) = \{ B_{i,b}, R_{i,r}, T_{i,B_i} \mid (i,b) \in V', (I,r) \in S', (i,B_i) \in U' \},$$

where $V' = \{ (i,b) \mid i \in V, b \in B_i \}$ is the set of variables,

$$B_{i,b} = t_b(B_i) = \{ t(b,x) \mid x \in B_i \}$$

are the domains,

$$S' = \{ (I,r) \mid I \in S, r \in R_I \}$$

$$U' = \{ (i,B_i) \mid i \in V \}$$

are sets of scopes where

$$(I,r) = \{ (i,r_i) \mid i \in I \} \quad \text{and} \quad (i,B_i) = \{ (i,b) \mid b \in B_i \},$$

and

$$R_{i,r} = t_r(R_I) = \{ (t(r_i,s_i) : i \in I) \mid s \in R_I \},$$

$$T_{i,B_i} = S_{B_i} \{ (t(b,c) : b \in B_i) \mid c \in B_i \}$$

are the relations where $B_i^* = \prod_{b \in B_i} B_{i,b}$.

Lemma 5. Let $A$ be an instance of $\text{CSP}(B)$ and $t$ be a binary term such that $t(x,t(x,y)) = t(x,y)$. If $A$ has a solution, then so does $t(A)$.

Proof. Let $f$ be a solution of the instance $A$. We define a solution $g$ of $t(A)$ as

$$g((i,b)) = t(b,f_i)$$

for all $(i,b) \in V'$. Clearly, $g((i,b)) \in B_{i,b}$. Take a scope $(I,r) \in S'$. By definition,

$$g|_r((i,r_i)) = (t(r_i,f_i) : i \in I) = t(r,f|_I).$$

However, $f$ is a solution, so both $r$ and $f|_I$ are in $R_I$ and therefore $t(r,f|_I) \in R_I$ as well. Clearly, $t(r,f|_I) \in \prod_{i \in I} B_{i,r_i}$, thus we have shown that $g|_r((i,r_i)) \in R_{i,r}$. Now take a scope $(i,B_i) \in U'$ of the second kind. Here

$$g|_r((i,B_i)) = (t(b,f_i) : b \in B_i),$$

that is, it is one of the generating elements of $T_{i,B_i}$. □

In the next lemma we will try to understand the structure of the $T_{i,B_i}$ relations in $t(A)$, so we focus on a single $B = B_i$ algebra for a moment.

Lemma 6. Let $B$ be an algebra, and $t$ be a binary term such that $t(x,t(x,y)) = t(x,y)$. For $b \in B$ let $B_b = t_b(B)$, and put $B^* = \prod_{b \in B} B_b$. Let

$$T = S_{B^*} \{ (t(b,c) : b \in B) \mid c \in B \}$$

and take a tuple $r \in T$. Then the following hold.
Let $b_1, b_2 \in B$ and $\theta$ be a congruence of $B$. If $t(b_1, x) \equiv_\theta t(b_2, x)$ for all $x \in B$, then $p(b_1) \equiv_\theta p(b_2)$.

Proof. Each generator tuple $(t(b, c) : b \in B)$ of $T$ is actually a map from $B$ to $B$ and it is a unary polynomial $B$ in the variable $b$ where $c$ is a constant. When we generate the subalgebra by these vectors, then we take a basic operation $f$ of $B$, some unary polynomials $p_1(b), \ldots, p_k(b)$ already generated and generate the tuple $p(b) = t(b, f(p_1(b), \ldots, f_k(b)))$, which is again a unary polynomial of $B$ in the variable $b$.

To prove the second claim it is enough to see that $s(b_1) \equiv_\theta s(b_2)$ for each generator tuple $s$ and verify that this property is preserved. So assume that the unary polynomials $p_1, \ldots, p_k$ are already generated and $p_i(b_1) \equiv_\theta p_i(b_2), \ldots, p_k(b_1) \equiv_\theta p_k(b_1)$. Thus $c_1 = f(p_1(b_1), \ldots, f_k(b_1)) \equiv_\theta f(p_1(b_2), \ldots, f_k(b_2)) = c_2$, and using again our assumption that $t(b_1, x) \equiv_\theta t(b_2, x)$, we get that $p(b_1) = t(b_1, c_1) \equiv_\theta t(b_1, c_2) \equiv_\theta t(b_2, c_2) = p(b_2)$ for the newly generated polynomial $p$.

**Lemma 7.** Let $A$ be an instance of CSP($B$) and $t$ be a binary term such that $t(x, t(x, y)) = t(x, y)$. If $t(A)$ has a solution, then there exists a consistent set $\{ p_i : i \in V \}$ of unary polynomials for the instance $A$ such that each polynomial $p_i$ of $B_i$ satisfies the conclusion of Lemma 6.

Proof. Let $g$ be a solution of $t(A)$. We define a consistent set $p = \{ p_i | i \in V \}$ of unary maps for $A$ as $p_i(b) = g((i, b))$ for $i \in V$ and $b \in B_i$. By Lemma 6, each map $p_i : B_i \to B_i$ is a unary polynomial of $B_i$. To see that it preserves the relations of $A$ take a scope $I \in S$ and a tuple $r \in R_I$. Since $g$ was a solution to $t(A)$ it respects the constraint $R_{I, r}$, that is the tuple $(g((i, r_i)) : i \in I)$ is in $R_{I, r} \subseteq R_I$. But this tuple is exactly $p_I(r)$, which shows that $p$ is consistent.

**Definition 8.** We say that an idempotent algebra $B$ can be eliminated, if whenever $B$ is a template such that $B \in B, \ B \setminus B$ is also a template, and CSP($B \setminus \{ B \}$) is tractable, then CSP($B$) is also tractable.

**Lemma 9.** Let $B$ be an algebra and $t$ be a binary term of $B$ such that for each $b \in B$ the map $t_b(x) = t(b, x)$ is idempotent and not surjective. Let $C$ be the set of elements $c \in B$ such that $x \mapsto t(x, c)$ is a permutation. If $C$ generates a proper subuniverse of $B$, then $B$ can be eliminated.

Proof. Let $B$ be a template containing $B$ and let $A$ be an instance of CSP($B$) containing at least one copy of $B$. Replace all occurrence of $B$ in $A$ with the subalgebra generated by the set $C$. Clearly, this new instance is an instance of CSP($B \setminus \{ B \}$) so it can be solved in polynomial time. If it has a solution, then we are done, so we can assume that it does not.

Since the maps $t_b$ are not surjective, $|t_b(B)| < |B|$ and therefore the decomposition $t(A)$ is an instance of CSP($B \setminus \{ B \}$). Thus it can be solved in polynomial time. If $t(A)$ has no solution, then $A$ has no solution either by Lemma 5. On the other hand if $t(A)$ has a solution, then by Lemma 7 we have a consistent set $p = \{ p_i : i \in V \}$ of unary polynomials for $A$. Let us assume for a moment that $p$ is not permutational. Now $p$ can be iterated to obtain an idempotent non-permutational consistent set $p'$ of unary polynomials for $A$. By Lemma 3 we know...
that $\mathcal{A}$ has a solution if and only if $p'(\mathcal{A})$ does. Also, since $p'$ is non-permutational, at least one of the domains of $p'(\mathcal{A})$ is smaller than that of $\mathcal{A}$. So by iterating this procedure we will get to a point when the algebra $\mathcal{B}$ no longer occurs in the instance $\mathcal{A}$.

Now we go back to the problem of making sure that $p$ becomes non-permutational. We know that if $\mathcal{A}$ has a solution $f$, then for at least one $i \in V$, $B_i = B$ and $f_i \not\in C$. Let us iterate through all variables $i \in V$ such that $B_i = B$ and all elements $d \in B \setminus C$. For each choice of $i$ and $d$ we create a new instance from $t(A)$ by adding new unary constraints stating that the solution $g|_{(i,B_i)} = \langle t(b,d) : b \in B_i \rangle$. This ensures that $p_i(b) = t(b,d)$, that is it is not permutational. If for any of these choices we find a non-permutational case, then we can reduce the instance as shown above. Otherwise we conclude that the instance has no solution. □

2. Application

Corollary 10. Let $\mathcal{B}$ be an idempotent algebra, and $\beta \in \text{Con} \mathcal{B}$ such that $\mathcal{B}/\beta$ is a semilattice (possibly with more operations) having more than one maximal element. Then $\mathcal{B}$ can be eliminated.

Proof. Take a binary term $t$ of $\mathcal{B}$ that is a semilattice operation on $\mathcal{B}/\beta$. We can assume, that $t(x,t(x,y)) = t(x,y)$ on $\mathcal{B}$. Since $\mathcal{B}/\beta$ has more than one maximal element, for all $b \in B$ the maps $x \mapsto t(b,x)$ and $x \mapsto t(x,b)$ are not permutations. Thus we can apply Lemma 9 with $C = \emptyset$. □

Corollary 11. Let $\mathcal{B}$ be an idempotent algebra, $\beta \in \text{Con} \mathcal{B} \setminus \{1\mathcal{B}\}$ and $t$ be a binary term such that $t$ is a semilattice operation on $\mathcal{B}/\beta$. If the largest $\beta$-block (with respect to the semilattice order) contains more than one element and satisfies $t(x,y) = x$, then $\mathcal{B}$ can be eliminated.

Proof. We can assume that $t(x,t(x,y)) = t(x,y)$ on $\mathcal{B}$, since we can iterate $t$ in the second variable without destroying the required properties stated in the lemma. By Corollary 10, $\mathcal{B}/\beta$ has a largest element $Q$. Suppose, that the $\beta$-block $Q$ has more than one element. Then the maps $t_b(x) = t(b,x)$ are not permutations. Moreover, for any $c \in B$ for which $x \mapsto t(x,c)$ is a permutation we must have $c \in Q$. However, $Q$ is a proper subuniverse of $\mathcal{B}$, thus we can apply Lemma 9 to finish the proof. □

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