MALTSEV ON TOP

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Abstract. Let $\mathbf{A}$ be an idempotent algebra, $\alpha \in \operatorname{Con} \mathbf{A}$ such that $\mathbf{A}/\alpha$ has few subpowers, and m be a fixed natural number. There is a polynomial time algorithm that can transform any constraint satisfaction problem over $\mathbf{A}$ with relations of arity at most m into an equivalent problem which is m consistent and in which each domain is inside an α block. Consequently if the induced algebras on the blocks of α generate an SD($\wedge$) variety, then CSP($\mathbf{A}$) is tractable.

1. Introduction

A wide variety of combinatorial problems can be expressed within the framework of *constraint satisfaction problems* (CSPs), where one searches for an assignment of values to variables that satisfies certain constraints. If the constraint relations used in the problem instances are restricted to a fixed finite set of relations, called the *template*, then we get a restricted class of (non-uniform) CSPs. Solving a system of equations over a finite algebra, satisfying Boolean formulae, or finding a 3-coloring of graphs falls into this category. The *dichotomy conjecture* of Feder and Vardi [6] states, that for every template the class of CSPs is solvable in polynomial time (in the number of variables) or NP-complete. The algebraic study of the CSP was initiated by Bulatov, Jeavons and Krokhin [4, 5], where they showed that the algorithmic complexity of the CSP depends only on the set of *polymorphisms* of the template. Under their approach the template is a finite set of subpowers of a finite algebra $\mathbf{A}$.

There are two algorithms that can solve the CSP in polynomial time for broad classes of algebras. One is the so called *bounded width algorithm* where local consistency checking is used to decide if a solution exists. This algorithm works if and only if the template $\mathbf{A}$ generates a congruence meet semi-distributive variety [1, 8]. The other algorithm is based on the *few subpowers property* [2, 7], where each subuniverse of $\mathbf{A}^n$ has a generator set of polynomial size in n. These algebras have a so called *edge* term. These two algorithms use fundamentally different approaches to the solution of CSP, although both of them handle the case when $\mathbf{A}$ has a near-unanimity term.

In this paper we combine these two algorithms, or rather we show how they can be executed in parallel to get an algorithm that can handle a larger class of algebras than either of original one. In particular, we show that if $\mathbf{A}$ is a finite algebra in a variety generated by idempotent algebras of few subpowers and of bounded width, then the corresponding CSP is solvable in polynomial time.

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2. Bounded width and few subpowers

Instead of restricting the number of relations in the template, we restrict their maximum arity. Adding new constraints to the template from the same relational clone does not change the complexity of the problem, so this does not cause any difficulty and will simplify our notation. Now we review the definitions and theorems we are going to extend in the next section.

Definition 2.1. Let $\mathbf{A}$ be a finite algebra. An *instance of* $\mathrm{CSP}(\mathbf{A})$ is a triple $(V, \mathcal{S}, \mathcal{R})$ where V is the set of *variables*, $\mathcal{S} \subseteq P(V)$ is the set of *constraint scopes*, and $\mathcal{R} = \{ R_I : I \in \mathcal{S} \}$ is an indexed set of *constraint relations* where R_I is a subuniverse of $\mathbf{A}^I$. A *solution* of the instance $(V, \mathcal{S}, \mathcal{R})$ is a function $f : V \to A$ such that $f|_I \in R_I$ for all $I \in \mathcal{S}$.

Definition 2.2. We say that an instance $(V, \mathcal{S}, \mathcal{R})$ of $\mathrm{CSP}(\mathbf{A})$ is *k-consistent*, if $\mathcal{S} = \{ S \subseteq V : |S| \leq k \}$, and for each $I, J \in \mathcal{S}$ with $I \subseteq J$ we have $R_I = R_J \cap \mathbf{A}^I$. The instance is *empty* if $R_I = \emptyset$ for all (any) $I \in \mathcal{S}$.

The following standard consistency checking algorithm of [6] can be used to preprocess the instance without loosing solutions. First we add the missing scopes with the full relation and then run the consistency algorithm. This algorithm checks for each pair $I \subseteq J$ of scopes whether $R_I = R_J \cap \mathbf{A}^I$ holds. If it fails, then it replaces R_I with $R_I \cap (R_J \cap \mathbf{A}^I)$ and R_J with $R_J \cap (R_I \times \mathbf{A}^{J \setminus I})$. Since the total number of tuples in all relations in $\mathcal{R}$ is bounded by a polynomial in the number $|V|$ of variables, this process must stop in polynomial time. It is quite easy to see, that we never loose (or add) a solution and that the resulting k-consistent instance is unique, i.e., it does not depend on the order we have considered the scope pairs.

Theorem 2.3 (Consistency Checking Algorithm [6])**.** *Let k be a fixed integer and $(V, \mathcal{S}, \mathcal{R})$ be an instance of* $\mathrm{CSP}(\mathbf{A})$ *with relations of arity at most k. Then there exists a unique k-consistent instance $(V, \mathcal{S}', \mathcal{R}')$ that can be computed in polynomial time and has the same set of solutions as $(V, \mathcal{S}, \mathcal{R})$.*

We do not define relational structures of bounded width, instead we just state the main theorem of [1] for algebras. Note, that if $\mathbf{A}$ does not generate a congruence meet semi-distributive variety, then the consistency checking algorithm with bounded degree is not enough to decide whether the instance has a solution [8].

Theorem 2.4 (Bounded Width Algorithm [1])**.** *Let k be a fixed integer and $\mathbf{A}$ be a finite algebra in a congruence meet semi-distributive variety. Then an instance of* $\mathrm{CSP}(\mathbf{A})$ *with relations of arity at most $2\lfloor \frac{k}{3} \rfloor$ has a solution if and only if the corresponding unique k-consistent instance is nonempty. Consequently, the problem can be solved in polynomial time.*

An operation t is called *idempotent*, if it satisfies the equation $t(x, \dots, x) \approx x$. An idempotent operation t is a *week near-unanimity* operation, if it is at least binary and satisfies the identities $t(y, x, \dots, x) \approx t(x, y, x, \dots, x) \approx \cdots \approx t(x, \dots, x, y)$. We will use the following characterization of congruence meet semi-distributivity with the existence of many weak near-unanimity terms.

Theorem 2.5 ([9])**.** *A locally finite variety $\mathcal{V}$ is congruence meet semi-distributive if and only if there exists an integer $m > 1$ such that $\mathcal{V}$ has weak near-unanimity terms of all arities greater than m.*

Now we turn our attention to the few subpowers algorithm. An n-ary operation t is called a *cube operation* [2, 10], if for each index $1 \leq i \leq n$ it satisfies a two-variable equation of the form $t(u_1, \dots, u_n) \approx x$ where $u_1, \dots, u_n \in \{x, y\}$ and $u_i = y$. Clearly, every cube operation is idempotent. A $k+1$-ary operation t is a k*-edge operation* if $k \geq 2$ and it satisfies the identities

$$\begin{aligned} t(y, y, x, x, x, x, \dots, x) &\approx x, \\ t(x, y, y, x, x, x, \dots, x) &\approx x, \\ t(x, x, x, y, x, x, \dots, x) &\approx x, \\ t(x, x, x, x, y, x, \dots, x) &\approx x, \\ \ddots \quad &\vdots \\ t(x, x, x, x, x, x, \dots, y) &\approx x. \end{aligned}$$

Clearly, every k-edge operation is a cube operation. The 2-edge operations are precisely the Maltsev operations, satisfying the identities $t(y, y, x) \approx x$ and $t(x, y, y) \approx x$. On the other hand, every k-ary near unanimity operation (satisfying the equations $t(y, x, \dots, x) \approx \cdots \approx t(x, \dots, x, y) \approx x$) can be turned into a k-edge operation by adding a dummy second variable.

We say that an algebra $\mathbf{A}$ has *few subpowers* [2], if the number of subalgebras of $\mathbf{A}^n$ is bounded by $2^{p(n)}$ for some polynomial function p. The following theorem characterizes algebras of few subpowers.

Theorem 2.6 ([2]). *For a finite algebra the following are equivalent: having few subpowers, having a cube term, and having an edge term.*

Let $\mathbf{A}$ be a fixed algebra with a k-edge term, and n be an arbitrary natural number. By an *index* we mean a triple $(i, a, b) \in \{1, \dots, n\} \times A \times A$. We do not define *minority indices* (the precise definition is Definition 3.6 in [7]), but for our purposes it is enough that the set of minority indices is a subset of the set of indices, and the exact defining property depends only on the pair $a, b \in A$ and the algebra $\mathbf{A}$. An index (i, a, b) is *witnessed* in a subset $Q \subseteq A^n$ if there exist elements $f, g \in Q$ such that $f(1) = g(1), \dots, f(i-1) = g(i-1)$, $f(i) = a$ and $g(i) = b$.

Definition 2.7. Let R be a subuniverse of $\mathbf{A}^n$. We say that $Q \subseteq R$ is a *compact representation* of R if

(1) the same minority indices are witnessed in R and Q,
(2) R and Q has the same $k-1$-element projections, and
(3) $|Q|$ is bounded by a $k-1$-degree polynomial in n.

Theorem 2.8 ([2, 7]). *If Q is a compact representation of a subpower $R \leq \mathbf{A}^n$, then the subuniverse generated by Q is R.*

The following theorem works for uniform instances of CSP where there is no bound on the arity of the constraint relations. A similar uniform (global tractability) result seem to hold for the bounded width case (through private communication with Libor Barto).

Theorem 2.9 (Few Subpowers Algorithm [7]). *Let $\mathbf{A}$ be a finite algebra with few subpowers. For every uniform instance $(V, \mathcal{S}, \mathcal{R})$ of* CSP($\mathbf{A}$) *the compact representation of the solution set can be computed in polynomial time in the size of the input.*

This algorithm is also known as the Dalmau Algorithm, since it is a slight modification of the one in [3]. The essential computational step in the proof is the *Next Procedure*, that takes a compact representation of a subuniverse $R \leq A^n$, a constraint scope $I \subseteq V$ and a constraint relation $R_I \leq \mathbf{A}^I$ and computes the compact representation of $\{ f \in R : f|_I \in R_I \}$. The running time of the Next Procedure is bounded by a fixed polynomial in $n|R_I|$.

Now we prove three lemmas on compact representations that can be derived from the original proofs quite easily but were not observed before. In particular, it is curious that Bulatov and Dalmau in [3] have not observed that the compact representation of the intersection $R_1 \cap R_2$ of two relations can be computed from the compact representations of R_1 and R_2.

Lemma 2.10. *Let $\mathbf{A}$ be a fixed finite algebra with few subpowers. If $R_1, \dots, R_m$ are subpowers of $\mathbf{A}$ and R is a relation defined by a primitive positive formula φ using $R_1, \dots, R_m$, then the compact representation of R can be computed from the compact representations of $R_1, \dots, R_m$ in polynomial time where the size of the input is the size of the formula φ.*

Proof. Assume, that each of the relations $R_1, \dots, R_m$ occurs exactly once in φ (otherwise we duplicate each relation as many times as needed). Let the arities of $R, R_1, \dots, R_m$ be $n, n_1, \dots, n_m$, respectively. Then the size of the input is at least $n + n_1 + \dots + n_m$ plus the number of equality predicates in φ.

First, we construct the compact representation of

$$S = A^n \times R_1 \times \dots \times R_m.$$

This can be done, since there are only polynomially many projections we have to consider and we can just freely combine projections coming from the compact representations of $R_1, \dots, R_m$. Second, we can go over all minority indices, and for each (i, a, b) we can find a witness by choosing a witness from the appropriate R_j where the coordinates of R_j contain i and arbitrary elements from all other relations.

Once we have the compact representation of S, then we apply the Next Procedure repeatedly with the equality constraint between pairs of coordinates as described in φ. We have only polynomially many equality predicates, so this part also runs in polynomial time.

Finally, we simply take the projection of this relation to the first n coordinates and we get R. The compact representation of the projection can be obtained from the projection of the compact representation by removing duplicate witnesses of projections and minority indices. □

Lemma 2.11. *Let $\mathbf{A}$ be a fixed finite algebra with few subpowers and m be a fixed integer. If $R_1, \dots, R_m$ are subuniverses of $\mathbf{A}^n$ such that $R = R_1 \cup \dots \cup R_m$ is also a subuniverse, then the compact representation of R can be computed in polynomial time (in n) from the compact representations of $R_1, \dots, R_m$.*

Proof. It is clear, that the union of the witnesses of small projections in $R_1, \dots, R_m$ are witnesses in R. For a minority index (i, a, b) and a pair $1 \leq j, k \leq m$ consider the relation

$$\{ (x_1, \dots, x_n, y_1, \dots, y_n) \in A^{2n} : (x_1, \dots, x_n) \in R_j, (y_1, \dots, y_n) \in R_k,$$
$$x_1 = y_1, \dots, x_{i-1} = y_{i-1}, x_i = a \text{ and } y_i = b \}$$

defined by a primitive positive formula. Clearly, this relation is nonempty if and only if the (i, a, b) minority index can be witnessed by $f \in R_j$ and $g \in R_k$. By trying out all possible choices of $1 \leq j, k \leq m$ and using Lemma 2.10 we can find witnesses of all minority indices that can be witnessed in polynomial time. □

To our knowledge, it is an open question whether the compact representation of the subuniverse generated by $R_1 \cup R_2$ can be computed in polynomial time from the compact representations of R_1 and R_2.

Lemma 2.12. *Suppose, that R_1 and R_2 are subuniverses of $\mathbf{A}^n$ with $R_1 \subseteq R_2$. If R_1 and R_2 witness the same number of $k-1$-element projections and minority indices, then $R_1 = R_2$.*

Proof. Take a compact representation Q of R_1. Since $R_1 \subseteq R_2$, Q witnesses some $k-1$-element projections and minority indices of R_2. But R_2 has the same number of witnesses as R_1, so Q witnesses all $k-1$-element projections and minority indices that can be witnessed in R_2. Thus by Theorem 2.8, Q generates both R_1 and R_2, and therefore $R_1 = R_2$. □

3. Combined algorithm

In this section let $\mathbf{A}$ be a fixed finite idempotent algebra and $\alpha \in \mathrm{Con}(\mathbf{A})$ be a congruence such that $\mathbf{A}/\alpha$ has an edge term. We will slightly modify the definitions of the previous section to get our new tractability result.

Definition 3.1. Let V be a set of variables and $I \subseteq V$ be a constraint scope. By a *Maltsev constraint on* I we mean a subuniverse M_I of $\mathbf{A}^I \times (\mathbf{A}/\alpha)^{V \setminus I}$. Thus for each $f \in M_I$ and $i \in I$, $j \in V \setminus I$ we have $f(i) \in A$ and $f(j) \in A/\alpha$.

As can be seen in the definition, each Maltsev constraint not only specifies the behavior of the solutions on the scope of variables, but also on the other variables modulo α. Since $\mathbf{A}$ is idempotent and $\mathbf{A}/\alpha$ has few subpowers, we can fully specify Maltsev constraints by compact representations.

Definition 3.2. By the *compact representation* of a Maltsev constraint $M_I \leq \mathbf{A}^I \times (\mathbf{A}/\alpha)^{V \setminus I}$ we mean the collection of compact representations of the subuniverses

$$M_I(g) = \{\, h \in (\mathbf{A}/\alpha)^{V \setminus I} : (g, h) \in M_I \,\}$$

for all choices of $g \in A^I$.

Note, that here we use that $\mathbf{A}$ is idempotent, otherwise $M_I(g)$ would not be a subuniverse of $(\mathbf{A}/\alpha)^{V \setminus I}$.

Definition 3.3. A *Maltsev instance* is a triple $(V, \mathcal{S}, \mathcal{M})$ where V is the set of variables, $\mathcal{S} \subseteq P(V)$ is the set of constraint scopes and

$$\mathcal{M} = \{\, M_I \leq \mathbf{A}^I \times (\mathbf{A}/\alpha)^{V \setminus I} : I \in \mathcal{S} \,\}$$

is an indexed set of Maltsev constraints given by their compact representation. A solution is a function $f : V \to A$ such that $(f|_I, (f|_{V \setminus I})/\alpha) \in M_I$ for all $I \in \mathcal{S}$.

We want to use a version of the consistency checking algorithm, but for that we need the following notion of notations of extension and projection of a Maltsev constraints.

Definition 3.4. Let $M_I \leq \mathbf{A}^I \times (\mathbf{A}/\alpha)^{V\setminus I}$ be a Maltsev constraint and $J \subseteq V$ be a constraint scope (neither $I \subseteq J$ nor $J \subseteq I$ is assumed). By the *projection* of M_I to J we mean the Maltsev constraint on J defined as

$$\mathrm{pro}_J(M_I) = \{\, (f|_J, (f|_{V\setminus J})/\alpha) \in \mathbf{A}^J \times (\mathbf{A}/\alpha)^{V\setminus J} : f \in \mathbf{A}^V, (f_I, (f_{V\setminus I})/\alpha) \in M_I \,\}.$$

Lemma 3.5. *Suppose, that $I, J \subseteq V$, $\max(|I|, |J|)$ is bounded by some constant and M_I is a Maltsev constraint on I. If $I \subseteq J$ or $J \subseteq I$, then the projection of M_I to J can be computed in polynomial time (in the number $|V|$ of variables).*

Proof. Since the sizes of I and J are bounded, we only have to verify that for all $g \in A^J$ the compact representation of the subpower $\mathrm{pro}_J(M_I)(g) \leq (\mathbf{A}/\alpha)^{V\setminus J}$ can be computed in polynomial time.

First consider the case when $I \subseteq J$. Then by definition,

$$\mathrm{pro}_J(M_I)(g) = \{\, h|_{V\setminus J} : h \in M_I(g|_I) \text{ and } h|_{J\setminus I} = (g_{J\setminus I})/\alpha \,\}.$$

This compact representation is defined by a primitive positive formula (using constant relations and projections) using $M_I(g|_I)$, thus by Lemma 2.10 it can be computed in polynomial time.

Now consider the case when $J \subseteq I$. In this case

$$\mathrm{pro}_J(M_I)(g) = \{\, (g'|_{I\setminus J}, h) : \exists g' \in A^{I\setminus J}, h \in M_I((g, g'|_{I\setminus J})) \,\}.$$

Clearly, if g' is fixed, then the compact representation of

$$\{\, (g'|_{I\setminus J}, h) : h \in M_I((g, g'|_{I\setminus J}) \,\}$$

can be computed from that of $M_I((g, g'|_{I\setminus J})$ (e.g., by using Lemma 2.10). Thus we know that $\mathrm{pro}_J(M_I)(g)$ is a subuniverse of $\mathbf{A}^{V\setminus J}$, and it is a disjoint union of subuniverses whose compact representations we can compute in polynomial time. Then by Lemma 2.11 we can compute the compact representation of $\mathrm{pro}_J(M_I)(g)$ in polynomial time. □

Definition 3.6. We say that a Maltsev instance $(V, \mathcal{S}, \mathcal{M})$ of $\mathrm{CSP}(\mathbf{A})$ is *k-consistent*, if $S = \{\, S \subseteq V : |S| \leq k \,\}$, and for each $I, J \in \mathcal{S}$ with $I \subseteq J$ we have $M_I = \mathrm{pro}_I(M_J)$. The instance if empty if $R_I = \emptyset$ for all (any) $I \in \mathcal{S}$.

Lemma 3.7. *Let k be a fixed integer and $(V, \mathcal{S}, \mathcal{R})$ be an instance of $\mathrm{CSP}(\mathbf{A})$ with relations of arity at most k. Then there exists a unique k-consistent Maltsev instance $(V, \mathcal{S}', \mathcal{M}')$ that can be computed in polynomial time and has the same set of solutions as $(V, \mathcal{S}, \mathcal{R})$.*

Proof. First, put $\mathcal{M} = \{\, M_I : I \in \mathcal{S} \,\}$ where $M_I = R_I \times (\mathbf{A}/\alpha)^{V\setminus I}$. Clearly $(V, \mathcal{S}, \mathcal{M})$ is a Maltsev instance and has the same set of solutions as $(V, \mathcal{S}, \mathcal{R})$. Next put $\mathcal{S}' = \{\, I \subseteq V : |I| \leq k \,\}$ and define $M_I = \mathbf{A}^I \times (\mathbf{A}/\alpha)^{V\setminus I}$ for $I \in \mathcal{S}' \setminus \mathcal{S}$ and put $\mathcal{M}' = \{\, M_I : I \in \mathcal{S}' \,\}$. Every function satisfies these new Maltsev constraints, so $(V, \mathcal{S}', \mathcal{M}')$ has the same set of solutions as $(V, \mathcal{S}, \mathcal{R})$.

The instance $(V, \mathcal{S}', \mathcal{M}')$ is not yet k-consistent, but if for $I, J \in \mathcal{S}'$, $I \subseteq J$ we have a failure $M_I \neq \mathrm{pro}_I(M_J)$ of consistency, then we can replace M_I with $M_I \cap \mathrm{pro}_I(M_J)$ and M_J with $M_J \cap \mathrm{pro}_J(M_I)$. From Definition 3.4 it is clear that any function that satisfies the constraint M_I (or M_J) also satisfies the constraint $\mathrm{pro}_J(M_I)$ ($\mathrm{pro}_I(M_J)$, respectively), therefore the smaller Maltsev instance has the same set of solutions as the original one.

Since in each modification step the total number witnessed small projects and minority indices in all Maltsev relations M_I, $I \in \mathcal{S}'$ must decrease by Lemma 2.12, and this total number was bounded by a polynomial in n, therefore this consistency failure correcting procedure must stop in polynomially many steps. Thus we have shown the existence of a k-consistent Maltsev instance $(V, \mathcal{S}', \mathcal{M}')$ that has the same set of solutions as $(V, \mathcal{S}, \mathcal{R})$.

The argument for the uniqueness of $(V, \mathcal{S}', \mathcal{M}')$ is analogous to the proof of Theorem 2.3. If $(V, \mathcal{S}'', \mathcal{M}'')$ is a k-consistent Maltsev instance inside of a Maltsev instance $(V, \mathcal{S}', \mathcal{M}')$, then in each consistency failure correcting step applied to $(V, \mathcal{S}', \mathcal{M}')$ the new Maltsev instance will still contain $(V, \mathcal{S}'', \mathcal{M}'')$. □

Lemma 3.8. *Let $(V, \mathcal{S}, \mathcal{M})$ be a k-consistent Maltsev instance of* CSP(A)*. Then for each element $h \in M_\emptyset$, the instance $(V, \mathcal{S}, \mathcal{R})$ with $\mathcal{R} = \{ R_I : I \in \mathcal{S} \}$ and*

$$R_I = \{ f|_I : f \in M_I, ((f|_I)/\alpha, f|_{V \setminus I}) = h \}$$

is a k-consistent instance of CSP$(\mathbf{A})$*. Moreover, $(V, \mathcal{S}, \mathcal{M})$ is empty if and only if $(V, \mathcal{S}, \mathcal{R})$ is empty.*

Proof. Follows easily from the definitions. □

Theorem 3.9. *Let $\mathbf{A}$ be an idempotent algebra, $\alpha \in \operatorname{Con} \mathbf{A}$ such that $\mathbf{A}/\alpha$ has few subpowers and the α-blocks (which are subalgebras of $\mathbf{A}$) generate a congruence meet semi-distributive variety. Then* CSP$(\mathbf{A})$ *can be solved in polynomial time for instance where the maximum arity of used relations is bounded.*

Proof. Take an instance $(V, \mathcal{S}, \mathcal{R})$ of CSP$(\mathbf{A})$ and let k be a bound on the maximum arity of relations in $\mathcal{R}$. By Lemma 3.7 we can convert the instance $(V, \mathcal{S}, \mathcal{R})$ into an equivalent Maltsev $2\lfloor \frac{k}{2} \rfloor$-consistent instance $(V, \mathcal{S}', \mathcal{M}')$. If $(V, \mathcal{S}', \mathcal{M}')$ is empty, then $(V, \mathcal{S}, \mathcal{R})$ have no solution by Lemmas 3.7 and 3.8. On the other hand, if $(V, \mathcal{S}', \mathcal{M}')$ is non-empty, then we can take an element $h \in M_\emptyset$. By Lemma 3.8 we have a nonempty $2\lfloor \frac{k}{2} \rfloor$-consistent instance $(V, \mathcal{S}', \mathcal{R}')$ strategy, such that for each $i \in V$ the domain of the i-th variable is a subuniverse of an α-class. We can encode this instance as an instance of CSP$(\mathbf{B})$ where

$$\mathbf{B} = \prod_{i \in i} h(i)$$

is the α-classes that occur. Since the α-classes generate congruence meet semi-distributive varieties, $\mathbf{B}$ is also generates an congruence meet semi-distributive variety, therefore by Theorem 2.4 we know this instance has a solution. Therefore $(V, \mathcal{S}, \mathcal{R})$ has a solution as well. □

Corollary 3.10. *Let $\mathcal{V}$ be a pseudo-variety (containing finite algebras closed under subalgebras, homomorphic images and finite direct products) generated by algebras of bounded width and Maltsev algebras. Then for every algebra $\mathbf{A} \in \mathcal{V}$ and finite set of relations $\Gamma \subset \mathsf{SP}(\mathbf{A})$ the problem* CSP(Γ) *can be solved in polynomial time.*

Proof. We know that CSP$(\mathbf{A})$ can be solved in polynomial time both for algebras of bounded width and Maltsev algebras. It is true in general that if CSP$(\mathbf{A})$ can be solved in polynomial time, then the same holds for any subalgebras and homomorphic images of $\mathbf{A}$. So the only problem we face is about finite products.

We know that finite products of bounded width algebras has bounded width as well (from the SD$(\wedge)$ description). The same result is true algebras of bounded width, by an observation of R. McKenize, that the direct product of algebras of few subpowers has few subpowers.

Finally, we have to show that the direct product of a Maltsev algebra and n algebra of bounded width is tractable. However, in the direct product we do have the α projection congruence which satisfies the conditions of Theorem 3.9, which finishes the proof. □

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