# TREE ON TOP OF MALTSEV

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Abstract. Let $\mathbf{B}$ be an idempotent algebra, $\beta \in \operatorname{Con}\mathbf{B}$ such that $\mathbf{B}/\beta$ is term equivalent to a semilattice whose order is a rooted tree, and each $\beta$-block is Maltsev. Then $\mathrm{CSP}(\mathbf{B})$ is tractable.

Let $\mathbf{B} = (B; F)$ be an algebra and $t$ be a unary idempotent polynomial of $\mathbf{B}$. The *retract* of $\mathbf{B}$ via $t$ is the algebra $t(\mathbf{B}) = (\, t(B);\ \{\, t \circ f : f \in F\,\})$. A template $\mathcal{B}$ for the constraint satisfaction problem is a set of finite idempotent algebras of similar type closed under taking subalgebras, homomorphic images and retracts via idempotent unary polynomials, but containing only one algebra of each isomorphism type. An *instance*

$$\mathcal{A} = \{\, \mathbf{B}_i \in \mathcal{B},\ \mathbf{R}_I \leq \prod_{i\in I} \mathbf{B}_i \mid i \in V,\ I \in S\,\}$$

of the constraint satisfaction problem $\mathrm{CSP}(\mathcal{B})$ consists of a set $V$ of variables, a domain set $\mathbf{B}_i \in \mathcal{B}$ for each variable $i \in V$, a set $S \subseteq P(V)$ of constraint scopes, and a constraint relation $\mathbf{R}_I \leq \prod_{i\in I} \mathbf{B}_i$ for each scope $I \in S$. A *solution* of $\mathcal{A}$ is a function $f \in \prod_{i\in V} B_i$ such that $f|_I \in R_I$ for each scope $I \in S$.

## 1. Consistent maps

**Definition 1.** Let $\mathcal{A}$ be an instance of $\mathrm{CSP}(\mathcal{B})$. A set $p = \{\, p_i \mid i \in V\,\}$ of maps is *consistent* with $\mathcal{A}$ if for all $i \in V$ the map $p_i$ is a unary polynomial of $\mathbf{B}_i$, and for every scope $I \in S$ and tuple $r \in R_I$ the tuple $p|_I(r) = \langle p_i(r_i) : i \in I\rangle$ is also in $R_I$. We say that $p$ is *permutational*, if each $p_i$ is a permutation, and it is *idemptent*, if $p_i(p_i(x)) = p_i(x)$ for all $i \in V$.

Clearly, every consistent set $p = \{\, p_i : i \in V\,\}$ of maps can be iterated to obtain an idempotent one $p' = \{\, p_i^k : i \in V\,\}$ where $k = (\max_{i\in I} |B_i|)!$ for example.

**Definition 2.** Let $\mathcal{A}$ be an instance of $\mathrm{CSP}(\mathcal{B})$ and $p = \{\, p_i \mid i \in V\,\}$ be a consistent set of idempotent unary polynomials. The *retraction* of $\mathcal{A}$ via $p$ is the new instance $p(\mathcal{A})$ of $\mathrm{CSP}(\mathcal{B})$ defined as

$$p(\mathcal{A}) = \{\, p_i(\mathbf{B}_i),\ p|_I(\mathbf{R}_I) \mid i \in V,\ I \in S\,\}.$$

It easily follows from the definitions that the relation

$$p|_I(\mathbf{R}_I) = \{\, p|_I(r) \mid r \in R_I\,\} = R_I \cap \prod_{i\in I} p_i(B_i)$$

is a subuniverse of $\prod_{i\in I} p_i(\mathbf{B}_i)$.

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**Lemma 3.** *Let $\mathcal{A}$ be an instance of* $\mathrm{CSP}(\mathcal{B})$ *and $p$ be a consistent set of idempotent unary polynomials. Then $\mathcal{A}$ has a solution if and only if $p(\mathcal{A})$ does.*

*Proof.* Since $p_i(B_i) \subseteq B_i$ and $p|_I(R_I) \subseteq R_I$, any solution of $p(\mathcal{A})$ is a solution of $\mathcal{A}$. Conversely, if $f$ is a solution of $\mathcal{A}$, then the function $p \circ f = \langle p_i(f_i) : i \in V \rangle$ is a solution of $p(\mathcal{A})$. □

**Definition 4.** Let $\mathcal{A}$ be an instance of $\mathrm{CSP}(\mathcal{B})$ and $t$ be a binary term such that $t(x, t(x, y)) = t(x, y)$. For an element $b \in B_i$ we put $t_b(x) = t(b, x)$, which is an idempontent polynomial of $\mathbf{B}_i$. The *decomposition* of $\mathcal{A}$ via $t$ is the new instance $t(\mathcal{A})$ of $\mathrm{CSP}(\mathcal{B})$ defined as

$$t(\mathcal{A}) = \{\, \mathbf{B}_{i,b},\ \mathbf{R}_{I,r},\ \mathbf{T}_{i,B_i} \mid (i,b) \in V',\ (I,r) \in S',\ (i,B_i) \in U' \,\},$$

where

$$V' = \{\, (i,b) \mid i \in V,\ b \in B_i \,\}$$

is the set of variables,

$$\mathbf{B}_{i,b} = t_b(\mathbf{B}_i) = \{\, t(b,x) \mid x \in B_i \,\}$$

are the domains,

$$S' = \{\, (I,r) \mid I \in S,\ r \in R_I \,\}$$
$$U' = \{\, (i,B_i) \mid i \in V \,\}$$

are sets of scopes where

$$(I,r) = \{\, (i,r_i) \mid i \in I \,\} \quad \text{and} \quad (i,B_i) = \{\, (i,b) \mid b \in B_i \,\},$$

and

$$\mathbf{R}_{I,r} = t_r(\mathbf{R}_I) = \{\, \langle t(r_i,s_i) : i \in I \rangle \mid s \in R_I \,\},$$
$$\mathbf{T}_{i,B_i} = \mathrm{Sg}_{\mathbf{B}_i^\star}\{\, \langle t(b,c) : b \in B_i \rangle \mid c \in B_i \,\}$$

are the relations where $\mathbf{B}_i^\star = \prod_{b \in B_i} \mathbf{B}_{i,b}$.

**Lemma 5.** *Let $\mathcal{A}$ be an instance of* $\mathrm{CSP}(\mathcal{B})$ *and $t$ be a binary term such that $t(x, t(x, y)) = t(x, y)$. If $\mathcal{A}$ has a solution, then so does $t(\mathcal{A})$.*

*Proof.* Let $f$ be a solution of the instance $\mathcal{A}$. We define a solution $g$ of $t(\mathcal{A})$ as

$$g((i,b)) = t(b, f_i)$$

for all $(i,b) \in V'$. Clearly, $g((i,b)) \in B_{i,b}$. Take a scope $(I,r) \in S'$. By definition,

$$g|_{(I,r)} = \langle t(r_i, f_i) : i \in I \rangle = t(r, f|_I).$$

However, $f$ is a solution, so both $r$ and $f|_I$ are in $\mathbf{R}_I$ and therefore $t(r, f|_I) \in \mathbf{R}_I$ as well. Clearly, $t(r, f|_I) \in \prod_{i \in I} \mathbf{B}_{i,r_i}$, thus we have shown that $g|_{(I,r)} \in \mathbf{R}_{i,r}$.

Now take a scope $(i, B_i) \in U'$ of the second kind. Here

$$g|_{(i,B_i)} = \langle t(b, f_i) : b \in B_i \rangle,$$

that is, it is one of the generating elements of $\mathbf{T}_{i,B_i}$. □

In the next lemma we will try to understand the structure of the $\mathbf{T}_{i,B_i}$ relations in $t(\mathcal{A})$, so we focus on a single $\mathbf{B} = \mathbf{B}_i$ algebra for a moment.

**Lemma 6.** *Let $\mathbf{B}$ be an algebra, and $t$ be a binary term such that $t(x, t(x, y)) = t(x, y)$. For $b \in B$ let $\mathbf{B}_b = t_b(\mathbf{B})$, and put $\mathbf{B}^* = \prod_{b \in B} \mathbf{B}_b$. Let*

$$\mathbf{T} = \mathrm{Sg}_{\mathbf{B}^*}\{\, \langle t(b,c) : b \in B \rangle \mid c \in B \,\}$$

*and take a tuple $r \in T$. Then the following hold.*

(1) *The map* $p : b \mapsto r_b$ *is a unary polynomial of* $\mathbf{B}$.
(2) *Let* $b_1, b_2 \in B$ *and* $\vartheta$ *be a congruence of* $\mathbf{B}$. *If* $t(b_1, x) \equiv_\vartheta t(b_2, x)$ *for all* $x \in B$, *then* $p(b_1) \equiv_\vartheta p(b_2)$.

*Proof.* Each generator tuple $\langle t(b, c) : b \in B\rangle$ of $\mathbf{T}$ is actually a map from $B$ to $B$ and it is a unary polynomial $\mathbf{B}$ in the variable $b$ where $c$ is a constant. When we generate the subalgebra by these vectors, then we take a basic operation $f$ of $\mathbf{B}$, some unary polynomials $p_1(b), \dots, p_k(b)$ already generated and generate the tuple $p(b) = t(b, f(p_1(b), \dots, f_k(b)))$, which is again a unary polynomial of $\mathbf{B}$ in the variable $b$.

To prove the second claim it is enough to see that $s(b_1) \equiv_\vartheta s(b_2)$ for each generator tuple $s$ and verify that this property is preserved. So assume that the unary polynomials $p_1, \dots, p_k$ are already generated and $p_1(b_1) \equiv_\vartheta p_1(b_2), \dots, p_k(b_1) \equiv_\vartheta p_k(b_1)$. Thus $c_1 = f(p_1(b_1), \dots, f_k(b_1)) \equiv_\vartheta f(p_1(b_2), \dots, f_k(b_2)) = c_2$, and using again our assumption that $t(b_1, x) \equiv_\vartheta t(b_2, x)$, we get that $p(b_1) = t(b_1, c_1) \equiv_\vartheta t(b_1, c_2) \equiv_\vartheta t(b_2, c_2) = p(b_2)$ for the newly generated polynomial $p$. □

**Lemma 7.** *Let* $\mathcal{A}$ *be an instance of* CSP($\mathcal{B}$) *and* $t$ *be a binary term such that* $t(x, t(x, y)) = t(x, y)$. *If* $t(\mathcal{A})$ *has a solution, then there exists a consistent set* $\{ p_i : i \in V \}$ *of unary polynomials for the instance* $\mathcal{A}$ *such that each polynomial* $p_i$ *of* $\mathbf{B}_i$ *satisfies the conclusion of Lemma 6.*

*Proof.* Let $g$ be a solution of $t(\mathcal{A})$. We define a consistent set $p = \{ p_i \mid i \in V \}$ of unary maps for $\mathcal{A}$ as

$$p_i(b) = g((i, b))$$

for $i \in V$ and $b \in B_i$. By Lemma 6, each map $p_i : B_i \to B_i$ is a unary polynomial of $\mathbf{B}_i$. To see that it preserves the relations of $\mathcal{A}$ take a scope $I \in S$ and a tuple $r \in \mathbf{R}_I$. Since $g$ was a solution to $t(\mathcal{A})$ it respects the constraint $\mathbf{R}_{I,r}$, that is the tuple $\langle g((i, r_i)) : i \in I\rangle$ is in $\mathbf{R}_{I,r} \subseteq \mathbf{R}_I$. But this tuple is exactly $p|_I(r)$, which shows that $p$ is consistent. □

**Definition 8.** We say that an idempotent algebra $\mathbf{B}$ can be *eliminated*, if whenever $\mathcal{B}$ is a template such that $\mathbf{B} \in \mathcal{B}$, $\mathcal{B} \setminus \mathbf{B}$ is also a template, and CSP($\mathcal{B} \setminus \{\mathbf{B}\}$) is tractable, then CSP($\mathcal{B}$) is also tractable.

**Lemma 9.** *Let* $\mathbf{B}$ *be an algebra and* $t$ *be a binary term of* $\mathbf{B}$ *such that for each* $b \in B$ *the map* $t_b(x) = t(b, x)$ *is idempotent and not surjective. Let* $C$ *be the set of elements* $c \in B$ *such that* $x \mapsto t(x, c)$ *is a permutation. If* $C$ *generates a proper subuniverse of* $\mathbf{B}$, *then* $\mathbf{B}$ *can be eliminated.*

*Proof.* Let $\mathcal{B}$ be a template containing $\mathbf{B}$ and let $\mathcal{A}$ be an instance of CSP($\mathcal{B}$) containing at least one copy of $\mathbf{B}$. Replace all occurence of $\mathbf{B}$ in $\mathcal{A}$ with the subalgebra generated by the set $C$. Clearly, this new instance is an instance of CSP($\mathcal{B} \setminus \{\mathbf{B}\}$) so it can be solved in polynomial time. If it has a solution, then we are done, so we can assume that it does not.

Since the maps $t_b$ are not surjective, $|t_b(\mathbf{B})| < |\mathbf{B}|$ and therefore the decomposition $t(\mathcal{A})$ is an instance of CSP($\mathcal{B} \setminus \{\mathbf{B}\}$). Thus it can be solved in polynomial time. If $t(\mathcal{A})$ has no solution, then $\mathcal{A}$ has no solution either by Lemma 5. On the other hand if $t(\mathcal{A})$ has a solution, then by Lemma 7 we have a consistent set $p = \{ p_i : i \in V \}$ of unary polynomials for $\mathcal{A}$. Let us assume for a moment that $p$ is not permutational. Now $p$ can be iterated to obtain an idempotent non-permutational consistent set $p'$ of unary polynomials for $\mathcal{A}$. By Lemma 3 we know that $\mathcal{A}$ has a solution if and only if $p'(\mathcal{A})$ does. Also, since $p'$ is non-permutational, at least one of the domains of $p'(\mathcal{A})$ is smaller than that of $\mathcal{A}$. So by iterating this procedure we will get to a point when the algebra $\mathbf{B}$ no longer occurs in the instance $\mathcal{A}$.

Now we go back to the problem of making sure that $p$ becomes non-permutational. We know that if $\mathcal{A}$ has a solution $f$, then for at least one $i \in V$, $\mathbf{B}_i = \mathbf{B}$ and $f_i \notin C$. Let us iterate through all variables $i \in V$ such that $B_i = B$ and all elements $d \in B \setminus C$. For each choice of $i$ and $d$ we create a new instance from $t(\mathcal{A})$ by adding new unary constraints stating that the solution $g|_{(i,B_i)} = \langle t(b,d) : b \in B_i \rangle$. This ensures that $p_i(b) = t(b,d)$, that is it is not permutational. If for any of these choices we find a non-permutational case, then we can reduce the instance as shown above. Otherwise we conculde that the instance has no solution. □

## 2. Application

**Corollary 10.** *Let $\mathbf{B}$ be an idempotent algebra, and $\beta \in \operatorname{Con} \mathbf{B}$ such that $\mathbf{B}/\beta$ is a semilattice (possibly with more operations) having more than one maximal element. Then $\mathbf{B}$ can be eliminated.*

*Proof.* Take a binary term $t$ of $\mathbf{B}$ that is a semilattice operation on $\mathbf{B}/\beta$. We can assume, that $t(x, t(x, y)) = t(x, y)$ on $\mathbf{B}$. Since $\mathbf{B}/\beta$ has more than one maximal element, for all $b \in B$ the maps $x \mapsto t(b, x)$ and $x \mapsto t(x, b)$ are not permutations. Thus we can apply Lemma 9 with $C = \emptyset$. □

**Corollary 11.** *Let $\mathbf{B}$ be an idempotent algebra, $\beta \in \operatorname{Con} \mathbf{B} \setminus \{1_{\mathbf{B}}\}$ and $t$ be a binary term such that $t$ is a semilattice operation on $\mathbf{B}/\beta$. If the largest $\beta$-block (with respect to the semilattice order) contains more than one element and satisfies $t(x, y) = x$, then $\mathbf{B}$ can be eliminated.*

*Proof.* We can assume that $t(x, t(x, y)) = t(x, y)$ on $\mathbf{B}$, since we can iterate $t$ in the second variable without destroying the required properties stated in the lemma. By Corollary 10, $\mathbf{B}/\beta$ has a largest element $Q$. Suppose, that the $\beta$-block $Q$ has more than one element. Then the maps $t_b(x) = t(b, x)$ are not permutations. Moreover, for any $c \in B$ for which $x \mapsto t(x, c)$ is a permutation we must have $c \in Q$. However, $Q$ is a proper subuniverse of $\mathbf{B}$, thus we can apply Lemma 9 to finish the proof. □

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